Semileptonic decays $B \to \rho l \nu$ and $D \to \rho l \nu$ and the heavy-quark symmetry

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(Received 7 December 1992)

We determine the first correction in inverse powers of heavy-quark masses to the ratio of the differential decay rates of $B \to \rho e \nu$ and $D \to \rho e \nu$ at the kinematical point of zero hadronic recoil (maximum q^2) within the quark model. In this particular case of heavy-to-light meson transitions, the heavy-quark limit gives a definite prediction, and theoretical uncertainties should be minimal. This result is potentially important for the extraction of the Kobayashi-Maskawa element V_{ub} from future measurements.

PACS number(s): 13.20.Jf, 11.30.Hv, 12.15.Ff, 13.20.Fc

I. INTRODUCTION

Experimental measurements of charmless semileptonic B decays [1] constitute a subject of great interest in physics of electroweak interactions, as these may help determine precise values of V_{ub}, the Kobayashi-Maskawa element that mixes the first and third families of quarks [2]. However, because of the smallness of the branching ratios associated with these decays, precise measurements require very large statistical samples of B decays, hopefully to be obtained in the next generation of accelerators [3]. On the other hand, theoretical calculations of the rates [4,5] are hindered by the nonperturbative character of strong interactions. Here one may resort to a model, but that introduces uncertainties that are inherent to the model itself, and thus hard to estimate. It is desirable to have a prediction that, at least to some degree, is model independent. In this work we make use of the heavy-quark symmetry [6] to determine a ratio of $B \to \rho e \nu$ to $D \to \rho e \nu$ differential rates which is model independent to leading order in inverse powers of large masses, and estimate each first-order correction to that result by means of a constituent quark model. As such, model uncertainties enter only at the level of the first corrections, and not at leading order.

The recently developed heavy-quark symmetry (HQS) provides a systematic expansion in inverse powers of the heavy-quark masses m_b and m_c, in which short-distance QCD effects can be included explicitly by the construction of a heavy-quark effective theory [7] (HQET). This approach has been particularly useful in treating heavy-to-heavy transitions [8]. For instance, in $B \to D^* e \nu$, with the use of the HQS one can find an estimate of the rate [9] which is model independent up to corrections of order $(1/m_c)^2$. For this process, the HQS features three important virtues: first, the four general form factors that enter the hadronic matrix element of the weak current reduce to a single one in the limit $m_Q \to \infty$ (Q standing for the heavy quark, $Q = b, c$); second, this single form factor has a known normalization at the kinematical point of no recoil (i.e., the point in phase space where the initial and final hadrons have equal velocity); and, finally, at this same kinematical point the corrections of first order in $1/m_Q$ vanish.

In this work, however, we consider $b \to u$ and $c \to d$, that is, heavy-to-light transitions. Unfortunately, here none of the above-mentioned virtues apply, and thus the HQS loses some predictive power as compared to $b \to c$ transitions. Indeed, it is not possible to find a model-independent prediction for the rate of $B \to \rho e \nu$ at any order in $1/m_b$. However, it is possible to find a prediction for the *ratio* of $B \to \rho e \nu$ to $D \to \rho e \nu$ differential rates at the point of no recoil that is model independent in leading order in $1/m_Q$. First corrections to this simple limit, however, do not vanish, as they do in $b \to c$.

The paper is organized as follows: in Sec. II, we define the problem and the quantities to be determined; Sec. III contains the calculation of the $1/m_Q$ corrections at the tree level in the HQET, with the use of a constituent quark model; Sec. IV incorporates the leading α_s corrections that originate at one loop in the HQET; finally, our numerical results and discussion are found in Sec. V.

