Semiclassical extremal black holes

Sandip P. Trivedi
Lauritsen Laboratory of High Energy Physics, California Institute of Technology, Pasadena, California 91125
(Received 13 November 1992)

Extremal black holes are studied in a two-dimensional model motivated by a dimensional reduction from four dimensions. Their quantum-corrected geometry is calculated semiclassically and a mild singularity is shown to appear at the horizon. Extensions of the geometry past the horizon are not unique but there are continuations free from malevolent singularities. A few comments are made about the relevance of these results to four dimensions and to the study of black hole entropy and information loss.

PACS number(s): 97.60.Lf, 04.60.+n, 11.17.+y

I. INTRODUCTION

Hawking's discovery of black hole radiance [1,2] raises several intriguing questions. It shows that black holes have an entropy which can be elegantly expressed in terms of their geometry but which remains mysterious in terms of any underlying microstates. It also suggests that because of the thermal nature of the outgoing radiation, a loss of quantum coherence might occur in processes involving black holes. Extremal black holes provide a convenient setting in which to address both these issues. Their zero temperature suggests that their entropy should be explained in terms of a degeneracy of ground states. It also makes them convenient toy laboratories in which to study scattering and a potential loss of quantum coherence.

In this paper we study a model consisting of dimensionally reduced gravity and electromagnetism coupled to two-dimensional scalar fields. Classically, this model has Reissner-Nordström black hole solutions, obtained from dimensionally reducing the usual four-dimensional charged black hole solutions. In this paper we concentrate for the most part on the extremal black holes which in Planck units have a mass equal to their charge and have zero Hawking temperature. We show that, contrary to expectations, the quantum stress tensor of a scalar field in the background of such an extremal black hole blows up at the horizon. This raises the possibility of their geometry being drastically modified in the vicinity of the horizon and their entropy being very different from what classical considerations would suggest. A semiclassical calculation of their quantum-corrected geometry shows, however, that this is not true. For large black holes, the value of the dilaton at the horizon and, hence, their entropy, [1] stays large. A singularity does appear at the horizon, but it is very mild. For example, tidal forces and the curvature stay finite at the horizon. This suggests that there should be a continuation of the geometry past the horizon. In fact, there is more than one such continuation, even when we restrict ourselves to static solutions. In one of these, the causal structure of spacetime is much like the classical extremal solution. But there is another continuation possible in which the causal structure is much different and in which there are no malevolent singularities. We conclude with a brief discussion of the relevance of our results to the study of four-dimensional extremal black holes and to the study of black hole entropy and information loss.

The study of quantum effects in two-dimensional black holes was first undertaken in the 1970s in some very interesting papers which include Refs. [3–7]. More recently, considerable interest was renewed by the discovery of a two-dimensional black hole solution in the work of Mandal, Sengupta, and Wadia [8], and Witten [9]. This solution was then used to study questions related to Hawking evaporation in the work of Callan, Giddings, Harvey, and Strominger (CGHS) [10]. Two recent papers which review the subsequent developments are Refs. [11, 12]. Papers especially relevant to the work presented here include Refs. [13–18]. Papers which discuss dimensionally reduced models include Refs. [19–21]. To our knowledge, the first published reference to the idea of using extremal black holes for studying issues related to Hawking radiation is in the paper of Preskill *et al.* [22].

II. THE MODEL

We start in four dimensions and make a spherical symmetric ansatz for the metric, which gives

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\phi} d\Omega \ . \tag{1}$$

Here $g_{\alpha\beta}$ is the two-dimensional metric in the r-t plane and $e^{-2\phi}$, which we call the dilaton, is the square of the radius of the two-sphere. The Einstein-Hilbert action then takes the form

$$S_G = \frac{1}{4G}\int d^2x \sqrt{g}\, e^{-2\phi}[R + 2(\nabla\phi)^2 + 2e^{2\phi}] \ . \tag{2}$$

We note that this differs from the action considered by CGHS [10] in the form of the dilaton potential, i.e., the last term above. The term considered here prevents the action S_G from scaling simply under the transformation $\phi \to \phi + c$.

