# Cauchy horizon singularity without mass inflation

Patrick R. Brady
*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1, Canada*

Darío Núñez
*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1, Canada*
*and Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,*
*Circuito Exterior, Ciudad Universitaria, Apartado Postal 70-543, México, Distrito Federal 04510, México*

Sukanya Sinha
*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,*
*Circuito Exterior, Ciudad Universitaria, Apartado Postal 70-543, México, Distrito Federal 04510, México*
(Received 19 November 1992)

A perturbed Reissner–Nordström–de Sitter solution is used to emphasize the nature of the singularity along the Cauchy horizon of a charged spherically symmetric black hole. For these solutions, conditions may prevail under which the mass function is bounded and yet the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges.

PACS number(s): 97.60.Lf, 04.20.Jb

## I. INTRODUCTION

The external gravitational field of a black hole formed by collapse settles down to the Kerr-Newman solution of general relativity. These generic stationary black hole solutions have a rather curious feature—the inner horizon acts as the boundary of predictability for external initial value problems. However, Penrose [1] pointed out that this inner (Cauchy) horizon is a surface of infinite blueshift, and thus partial absorption of the radiative tail of the gravitational collapse leads to a divergent flux of energy along it. This suggests that the back reaction of perturbations on the geometry may generate unbounded curvature along the Cauchy horizon (CH). Detailed calculations, which analyzed the evolution of gravitational perturbations, confirmed the divergence of the energy flux [2], and for charged spherical black holes Poisson and Israel (PI) [3] showed that a scalar curvature singularity does form at the CH when the above influx is accompanied by the outflux emitted from the collapsing star. Since they found the mass function diverges, they called it a *mass inflation* singularity. An exact solution, based on the work of PI, was used by Ori to examine the nature of the singularity in some detail [4], and he showed that an observer who falls into the black hole experiences finite tidal distortion at this singularity.

The presence of a cosmological constant (of arbitrary magnitude) modifies this picture in a nontrivial manner, and serves to emphasize an aspect of the singularity along the CH which has so far been ignored. The leading divergence of the curvature scalars is not contained in the Weyl scalar, but rather in $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.

For a generic perturbation of a Reissner-Nordström black hole in de Sitter space, the CH can be stable for some values of the charge ($e$) and the mass ($m$) [5, 6]. These configurations are quite rare though, and require $|e| > m$. In [5] it was speculated that this spacetime might provide a counterexample to the conjecture that nonscalar curvature singularities are catalyzed into stronger scalar curvature singularities, once perturbed. This was based on the observation that the Weyl curvature of the solution to the Einstein field equations (for simple spherical perturbations) is dominated by the proper time integral of the energy flux along the horizon. It was shown in [5] that this flux behaves like $\mathcal{F}(v) \sim e^{2(\kappa_i - \kappa_c)v}$ and

$$\Psi_2 \propto \int \mathcal{F}(v) e^{-\kappa_i v} dv \sim e^{(\kappa_i - 2\kappa_c)v} , \tag{1.1}$$

where $v \to \infty$ on the CH. The constants $\kappa_i$ and $\kappa_c$ are the surface gravities of the inner and cosmological horizons, respectively. Thus one finds values of the charge and the external mass of the black hole for which $\kappa_c < \kappa_i < 2\kappa_c$, and the observed energy density grows without bound, but $\Psi_2$ is finite.

Although these observations are correct, a scalar curvature singularity does form along the CH contrary to the speculation in [5]. We will demonstrate this fact using a model in which perturbations of a Reissner–Nordström–de Sitter black hole are modeled by crossflowing streams of null dust, as in [3]. The singularity is characterized by the divergence of $R_{\alpha\beta}R^{\alpha\beta}$, where the dominant behavior is proportional to the influx of blueshifted radiation. We wish to emphasize that this behavior should be present in all spherically symmetric solutions, where the outflux from the star is continuous, and does not require a nonzero cosmological constant. However, the presence of a cosmological term shows clearly that the divergence of the mass function is not *necessary* for the presence of a scalar curvature singularity.