II. OUTLINE OF THE PROBLEM

Consider the hadronic matrix element of the weak current entering the decay of a heavy meson $M (M = B, D)$ into $\rho e \nu$:

$$\langle \rho(v') | J^\nu(0) | M(v) \rangle = \sqrt{4 m_M m_\rho} \{ i f_V^{[M]}(y) \epsilon^{\nu\rho\delta\sigma} v_\rho v'_\delta e_\sigma - f_{A_1}^{[M]}(y) e^\nu - f_{A_2}^{[M]}(y) (e \cdot v) v^\nu - f_{A_3}^{[M]}(y) (e \cdot v) v'^\nu \} , \quad (1)$$

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with v and m_M the four-velocity and mass of the heavy meson, and v' and e the four velocity and polarization of the ρ meson, respectively. We have expressed this amplitude in terms of the velocities of the mesons instead of their momenta, and extracted a factor $\sqrt{4m_M m_\rho}$ to take into account the usual normalization of the meson states, thus ensuring that the form factors are dimensionless and at the same time remain finite in the limit $m_Q \to \infty$ ($Q=b,c$). $f_V^{[M]}$ is the form factor of the vector current, and $f_{A_1}^{[M]}$, $f_{A_2}^{[M]}$, and $f_{A_3}^{[M]}$ are those of the axial-vector current. Notice that the form factors are also chosen to depend on the kinematical variable. $y \equiv v \cdot v'$ (i.e., the energy of the ρ meson in the center-of-momentum frame in units of its mass). This quantity ranges from 1 to approximately $m_M/(2m_\rho)$, the upper limit assuming negligible lepton masses. In general, the form factors for B and D decays, i.e., $f^{[B]}$'s and $f^{[D]}$'s are different functions of y, and, moreover, y varies over a different range, since $m_b \neq m_c$. However, in the limit of m_b and m_c infinitely larger than the hadronization scale, these form factors should be equal for equal (and near unity) values of y. In this limit, and from the viewpoint of strong interactions, the heavy quark of the initial state is a static, flavor independent, color source; the actual value of the heavy mass determines the size of the final phase space, but the strong interaction dynamics is determined only by the relative motion of the final hadron with respect to the initial one, that is, by y. We therefore assume an expansion of the form factors in powers of $1/m_Q$ as

$$f^{[M]}(y) = f^{(0)}(y) + \frac{1}{m_Q} f^{(1)}(y) + \cdots , \tag{2}$$

for $M=B,D$ and $Q=b,c$, respectively. Here $f^{(1)}(y)$ is a dimensionful quantity of order $\Lambda_{\rm QCD}(\ll m_Q)$.

We want to estimate the ratio between the differential rates of $B \to \rho e \nu$ and $D \to \rho e \nu$, so that the leading term in a $1/m_Q$ expansion is model independent. Clearly this would be the case if there were only one form factor entering Eq. (1). In that case, the sought ratio would be of the form

$$\frac{d\Gamma(B \to \rho e \nu)/dy}{d\Gamma(D \to \rho e \nu)/dy} \sim \left| \frac{V_{ub}}{V_{cd}} \right|^2 \left[1 + 2 \left[\frac{1}{m_b} - \frac{1}{m_c} \right] \frac{f^{(1)}(y)}{f^{(0)}(y)} + \cdots \right] . \tag{3}$$

However, since there are actually four form factors, this ratio generally contains unknown terms of the form $f_i^{(0)}/f_j^{(0)}$, introducing model dependence even at leading order.

One way to select a single form factor in Eq. (1) is to extract the vector part of the current, thus having only f_V in the ratio (3). The determination of $|V_{ub}/V_{cd}|$ would then require experimentally an angular analysis of the decays, and theoretically the prediction of the full shape of the form factor.

Another way to select a single form factor is to consider the full decay, but only at the kinematical point of zero recoil, $y=1$, where only the axial form factor f_{A_1} contributes to the decay amplitude. However, from the expression of the differential rate for $M \to \rho e \nu$,

$$\frac{d\Gamma(M \to \rho e \nu)}{dy} = \frac{G_F^2}{48\pi^3} |V_{qQ}|^2 m_M^2 m_\rho^3 (y^2-1)^{1/2} (y+1)^2 \times \left[2(1+r^2-2ry) \left[\frac{1}{(y+1)^2} f_{A_1}^2 + \frac{y-1}{y+1} f_V^2 \right] + \left[\frac{y-r}{y+1} f_{A_1} + (y-1)(r f_{A_2} + f_{A_3}) \right]^2 \right] , \tag{4}$$

with $r = m_\rho/m_M$, one can see that it still vanishes due to phase space:

$$\left. \frac{d\Gamma(M \to \rho e \nu)}{dy} \right|_{y \to 1} = \frac{G_F^2 |V_{qQ}|^2}{4\pi^3} [f_{A_1}^{[M]}(1)]^2 (m_M - m_\rho)^2 m_\rho^2 |\mathbf{p}_\rho| . \tag{5}$$