[1]The entropy of these black holes depends on the dilaton and is large if the value of the dilaton at the horizon is large.

Proceeding similarly, the Maxwell field can be dimensionally reduced to give an action

$$S_{\rm EM} = -\frac{1}{4G}\int d^2x\sqrt{g}\,e^{-2\phi}F^2 \;, \tag{3}$$

where F^2 now refers to the field strength of a two-dimensional gauge field.

Spherically symmetric charged black hole solutions of the original four-dimensional theory continue to be solutions of this theory. They are given by a metric

$$ds^2 = -\left[1-\frac{2M}{r}+\frac{Q^2}{r^2}\right]dt^2+\frac{1}{1-2M/r+Q^2/r^2}dr^2 \tag{4}$$

and a dilaton

$$e^{-2\phi}=r^2 \;. \tag{5}$$

The corresponding field strength is

$$F_{rt}=\frac{Q}{r^2} \;. \tag{6}$$

Here M is the mass and Q the charge of the black hole. We will be mainly interested here in extremal black holes for which $M=Q$.

In order to incorporate quantum effects in a manageable way, we couple the above theory to N conformaly coupled scalar fields in two dimensions. In total, the action of the two-dimensional theory is then

$$S=\int d^2x\sqrt{-g}\left[\frac{1}{4G}\left[e^{-2\phi}R+e^{-2\phi}2(\nabla\phi)^2+2-\frac{1}{4}F^2\right] -\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right] \;. \tag{7}$$

The parameter N allows us to consider the theory in the large-N limit in which $N\to\infty$ and $\hbar\to 0$ while keeping $N\hbar$ fixed. This allows us to systematically incorporate the quantum effects due to scalar loops, which go as $N\hbar$, while keeping the other fields classical. The scalar fields are taken to be conformally coupled in the interest of tractability, but, as a result, the model we consider here with electrically charged black holes does not retain an obvious four-dimensional interpretation. However, all our conclusions will go through, unchanged, for a closely related model obtained by dimensionally reducing a system consisting of fermions coupled to a magnetically charged black hole in four dimensions.[2] The scalar fields will then correspond to the bosonized version of the Callan-Rubakov modes of the fermions [13,17]. We should add, though, that from a strictly four-dimensional point of view, even in this model, the other modes of the fermion fields will contribute to the quantum stress tensor and their neglect cannot be justified.

[2]I would like to thank A. Strominger for pointing this out to me.

III. QUANTUM STRESS TENSOR

For a black hole with a mass much bigger than the Planck mass, the curvature, as measured, for example, by a nonvanishing invariant such as $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, is small all the way from infinity to the horizon. Thus we would expect quantum effects associated with curved spacetime to be small in this region. In fact, this expectation is not met for an extremal black hole. As we show below, the quantum stress tensor of a massless scalar field diverges at the horizon of an extremal black hole no matter how large its mass.

We work in Schwarzschild gauge where the metric is given by

$$ds^2=-f dt^2+\frac{1}{f}dr^2 \;. \tag{8}$$

The conservation equations for the stress tensor then imply that [23]

$$T_t^r=c_1 \tag{9}$$

and that

$$T_r^r=\frac{1}{2f}\int_{r_h}^{r} f' T_\mu^\mu dr+\frac{c_2}{f} \;, \tag{10}$$

where r_h refers to the position of the horizon. An ambiguity in state of the scalar field is reflected in the two arbitrary constants c_1 and c_2. Consider now an observer freely falling into the black hole with a four-velocity $(p_0/f, -(p_0^2-f)^{1/2})$. She sees an energy density

$$T_{\mu\nu}U^\mu U^\nu = T_r^r\left[\frac{p_0^2}{f}-1\right]-T_t^t\frac{p_0^2}{f}-2T_t^r\frac{p_0}{f^2}(p_0^2-f)^{1/2} \;. \tag{11}$$

Substituting from Eqs. (9) and (10), we see that

$$T_{\mu\nu}U^\mu U^\nu=\frac{2(c_2-c_1)p_0^2}{f^2}+\left[-T_\mu^\mu+\frac{1}{f}\int_{r_h}^{r} f' T_\mu^\mu dr\right]\frac{p_0^2}{f} -\frac{c_2-c_1}{f}-\frac{1}{2f}\int_{r_h}^{r} f' T_\mu^\mu dr \;, \tag{12}$$

where f' refers to the derivative of f with respect to r.