## II. AN APPROXIMATE SOLUTION

The spherically symmetric line element can be written as

 

$$ds^2 = d\sigma^2 + r^2 d\Omega^2, \qquad d\sigma^2 = g_{ab}dx^a dx^b, \tag{2.1}$$

where $d\Omega^2$ is the line element on the unit two-sphere, and latin indices $a, b, \ldots$, range over (0,1). Along with the scalar function $r(x^a)$ we introduce $m(x^a)$, $f(x^a)$, and $\kappa(x^a)$ defined by

$$g^{ab}r_{,a}r_{,b} = f(x^a) = 1 - \frac{2m(x^a)}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3}r^2, \tag{2.2}$$

$$\kappa = -\tfrac{1}{2}\partial_r f. \tag{2.3}$$

It is also convenient to use null coordinates $U$, $V$ on the "radial" two spaces so that $d\sigma^2 = -2e^{2\nu}dUdV$ and $\nu = \nu(U,V)$. It is now easy to generalize the derivation of the field equations in [3] to include a cosmological constant. The result is

$$r_{;ab} + \kappa g_{ab} = -4\pi r T_{ab}\,, \tag{2.4}$$

$$\Box\nu = -\partial_r\kappa = -\frac{1}{r^3}\left(2m - \frac{3e^2}{r} + \frac{\Lambda}{3}r^3\right), \tag{2.5}$$

$$\Box m = -(4\pi)^2 r^3 T_{ab}T^{ab}\,, \tag{2.6}$$

where a semicolon indicates the covariant derivative associated with $d\sigma^2$, and $T_{ab}$ is the 2×2 submatrix of the stress-energy tensor.

For crossflowing null dust, the stress-energy tensor is

$$T_{\mu\nu} = \rho_{\rm in} l_\mu l_\nu + \rho_{\rm out} n_\mu n_\nu\,, \tag{2.7}$$

where $l_\mu = -\partial_\mu V$ and $n_\mu = -\partial_\mu U$ are radial null vectors pointing inward and outward, respectively, and, $\rho_{\rm in}$ and $\rho_{\rm out}$ represent the energy densities of the inward and outward fluxes. Covariant conservation requires that they are given by

$$\rho_{\rm in} = \frac{L_{\rm in}(V)}{4\pi r^2}\,, \qquad \rho_{\rm out} = \frac{L_{\rm out}(U)}{4\pi r^2}\,. \tag{2.8}$$

Let us imagine that the inflow is turned on at a finite advanced time $V = V_0$ and the outflow is turned on at $U = U_0$. In the pure inflow (outflow) regime the solution is an ingoing (outgoing) Vaidya–Reissner–Nordström–de Sitter spacetime with mass function $m(V)$ [$m(U)$]. The global structure of the spacetime with crossflow is shown in the Penrose diagram of Fig. 1. The loci of the three apparent horizons, in the pure inflow region, are given by the positive roots of $f = 0$. The ingoing sheet of the inner horizon is $r = r_i$, and of the cosmological horizon is $r = r_c$. The outer apparent horizon settles down to the constant radius $r = r_e$, for asymptotically large $v$, and the function $\kappa$ evaluated at a particular horizon gives the surface gravity of that horizon.

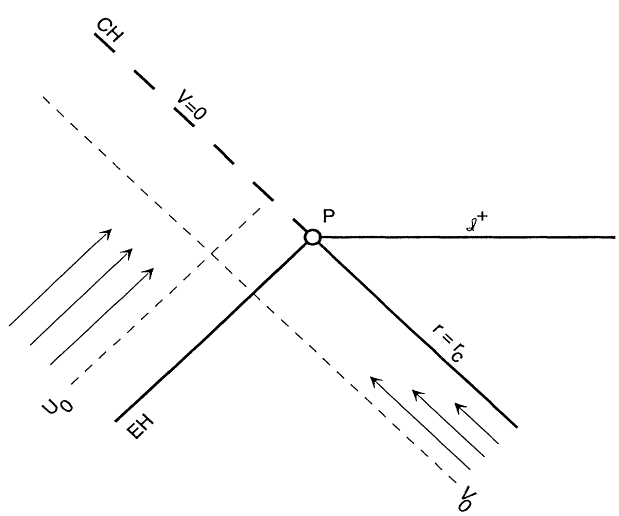


FIG. 1. The global structure of the spacetime, with crossflowing null dust. EH is the event horizon, CH is the Cauchy horizon, and the cosmological horizon is at $r = r_c$. The lines $U = U_0$ and $V = V_0$ at which the fluxes are switched on are also shown.