Here, $|\mathbf{p}_\rho|$ is the magnitude of the (vanishing) momentum of the ρ meson. Since the rates vanish at this point, experimentally one should access the region nearby and extrapolate to the point of $y=1$, somewhat affecting the statistics. The ratio of the differential rates at $y \to 1$ is

$$\left. \frac{d\Gamma(B \to \rho e \nu)/dy}{d\Gamma(D \to \rho e \nu)/dy} \right|_{y \to 1} = \left| \frac{V_{ub}}{V_{cd}} \right|^2 \left[\frac{f_{A_1}^{[B]}(1)}{f_{A_1}^{[D]}(1)} \right]^2 \left[\frac{m_B - m_\rho}{m_D - m_\rho} \right]^2 . \tag{6}$$

In this paper we address this latter approach. Thus, the core of the calculation consists in determining $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$. Here the constituent quark model seems particularly appropriate because it works best where the relative motion of the constituents is as nonrelativistic as possible and where the overlap of wave functions is maximal, both conditions occurring precisely at the point of no recoil $y=1$.

III. TREE-LEVEL CALCULATION

In the constituent quark model, the state of a meson X is given in terms of the wave function for a bound state of a (constituent) quark and antiquark. For an s wave with total angular momentum j in a radial excitation n, the meson state is

$$|X(P)\rangle=\sqrt{2m_X}\int\frac{d^3k}{(2\pi)^3}\tilde{\psi}_n^{[X]}(k)e^m C(j,m,s_1,s_2)\times q_1^\dagger(\mathbf{p}_1,s_1)\bar{q}_2^\dagger(\mathbf{p}_2,s_2)|0\rangle\ , \tag{7}$$

where $\tilde{\psi}_n^{[X]}(k)$ is the Fourier transform of the wave function, e^m the polarization (irrelevant for $j=0$), $C(j,m,s_1,s_2)$ the Clebsch-Gordan coefficient for the combination of the two spin $\frac{1}{2}$ into spin j, and $q_1^\dagger$ and $\bar{q}_2^\dagger$ the creation operators for the quark and antiquark, respectively. Using wave functions ψ normalized to unity and constituent quark states normalized as

$$\langle q(\mathbf{k},s)|q^\dagger(\mathbf{k}',s')\rangle=(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'}$$

the factor $\sqrt{2m_X}$ keeps the normalization of the meson state consistent with that of Eq. (1). The quarks three-momenta are simply decomposed as $\mathbf{p}_{1,2}=m_{q_{1,2}}\mathbf{v}\pm\mathbf{k}$, where $\mathbf{v}\equiv\mathbf{P}/m_X$ is the velocity of the meson. For a heavy meson, one of the quarks is taken as infinitely heavy (corrections to this limit are taken perturbatively, in accordance to the HQET), while for the ρ meson both quarks are taken as light constituents. Although the use of a constituent quark model for the ρ meson, a system far from being nonrelativistic, is a strong assumption, it works reasonably well in meson spectroscopy, and in the same way it should work well in this treatment, where the results are only sensible to the overall integrated wave function of ρ, and not to particular details of its shape.