This shows that the leading divergence goes as $[(c_2-c_1)/f^2]p_0^2$. So we restrict ourselves to states in which $c_2=c_1$. Then, using the trace anomaly

$$T_\mu^\mu=-\frac{f''}{24\pi} \;, \tag{13}$$

and l'Hôpital's rule, we see that, close to the horizon,

$$T_{\mu\nu}U^\mu U^\nu \simeq \text{const}\times\frac{f'''}{f'} \;. \tag{14}$$

And this diverges since for an extremal black hole f has a double zero at the horizon. In other words, if $\delta\tau$ is the proper time taken to reach the horizon, the energy density diverges like

$$T_{\mu\nu}U^\mu U^\nu \sim 1/\delta\tau \;. \tag{15}$$

We note that the analysis above was very general without restrictions to any particular state of the scalar field. In fact, for these black holes, the Boulware, Unruh, and Hartle-Hawking states are the same and considerations similar to those above at the past horizon would single it out as having the minimal divergence.

This divergence can be better understood by regarding an extremal black hole as the limit of a nonextremal one. A nonextremal black hole has an outer and an inner horizon, and these come together in the extremal limit. Let us take r_h in Eq. (10) to refer to the outer horizon. Then, as before, we see that c_1 must equal c_2 for the leading divergence to vanish at the outer horizon. Furthermore, Eq. (14) shows that the stress tensor stays finite at the outer horizon, since f has a single zero. Now let us focus on the inner horizon. Since r_h in Eq. (10) was taken to mean the outer horizon and $c_1=c_2$, we see from Eq. (12) that the leading divergence goes as

$$T_{\mu\nu}U^{\mu}U^{\nu}\simeq-\frac{p_0^2}{48\pi}\frac{[f'(r_{\text{inner}})^2-f'(r_{\text{outer}})^2]}{f^2} . \qquad (16)$$

And this diverges since the quantity within square brackets does not cancel and f is zero, albeit a simple zero, at the inner horizon. If $\delta\tau$ is the proper time taken by a freely falling observer to reach the horizon, this implies that

$$T_{\mu\nu}U^{\mu}U_{\nu}\simeq\frac{\text{const}}{(\delta\tau)^2} . \qquad (17)$$

In summary, we find that if we adjust the quantum state of the scalar field so that the stress tensor is finite at the outer horizon, it diverges at the inner horizon. It is perhaps not so surprising then that in the extremal case when the two horizons come together the divergence persists, although in a softened form.

IV. QUANTUM-CORRECTED GEOMETRY

We turn now to calculating the quantum-corrected geometry of the extremal black hole. We expect the conformal gauge to be most convenient for this purpose since in this gauge the quantum stress tensor of the scalar fields can be expressed as a local function of the metric. But the extremal solution is static and the related Killing vector is most easily expressed in the Schwarzschild gauge. It is in this gauge also that the classical metric is most easily described, whereas its description in the conformal gauge is at best implicit. Our strategy will therefore be to derive the equations in the conformal gauge, but then to do a coordinate transformation and express them in the Schwarzschild gauge. The conformal coordinates we use for this purpose are defined as follows. We first start with the Schwarzschild coordinates

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2$$

and then define a new coordinate r_* as

$$\frac{dr_*}{dr}=f(r) . \qquad (18)$$

The metric then becomes

$$ds^2=-f(r)dv\,du , \qquad (19)$$

where

$$v=t+r_* ,\quad u=t-r_* \qquad (20)$$

are the required conformal coordinates. To illustrate what we mean by a change of coordinates, we consider the T_{vv} constraint equation. This is given as

$$-\left[\partial_v^2e^{-2\phi}-\frac{1}{f}\partial_v f\partial_v e^{-2\phi}\right]+\frac{1}{2}e^{2\phi}\partial_v e^{-2\phi}\partial_v e^{-2\phi}=-\frac{2G}{12\pi}[f^{1/2}\partial_v^2f^{-1/2}+t_+(v)] . \qquad (21)$$