We choose $V$ such that $V = 0$ at the Cauchy horizon. Note that this does not specify the coordinate completely, but ensures that it may be regularly related to the Kruskalized advanced time associated with the inner apparent horizon. Our eventual aim is to construct an approximate solution which is valid as $V \to 0$.

Taking the trace of (2.4) we find that $r(U,V)$ also satisfies

$$\Box r = -2\kappa\,. \tag{2.9}$$

Using (2.2), (2.5), and (2.9) we obtain the wave equation

$$\Box[\ln(re^{2\nu})] = \frac{1}{r^4}(3e^2 - r^2 - \Lambda r^4)\,, \tag{2.10}$$

which can be integrated formally to

$$re^{2\nu} = \alpha g_1(U) g_2(V) \exp[F(U,V)]\,, \tag{2.11}$$

with

$$F(U,V) = \frac{1}{2}\int_{U_0}^{U}\int_{V_0}^{V} dU'dV' \frac{e^{2\nu'}}{(r')^4}\left[(r')^2 - 3e^2 + \Lambda(r')^4\right]. \tag{2.12}$$

The functions $g_1(U)$ and $g_2(V)$ are determined by the boundary conditions along the null rays $V = V_0$ and $U = U_0$, and $\alpha$ is a constant with the dimension of length. In [7] and [3] it was shown that null coordinates exist such that $g_2(V)$ is well behaved as $V \to 0$; in particular, we can rescale the coordinates so that $g_2(0) = 1$. This guarantees that $V = -(\text{const}) \times e^{-\kappa_i v}$, where $\kappa_i$ is the surface gravity of the static portion of the inner horizon. Similarly, provided we limit ourselves to values of $U$ near to $U_0$, we may set $g_1(U) = 1$ without loss of generality.

In order to proceed, we must estimate the behavior of the integral (2.12). The crucial observation (which was made in [3] but not carried to its full consequences) is that the integrand is negative for $r \leq r_i$, and we expect the leading behavior to come from near the CH. Thus, if it diverges, the integral must diverge to negative values and $re^{2\nu} \to 0$. As an immediate consequence of this, we see from (2.12) that this would require that $r$ go to zero. Since we expect $r(U,V)$ to be a well-behaved function, with the *slow* contraction of ingoing null rays governed primarily by the outflux from the star, we conclude that a good approximation to the metric coefficient is

$$e^{2\nu} \simeq \frac{\alpha \exp[F(U,0)]}{r}\,, \tag{2.13}$$

near to $V = 0$.

With this ansatz we can rewrite (2.4) as

$$(r^2)_{,UU} - F_{,U}(r^2)_{,U} \simeq -2L_{\rm out}(U), \qquad (2.14)$$

$$(r^2)_{,VV} \simeq -2L_{\rm in}(V)\ , \qquad (2.15)$$

where $F$ is treated as a function of $U$ alone from here on. The solution of these equations is then given by

$$r^2 \simeq r^2(U_0,V_0) - 2\int_{V_0}^{V} dV' \left[\int_{V_0}^{V'} dV'' L_{\rm in}(V'')\right] + \int_{U_0}^{U} e^{F'}\left[V - 2\int_{U_0}^{U'} e^{-F''}L_{\rm out}(U'')dU''\right] dU'\ . \qquad (2.16)$$

We still have not determined in detail what the function $F(U,0)$ is, but it is unimportant in the ensuing analysis.