In this calculation we employ the model of Ref. [5], for which the ground state and first radial excitation wave functions are

$$\psi_{1s}(r)=\frac{\beta^{3/2}}{\pi^{3/4}}e^{-\beta^2r^2/2}\ , \tag{8}$$

$$\psi_{2s}(r)=\left[\frac{2}{3}\right]^{1/2}\frac{\beta^{3/2}}{\pi^{3/4}}(\beta^2r^2-\tfrac{3}{2})e^{-\beta^2r^2/2}\ , \tag{9}$$

with β_M and β_ρ determined variationally in a potential $V(r)=-a/r+br+c$. Adjusting the potential to fit the spectroscopy of mesons, the authors of Ref. [5] find $a=0.67$, $b=0.18$ GeV^2, and $c=-0.84$ GeV, and then $\beta_M=0.42$ GeV for the heavy meson (i.e., $m_Q\to\infty$), and $\beta_\rho=0.31$ GeV for the ρ meson.

In the $1/m_Q$ expansion given by the HQET [10] there is one leading contribution that enters $f_{A_1}^{(0)}$, while several $O(1/m_Q)$ contributions enter $f_{A_1}^{(1)}$: namely, the $1/m_Q$ correction to the weak current (which is the modification of the current due to the motion of the heavy quark) and the $1/m_Q$ corrections of the effective Lagrangian (which correspond to the modification of the heavy meson wave function due to the kinetic energy and color magnetic moment of the heavy quark).

The expansion to order $1/m_Q$ of the axial-vector current matrix element at the tree level in the HQET is then of the form

$$\langle\rho|A_\nu(0)|M\rangle=\langle\rho|A_\nu^{(0)}(0)|M_\infty\rangle+\langle\rho|A_\nu^{(1)}(0)|M_\infty\rangle+\langle\rho|\mathcal{T}_\nu^{\rm kin}(0)|M_\infty\rangle+\langle\rho|\mathcal{T}_\nu^{\rm mag}(0)|M_\infty\rangle\ , \tag{10}$$

where $|M_\infty\rangle$ on the right-hand side (RHS) is the state of a pseudoscalar meson containing one infinitely heavy quark,

$$A_\nu^{(0)}(x)=\bar{q}(x)\gamma_\nu\gamma_5h_v(x)\ , \tag{11}$$

$$A_\nu^{(1)}(x)=\bar{q}(x)\gamma_\nu\gamma_5\frac{i\not{D}}{2m_Q}h_v(x) \tag{12}$$

are the leading term and $O(1/m_Q)$ correction to the current, respectively, expressed in terms of the light quark field $q(x)$ and the effective heavy-quark field $h_v(x)$ of the HQET, while $\mathcal{T}^{\rm kin}$ and $\mathcal{T}^{\rm mag}$, which are given by expressions such as

$$\mathcal{T}_\nu^{\rm kin}(x)=i\int d^4x'TA_\nu^{(0)}(x)\mathcal{L}^{\rm kin}(x'),\ \text{etc.}, \tag{13}$$

are the corrections due to the $O(1/m_Q)$ pieces of the HQET effective Lagrangian density:

$$\mathcal{L}^{\rm kin}(x)=-\bar{h}_v(x)\frac{D^2}{2m_Q}h_v(x)\ , \tag{14}$$

$$\mathcal{L}^{\rm mag}(x)=-\bar{h}_v(x)\frac{g_s\sigma_{\mu\nu}G^{\mu\nu}}{4m_Q}h_v(x)\ , \tag{15}$$

namely, a kinetic energy and a color magnetic moment interaction of the heavy quark.

We can now calculate the ratio $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ of Eq. (6) in terms of the parameters of the model. Since at $y=1$ only the axial form factor f_{A_1} survives, all the terms in Eq. (10) are proportional to the polarization e_ν. For the first two terms of the expansion (10) we obtain

$$\langle\rho|A_\nu^{(0)}(0)|M_\infty\rangle_{y=1}=\sqrt{4m_Mm_\rho}I_0e_\nu\ , \tag{16}$$

$$\langle\rho|A_\nu^{(1)}(0)|M_\infty\rangle_{y=1}=\sqrt{4m_Mm_\rho}\frac{1}{m_Q}I_1e_\nu\ , \tag{17}$$

where I_0 and I_1 stand for the overlap integrals:

$$I_0\equiv\langle\psi^{[\rho]}|\psi_{1s}^{[M]}\rangle=\left[\frac{2\beta_M\beta_\rho}{\beta_M^2+\beta_\rho^2}\right]^{3/2}\ , \tag{18}$$

$$I_1\equiv\left\langle\psi^{[\rho]}\left|\frac{\nabla^2}{4m_q}\psi_{1s}^{[M]}\right.\right\rangle=-\frac{3}{8}\frac{\beta_M\beta_\rho}{m_q}\left[\frac{2\beta_M\beta_\rho}{\beta_M^2+\beta_\rho^2}\right]^{5/2}\ . \tag{19}$$