Here $t_+(v)$ is an arbitrary function of v, which represents the ambiguity in the quantum state of the scalar fields, and

$$\xi=G\frac{N\hbar}{12\pi} . \qquad (22)$$

Now, since the solution is static, we convert, by using Eq. (20), the derivatives with respect to v to those with respect to r. This gives

$$-\frac{1}{4}f^2\left[(e^{-2\phi})''-\frac{1}{2}\frac{(e^{-2\phi})'^2}{e^{-2\phi}}\right]=\frac{\xi}{4}\left[ff''-\frac{1}{2}(f')^2\right]+C . \qquad (23)$$

Here the primes denote derivatives with respect to r, and since the dilaton and metric are functions of r alone, we have replaced $t_+(v)$ by a constant C. Proceeding similarly, the equation of motion of the Maxwell field gives

$$F^2=-2\frac{Q^2}{e^{-4\phi}} . \qquad (24)$$

The equation obtained by varying the trace of the metric then becomes

$$[f(e^{-2\phi})']'-2+2\frac{Q^2}{e^{-2\phi}}=-\xi f'' . \qquad (25)$$

The equation obtained by varying the dilaton is

$$-f''=\frac{1}{2}f\frac{[(e^{-2\phi})']^2}{(e^{-2\phi})^2}+\left[\frac{f(e^{-2\phi})'}{e^{-2\phi}}\right]'-\frac{2Q^2}{(e^{-2\phi})^2} . \qquad (26)$$

Finally, the T_{uu} constraints gives back Eq. (23).

We expect extremal black holes to have zero temperature and seek solutions with $C=0$ and $f'=0$ at the horizon (where $f=0$). If x represents the coordinate distance from the horizon, this suggests that, close to the horizon,

$$f\simeq\alpha_1x^2+\alpha_2x^{3+\delta} \qquad (27)$$

and

$$e^{-2\phi}\simeq d_h+d_2x^{1+\delta} . \qquad (28)$$

Equations (25) and (26) then show that

$$Q^2=\frac{d_h^2}{\xi+d_h} \tag{29}$$

and that

$$\alpha_1=\frac{1}{\xi+d_h} . \tag{30}$$

Further, Eq. (25) then shows that

$$d_2=-\xi\frac{\alpha_2}{\alpha_1}\frac{\delta+2}{\delta} . \tag{31}$$

Finally, Eq. (26) shows that δ is given by the equation

$$\delta=\frac{-3+\sqrt{9+24\xi/(d_h-\xi)}}{2} . \tag{32}$$

These values of the parameters can also be shown to be consistent with Eq. (23) (with C set equal to 0). Like their classical counterparts, these solutions have only one free parameter, which we can take to be d_h, the value of the dilaton at the horizon. $|\alpha_2|$ can be set equal to 1 by rescaling x. The other parameters are then uniquely determined. Numerical computations show that with $\alpha_2=-1$ the solution evolves to an asymptotically flat geometry as $x\to\infty$. As a special case of Eq. (32), note that for large black holes, where $d_h \gg \xi$, $\delta\simeq 2\xi/d_h$.

V. DISCUSSION

What do we learn from these solutions? Extremal black holes are dangerously close to becoming naked singularities. And we might have thought that the diverging stress tensor would cause a singularity to appear at the horizon and even drive the dilaton to a small value, thereby changing the entropy of the black hole dramatically. However, this does not happen. The value of the dilaton field at the horizon remains a free parameter and for large black holes ($d_h \gg \xi$), the entropy stays large and remains as mysterious as ever.