Before proceeding to analyze the curvature singularity which forms due to the infinitely blueshifted influx, we present the solution of Eq. (2.6) for the mass function. In these coordinates and using (2.13) it is easy to integrate this to

$$m(U,V) \simeq \alpha^{-1}\int_{U_0}^{U}\left[e^{-F'}L_{\rm out}(U')\right] dU' \int_{V_0}^{V}[L_{\rm in}(V')]\, dV' + m(U_0,V_0) - \beta\int_{V_0}^{V}[V'L_{\rm in}(V')]\, dV' + \gamma\int_{U_0}^{U}[U'L_{\rm out}(U')]\, dU'\ , \qquad (2.17)$$

where $\beta$ and $\gamma$ are constants. The contribution from the pure inflow (outflow) region is finite for realistic perturbations, and it is only when both fluxes are present that the first term can lead to a divergence of the mass function.

The functional form of the luminosities $L_{\rm out}(U)$ and $L_{\rm in}(V)$ is as yet unspecified, so we now specialize to the case of interest. In [5] and [6] it was shown that for a generic perturbation the initial conditions on the influx should be

$$L_{\rm in}(V) \sim K(V)[-V/\alpha]^{\frac{2(\kappa_c-\kappa_i)}{\kappa_i}}\ , \qquad (2.18)$$

near to the CH. This is proportional to the energy density of the influx, as measured by a free-falling observer crossing the CH in the pure inflow region. Notice that it diverges provided $\kappa_i > \kappa_c$. The behavior of the prefactor in (2.18) is unimportant, so we take $K(V) \to$ const as $V \to 0$ (if the cosmological constant is zero, then $\kappa_c = 0$ and $K(V)$ is given by the analysis of Price [8] as $K(V) \sim |\ln(-V/\alpha)|^{-p}$). Since it is slowly varying compared to the other factor in (2.18), we can write

$$\int dV L_{\rm in}(V) \sim K(V)[-V/\alpha]^{\frac{2\kappa_c-\kappa_i}{\kappa_i}}\ . \qquad (2.19)$$

The mass function is proportional to (2.19), so it will diverge as $V \to 0$ unless $2\kappa_c > \kappa_i$.

## III. THE CAUCHY HORIZON SINGULARITY

Our solution, given by (2.13), (2.16), and (2.17), gives conditions under which the mass function is finite, and shows that the metric is well behaved provided $r$ is bounded away from zero. In this section we examine the nature of the singularity that is present for a general outflux. There are three essentially different possibilities: (i) complete stability when $\kappa_c > \kappa_i$, (ii) divergent influx but finite-mass function when $2\kappa_c > \kappa_i > \kappa_c$, and (iii) both divergent influx and mass function when $\kappa_c < \kappa_i$. Defining $a = e^2/m_1^2$ and $b = \Lambda m_1^2/3$, where $m_1$ is the asymptotic value of the mass function in the pure inflow region, we have plotted these conditions in Fig. 2. Some useful relations involving the surface gravities are also given in the Appendix.

We wish to focus our analysis on case (ii), where no mass inflation has occurred despite the presence of the infinite energy density. At first sight, one is tempted to conclude that the spacetime is regular, since we know that the mass function characterizes the Weyl curvature scalar $\Psi_2 \propto m/r^3$ within spherical symmetry. However, with a moment's thought one realizes that Eq. (2.7) implies that $R_{\alpha\beta}R^{\alpha\beta}$ diverges on the CH. Thus it should come as no surprise that the Kretchsmann invariant, given by

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \simeq 8r_{,VV}\left[\frac{(r^2)_{,UU} - F_{,U}(r^2)_{,U}}{\alpha^2 e^{2F} r}\right] + \dots \qquad (3.1)$$