The right-hand side of these equations displays the values of I_0 and I_1 for the specific wave functions of Eq. (8), m_q being the constituent mass of the light quark (u or d).

The last two terms of expansion (10) can be easily calculated in the model by realizing that these terms are corrections to the wave function of the heavy meson, that is, the matrix elements involving $\mathcal{T}_v^{\rm kin}$ and $\mathcal{T}_v^{\rm mag}$ are

$$\langle\rho|\mathcal{T}_v^{\rm kin}(0)|M_\infty\rangle=\langle\rho|A_v(0)|M^{\rm kin}\rangle_{\rm quark\ model} \quad (20)$$

(and similarly for $\mathcal{T}_v^{\rm kin}$), where $|M^{\rm kin}\rangle$ and $|M^{\rm mag}\rangle$ are the first corrections to the state of the heavy meson due to the perturbations (14) and (15). Using time-independent perturbation theory one can expand $|M^{\rm kin}\rangle$ and $|M^{\rm mag}\rangle$ in terms of all the bound-state excitations orthogonal to $|M_\infty\rangle$. For an $l=0$ meson, and to the order of approximation of this work, it suffices to expand up to the first radial excitation $|2s\rangle$ (in this spectroscopic notation, $|1s\rangle\sim|M_\infty\rangle$). $|M^{\rm kin}\rangle$ is then

$$|M^{\rm kin}\rangle=|2s\rangle\frac{\langle 2s|H^{\rm kin}|M_\infty\rangle}{E^{(1s)}-E^{(2s)}}\equiv|2s\rangle\sqrt{2m_M}\frac{\Delta^{\rm kin}}{m_Q}\ , \quad (21)$$

where $H^{\rm kin}$, given below, is in direct correspondence with $\mathcal{L}^{\rm kin}$, $E^{(n)}$ are the energy eigenvalues of the states $|n\rangle$, and $\Delta^{\rm kin}$ is a quantity of order $\Lambda_{\rm QCD}$. $|M^{\rm mag}\rangle$ is given by an analogous expression.

The coefficients $\Delta^{\rm kin}$ and $\Delta^{\rm mag}$ can be calculated within the quark model using the perturbation Hamiltonian that corresponds to Eqs. (14) and (15) in the HQET:

$$H^{\rm kin}+H^{\rm mag}=-\frac{\nabla^2}{2m_Q}+\frac{g}{2m_Qm_q}\mathbf{S}_1\cdot\mathbf{S}_2\delta^3(x)\ . \quad (22)$$

In the second (i.e., "magnetic") term, the constant g must be chosen to fit the B^*-B mass difference Δm_B, thus getting $g/(2m_bm_q)=\Delta m_B/|\psi_B(0)|^2$. In addition, the energy difference $E^{(1s)}-E^{(2s)}$ that enters the definition of Δ can be easily calculated within the quark model in terms of the parameters of the potential $V(r)=-a/r+br+c$. One thus finds $\Delta^{\rm kin}$ and $\Delta^{\rm mag}$ to be given respectively by

$$\frac{\Delta^{\rm kin}}{m_Q}=\left[\frac{3}{2}\right]^{1/2}\frac{\beta_M^2}{2m_Q(E^{(2s)}-E^{(1s)})}\ , \quad (23)$$

$$\frac{\Delta^{\rm mag}}{m_Q}=-\left[\frac{3}{2}\right]^{3/2}\frac{\beta_M^2}{2m_Q(E^{(2s)}-E^{(1s)})}\frac{m_b\Delta m_B}{\beta_M^2}\ , \quad (24)$$

with

$$E^{(2s)}-E^{(1s)}=\beta_M\left[\frac{\beta_M}{m_q}+\frac{a}{3\pi^{1/2}}+\frac{b}{\beta_M^2\pi^{1/2}}\right]\ . \quad (25)$$

Notice that there appears a factor of the large mass m_b in the numerator of Eq. (24). Nevertheless, the full expression is still of order $1/m_Q$ because Δm_B is a quantity of order $\Lambda_{\rm QCD}^2/m_b$.