For generic values of d_h, δ is not an integer and the solution is nonanalytic in x. This nonanalyticity implies a very mild singularity at the horizon. The second derivative of the curvature as seen by a freely falling observer diverges as she crosses the horizon, but the tidal forces and the curvature stay finite. Thus there should be an extension of the geometry past the horizon. In fact, there are two obvious static extensions. These correspond to replacing $x^{3+\delta}$ in Eq. (27) above by $|x|^{3+\delta}$ or by $x^3|x|^\delta$ and correspondingly extending the dilaton field. We will call these the even and odd extensions, respectively. It can be shown that both of these satisfy the junction conditions which arise from Eqs. (23), (25), and (26).[3] The odd continuation results in a spacetime much like the classical extreme black hole spacetime in which

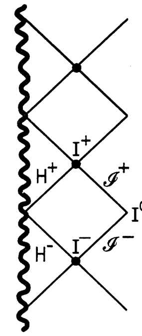

FIG. 1. Penrose diagram of the odd extension.

we can hit a timelike singularity within a finite proper time of crossing the horizon. The even continuation, though, results in an entirely different spacetime, in which we pass from one asymptotically flat universe to another without encountering any malevolent singularities at all.[4] The corresponding Penrose diagrams are shown in Figs. 1 and 2, respectively.

We have not investigated, as yet, whether any of these extensions are relevant for a black hole formed from collapse; but it is not inconceivable that at least some part of spacetime outside a collapsing "star" is described by either of these. In this case it should be possible to begin the collapse from one asymptotically flat universe and open out into another. Information thrown in from one asymptotically flat universe could then end up in another. Indeed, such black holes would be the "ultimate" remnants [10,17,25]. Information would not just be hiding in some long tube, but would have found its way into another universe and be lost forever. This model might also furnish a simple context in which to explore the production of such remnants in external fields.

We cannot say much about four-dimensional black holes, since we do not know how the stress tensor of a four-dimensional scalar field behaves.[5] If we do interpret

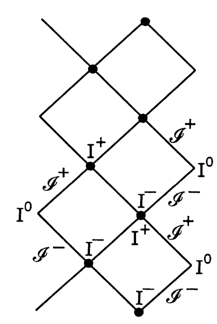

FIG. 2. Penrose diagram of the even extension.

[3]For special values of d_h (within the semiclassical regime), δ becomes an integer and depending on its value either of these can become the analytic continuation, in which even the mild singularity disappears.

[4]A spacetime with the same causal properties has been found by Horne and Horowitz [24].

[5]These calculations are currently in progress [26].

the two-dimensional metric and the dilaton as components of a spherically symmetric four-dimensional metric, we find that the curvature and hence components of the tidal force felt by a freely falling observer blow up at the horizon. However, the divergence is quite mild and the tidal impulse stays finite, which suggests that there should be an extension of the geometry past the horizon. If the four-dimensional stress tensor behaves similarly to the two-dimensional case and blows up, for example, no faster than $1/\delta\tau$ [Eq. (15)], we would expect a mild singularity to form at the horizon. In particular, the area of the horizon would stay large and so would the entropy.

Is there a loss of information in scattering quanta off these extremal black holes? Unfortunately, we cannot answer this question conclusively here. It is clear that some information regarding the kinds of scalar quanta thrown in will be lost in scattering. But in the large-N limit considered here, the entropy of the black hole, which goes as $1/\hbar$, is large.[6] If this entropy has an explanation in terms of underlying microstates, this would suggest a large degeneracy of ground states. And we cannot exclude the possibility that the lost information is hiding in correlations between the ground states and the outgoing radiation. To settle this issue would require keeping track of very subtle correlations (beyond leading order in N) in the outgoing radiation. We hope in subsequent work to return to this problem.

Finally, a few comments regarding nonextremal black holes in contact with a heat bath. Classically, these black holes have an outer and inner horizon. Numerical calculations show that the behavior of the geometry inside the outer horizon depends, for a given value of the dilaton field, on the electric charge. If the electric charge is small, the dilaton decreases until it reaches its critical value[7]

$$e^{-2\phi_c}=G\frac{N\hbar}{12\pi}\ , \tag{33}$$

and a spacelike singularity forms. However, once the charge is large enough, the dilaton does not go to its critical value. Instead, an inner horizon forms at which the metric component f [Eq. (8)] behaves like