$$\sim 16\frac{L_{\rm out}(U)L_{\rm in}(V)}{\alpha^2 e^{2F} r^2} \qquad (3.2)$$

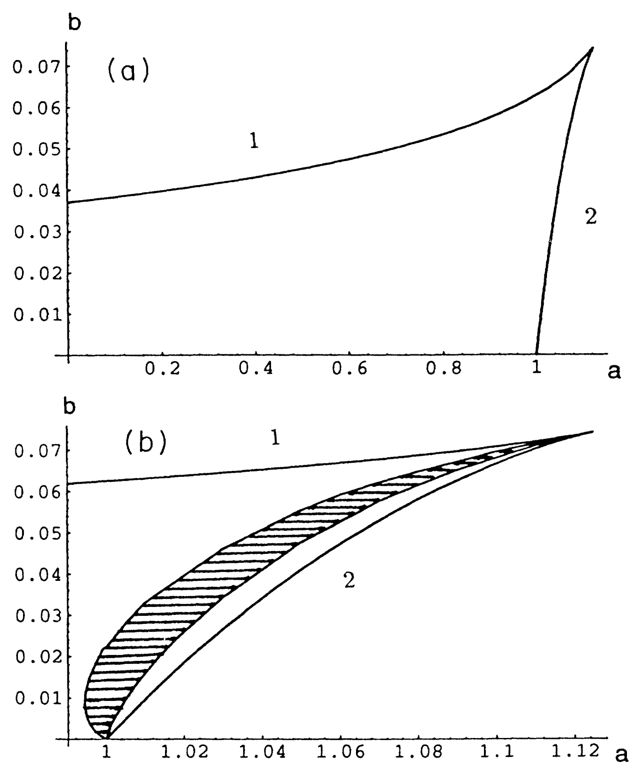


FIG. 2. The $ab$ plane. Along curve 1 $r_c = r_e$, and along curve 2 $r_i = r_c$. (a) The region of physical interest is bounded by the axes and curves 1 and 2. (b) A close up of the physical region. In the shaded portion no mass inflation occurs, but energy densities are divergent on the CH. Above this, both the mass function and the energy density diverge, and below is the region of complete stability.

to leading order, also diverges like the measured energy density provided that the outflux from the star is nonvanishing. This shows that a scalar curvature singularity forms along the CH whenever a blueshifted influx and continuous outflux from the star are present.

## IV. CONCLUSION

For completeness, we will take the Ori model limit of our solution. This is achieved by replacing the continuous outflux by a $\delta$-function source at $U = U_0$ and then integrating over $U$ where possible. The result is

$$r^2 \simeq r^2(U_0, V_0) - 2\int_{V_0}^{V} dV' \left[\int_{V_0}^{V'} dV'' L_{\rm in}(V'')\right] + \int_{U_0}^{U} e^{2F'}\,[V - 2b]\,dU' , \tag{4.1}$$

$$m(V) \simeq \text{const} \times \int_{V_0}^{V} [L_{\rm in}(V')]\,dV' - \beta\int_{V_0}^{V} [V' L_{\rm in}(V')]\,dV' + \cdots , \tag{4.2}$$

where $b = e^{-F}\big|_{U=U_0}$, and is in complete agreement with [4]. Since $L_{\rm out}(U) \equiv 0$ for $U > U_0$, it is clear that the leading term in (3.2) is absent, and the divergence of the curvature scalars is proportional to the mass function. Therefore this model misses the leading behavior discussed above, but we wish to stress that the tidal forces (and hence tidal distortion) experienced by a freefalling observer in both models are almost identical.

In our opinion the result (3.2) is rather important. It shows that $L_{\rm out}(U)$ must be continuous in order to see the full nature of the singularity which forms along the CH. Based on this observation it is tempting to speculate that asphericities may act to change the nature of the singularity by pushing divergences [proportional to $L_{\rm in}(V)$] into the Weyl curvature. Indeed, asymptotic and perturbative analyses [9] of the singularity inside a more realistic black hole suggest that the leading terms are in the radiative part of the Weyl curvature. It seems to us very important, then, to further investigate the internal structure of black holes in aspherical models.