Accordingly, at $y=1$ the term in Eq. (10) involving $\mathcal{T}_v^{\rm kin}$ becomes

$$\langle\rho|\mathcal{T}_v^{\rm kin}(0)|M\rangle_{y=1}=\sqrt{4m_Mm_\rho}\frac{\Delta^{\rm kin}}{m_Q}I_2e_v\ , \quad (26)$$

and similarly for the term involving $\mathcal{T}_v^{\rm mag}$. In both cases, I_2 is the overlap integral:

$$I_2\equiv\langle\psi^{[\rho]}|\psi_{2s}^{[M]}\rangle=\left[\frac{3}{2}\right]^{1/2}\left[\frac{2\beta_M\beta_\rho}{\beta_M^2+\beta_\rho^2}\right]^{3/2}\left[\frac{\beta_M^2-\beta_\rho^2}{\beta_M^2+\beta_\rho^2}\right]\ . \quad (27)$$

Consequently, to order $1/m_Q$ and at tree level in the HQET one finds

$$\frac{f_{A_1}^{[B]}(1)}{f_{A_1}^{[D]}(1)}\simeq1+\left[\frac{1}{m_b}-\frac{1}{m_c}\right]\times\left[\frac{I_1}{I_0}+\frac{I_2\Delta^{\rm kin}}{I_0}+\frac{I_2\Delta^{\rm mag}}{I_0}\right]\ . \quad (28)$$

The $1/m_Q$ terms in Eq. (28) are then decomposed into three contributions: the contribution from the correction to the current, $A_v^{(1)}$, that due to the "kinetic-energy" Lagrangian $\mathcal{L}^{\rm kin}$, and that due to the "magnetic moment" Lagrangian $\mathcal{L}^{\rm mag}$, respectively.

IV. LOOP CORRECTIONS IN THE HQET

The previous calculation is done at tree level in the HQET. As such, it takes strong interactions into account only at the level of hadronization by means of the quark model, and neglects QCD effects at higher energies. However, although gluons of high invariant mass are not crucial in the bound-state formation, they do modify the short-distance behavior of the decay process. The perturbative treatment of short-distance QCD within the HQET can be found elsewhere [11], the net result being the appearance of additional effective operators in the expansion of the current, and of coefficients containing $\ln m_Q$, thus breaking the purely analytic behavior of the expansion in $1/m_Q$. In a general Lorentz frame, Falk and Grinstein [11] find 12 effective local operators of order $1/m_Q$ in the expansion of the vector current, and a similar set in the expansion of the axial-vector current. One of these operators, namely, $A_v^{(1)}$ of Eq. (12), is found at tree level, while the other 11 operators appear at order α_s. However, only one of the latter has a nonvanishing matrix element at $y=1$:

$$A_v^{(2)}=\frac{m_q}{m_Q}\bar{q}(x)\gamma_v\gamma_5h_v(x)\ . \quad (29)$$

The matrix element of the axial current at $y=1$, to order $1/m_Q$ and including short-distance QCD, can then be expanded as

$$\langle\rho|A_v|M\rangle=c_0^{[M]}(\mu)\langle A_v^{(0)}\rangle+c_1^{[M]}(\mu)\langle A_v^{(1)}\rangle+c_2^{[M]}(\mu)\langle A_v^{(2)}\rangle+c_{\rm kin}^{[M]}(\mu)\langle\mathcal{T}_v^{\rm kin}\rangle+c_{\rm mag}^{[M]}(\mu)\langle\mathcal{T}_v^{\rm mag}\rangle\ , \quad (30)$$