$$f\simeq\alpha_1 x-\frac{\alpha_1}{2}\frac{x}{\ln(x)} \tag{34}$$

and the dilaton behaves like

$$e^{-2\phi}\simeq-\text{const}\times\ln(x)\ , \tag{35}$$

where x is the coordinate distance from the horizon. This shows that the curvature (which is related to the second derivative of f) and hence the tidal forces as felt by a freely falling observer blow up at the inner horizon. But it is straightforward to show that the divergence is mild enough for the tidal impulse to stay finite, which suggests[8] that there should be an extension of the geometry past the inner horizon. If the metric and dilaton above are regarded as components of a spherically symmetric metric in four dimensions, though, there are components of the tidal impulse that blow up, which suggests that the four-dimensional black holes might behave differently.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the help M. Bucher, G. Horowitz, J. March-Russell, E. Poisson, A. Ridgway, K. Thorne, E. Verlinde, E. Witten, and P. Yi have given. This work greatly benefited from conversations with J. Preskill, A. Strominger, and F. Wilczek, and it is a special pleasure to thank them for their generous insight and encouragement. This research was supported in part by the U.S. Department of Energy under Contract No. DEAC-03-81ER4050.

[6] I would like to thank J. Preskill for pointing this out and for the subsequent argument about the large-N limit being inadequate.

[7] At this value the kinetic-energy term for the metric becomes degenerate in conformal gauge.

[8] A. Ori has found a similar divergence in his study of mass inflation in four dimensions [27].

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] S. W. Hawking, Phys. Rev. D **14**, 2460 (1976).
[3] P. C. W. Davis, Proc. R. Soc. London **A351**, 129 (1976).
[4] P. C. W. Davis, S. A. Fulling, and W. G. Unruh, Phys. Rev. D **13**, 2720 (1976).
[5] W. A. Hiscock, Phys. Rev. D **15**, 3054 (1977).
[6] W. A. Hiscock, Phys. Rev. D **16**, 2673 (1977).
[7] N. D. Birrell and P. C. W. Davis, Phys. Rev. D **18**, 4408 (1978).
[8] G. Mandal, A. Sengupta, and S. Wadia, Mod. Phys. Lett. A **6**, 1685 (1991).
[9] E. Witten, Phys. Rev. D **44**, 314 (1991).
[10] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, 1005 (1992).
[11] J. A. Harvey and A. Strominger, "Quantum aspects of black holes," Enrico Fermi Institute report, 1992 (unpublished).
[12] S. B. Giddings, "Toy models for black hole evaporation," Report No. UCSBTH-92-36 (unpublished).
[13] S. B. Giddings and A. Strominger, Phys. Rev. D **46**, 627 (1992).
[14] B. Birnir, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **46**, 638 (1992).
[15] J. G. Russo, L. Susskind, and L. Thorlacius, "Black hole evaporation in 1+1 dimensions," Report No. SU-ITP-92-4 (unpublished).
[16] L. Susskind and L. Thorlacius, Nucl. Phys. **B382**, 123 (1992).
[17] T. Banks, A. Dabholkar, M. R. Douglas, and M. O'Loughlin, Phys. Rev. D **45**, 3607 (1992).

[18] S. W. Hawking, Phys. Rev. Lett. **69**, 406 (1992).
[19] M. McGuigan, C. R. Nappi, and S. A. Yost, "Charged black holes in two dimensional string theory," Report No. IASSNS-HEP-91/57 (unpublished).
[20] O. Lechtenfeld and C. Nappi, "Dilaton gravity and no hair theorem in two dimensions," Report No. IASSNS-HEP-92-22 (unpublished).
[21] D. A. Lowe, Phys. Rev. D **47**, 2446 (1993).
[22] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2353 (1991).
[23] S. M. Christenson and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977).
[24] J. H. Horne and G. T. Horowitz, Nucl. Phys. **B368**, 444 (1992).
[25] T. Banks and M. O'Loughlin, Phys. Rev. D **47**, 540 (1993).
[26] P. Anderson and W. Hiscock (private communication).
[27] A. Ori, Phys. Rev. Lett. **67**, 789 (1992).