At the present time, the indications are that the singularity which forms inside a realistic black hole is null. A question which deserves some attention is, how generic is this null picture? Within spherical symmetry, we see that the singularity becomes spacelike when $r(U,V) \to 0$. The decrease in $r(U,V)$ (near the CH) is governed primarily by the outflux from the star [Eq. (2.16)], and thus the null portion of the singularity can be quite large. It is possible (likely) that the presence of shear in more realistic models will change this, provoking a spacelike singularity which is asymptotically null near $P$ in Fig. 1. This would be consistent with the perturbative analysis in [9] and, as pointed out by Yurtsever [10], would be similar to the situation in plane-wave spacetimes where perturbations do not capture the generic structure for colliding waves. Finally we suggest that it is rather premature to discuss the strength of the singularity which forms, or the possible fate of an astronaut who falls into a black hole, since it is likely that tidal forces will be substantially enhanced if the singularity in more realistic models is spacelike.

Some of these issues are currently under investigation and details will be presented elsewhere.

## ACKNOWLEDGMENTS

It is a great pleasure to thank Werner Israel for fruitful comments and encouragement, and also Eric Poisson for constructive criticism which helped us clarify the exposition. D.N. is grateful for the financial support of the External Affairs and International Trade Canada, Government of Canada, administrated by the International Council for Canadian Studies. He also thanks DGAPA and UNAM for support. This work was also partly supported by the Natural Sciences and Engineering Research Council of Canada. Figures 2(a) and 2(b), and some calculations in this work, were done using MATHEMATICA©.

## APPENDIX: THE SURFACE GRAVITIES

In order to obtain the physical regions in the $ab$ plane in Fig. 2, and to examine the conditions on the surface gravities we found the following useful.

Lake [11] introduced a rescaled quantity $x = r/m_1$, where $m_1$ is the mass of the black hole, so that the scalar function $f$ is given by

$$f = \frac{-b}{m_1^2 x^2}\left(x^4 - \frac{x^2}{b} + \frac{2x}{b} - \frac{a}{b}\right) . \tag{A1}$$

The surface gravity at any horizon $r$ is similarly given by

$$\kappa_r = \frac{1}{m_1 b {x^3}_r}\left|{x^2}_r - 3x_r + 2a\right| . \tag{A2}$$

By construction, the roots of (A1) satisfy $0 \le x_i \le x_e \le x_c$, and we know that the surface gravities are always greater than zero. These facts and (A2) tell us that

$$x_i \le \frac{3 - \sqrt{9 - 8a}}{2} \le x_e \le \frac{3 + \sqrt{9 - 8a}}{2} \le x_c . \tag{A3}$$

We now observe that for $a = 9/8$ we have $x_e = 3/2$ and, since this satisfies (A1), $b = 2/27$. In fact, for these values of $a$ and $b$ we have $x_i = x_e = x_c$ [11].

Finally, the coalescence of the two or more roots occurs when $\kappa_r = 0$. Using (A1) gives

$$b = \frac{y^2 - 2y + a}{y^4} , \quad y = \frac{3 \pm \sqrt{9 - 8a}}{2} , \tag{A4}$$

which allows us to plot the boundary of the regions of interest.

[1] R. Penrose, in *Battelle Rencontres,* edited by C.M. DeWitt and J.A. Wheeler (Benjamin, New York, 1968), p. 222.

[2] M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973); S. Chandrasekhar and J.B. Hartle, Proc. R. Soc. London **A284**, 301 (1982).

[3] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990), and references therein.

[4] A. Ori, Phys. Rev. Lett. **67**, 789 (1991).

[5] P.R. Brady and E. Poisson, Class. Quantum Grav. **9**, 121 (1992).

[6] F. Mellor and I. Moss, Class. Quantum Grav. **6**, 1379 (1990); Phys. Rev. D **41**, 403 (1990); Class. Quantum Grav. **9**, L43 (1992).

[7] B. Waugh and K. Lake, Phys. Rev. D **34**, 2978 (1986).

[8] R.H. Price, Phys. Rev. D **5**, 2419 (1972).

[9] W. Israel (private communication); A. Ori, Phys. Rev. Lett. **68**, 2117 (1992).

[10] U. Yurtsever, Class. Quantum Grav. **10**, L17 (1993).

[11] K. Lake, Phys. Rev. D **19**, 421 (1979).