which resembles Eq. (10), except for the additional operator $A_v^{(2)}$ and the coefficients $c_i(\mu)^{[M]}$:

$$c_0(\mu)=c_{\rm kin}(\mu)=w^6\ , \quad (31)$$

$$c_{\rm mag}(\mu)=w^{-3}\ , \quad (32)$$

$$c_1(\mu)=\tfrac{10}{9}-\tfrac{8}{27}w^{-3}+\tfrac{10}{54}w^6-4w^6\ln w\ , \quad (33)$$

$$c_2(\mu)=\tfrac{1}{18}w^{-6}-\tfrac{8}{27}w^{-3}+\tfrac{13}{54}w^{6}+2w^{6}\ln w\ . \qquad (34)$$

Here, $w=[\alpha_s(\mu)/\alpha_s(m_c)]^{1/27}$ for $M=D$, and

$$w=[\alpha_s(\mu)/\alpha_s(m_c)]^{1/27}[\alpha_s(m_c)/\alpha_s(m_b)]^{1/25}$$

for $M=B$. These coefficients take into account the renormalization-group-improved leading logarithmic corrections. Two remarks are now in order concerning Eq. (30). First, the coefficients $c_i^{[M]}(\mu)$, as indicated, depend on the heavy-quark content of the decaying meson as $\ln m_Q$, thus destroying the simple analytical expansion of Eq. (28); instead, one finds

$$\frac{f_{A_1}^{[B]}(1)}{f_{A_1}^{[D]}(1)}=\frac{c_0^{[B]}(\mu)}{c_0^{[D]}(\mu)}\{1+R_{J_1}+R_{J_2}+R_{\rm kin}+R_{\rm mag}\}\ , \qquad (35)$$

where

$$R_{J_1}=\left[\frac{c_1^{[B]}(\mu)/c_0^{[B]}(\mu)}{m_b}-\frac{c_1^{[D]}(\mu)/c_0^{[D]}(\mu)}{m_c}\right]\frac{I_1}{I_0}\ , \qquad (36)$$

$$R_{J_2}=\left[\frac{c_2^{[B]}(\mu)/c_0^{[B]}(\mu)}{m_b}-\frac{c_2^{[D]}(\mu)/c_0^{[D]}(\mu)}{m_c}\right]m_q\ , \qquad (37)$$

$$R_{\rm kin}=\left[\frac{c_{\rm kin}^{[B]}(\mu)/c_0^{[B]}(\mu)}{m_b}-\frac{c_{\rm kin}^{[D]}(\mu)/c_0^{[D]}(\mu)}{m_c}\right]\times\frac{I_2\Delta^{\rm kin}}{I_0}\ , \qquad (38)$$

$$R_{\rm mag}=\left[\frac{c_{\rm mag}^{[B]}(\mu)/c_0^{[B]}(\mu)}{m_b}-\frac{c_{\rm mag}^{[D]}(\mu)/c_0^{[D]}(\mu)}{m_c}\right]\times\frac{I_2\Delta^{\rm mag}}{I_0}\ , \qquad (39)$$

are the corrections due to $A_v^{(1)}$, $A_v^{(2)}$, $\mathcal{T}_v^{\rm kin}$, and $\mathcal{T}_v^{\rm mag}$, respectively. Second, these same coefficients introduce the artificial low-energy scale μ, which should properly cancel in Eq. (30) by a corresponding μ dependence in the matrix elements. Actually, in the ratio of Eq. (35), the leading term is already μ independent, as

$$c_0^{[B]}(\mu)/c_0^{[D]}(\mu)=[\alpha_s(m_c)/\alpha_s(m_b)]^{6/25}$$

is μ independent; however, the terms of order $O(1/m_Q)$ are not. The exact cancellation of the μ dependence can only be done by solving the theory to all orders, a task we cannot yet accomplish. By estimating the matrix elements with the use of a model, we can only hope to choose a scale μ sufficiently low as to be close to the regime at which the quark model is valid, and sufficiently high as to be in the perturbative regime. These two requirements are somewhat opposite, and so there will remain an uncertainty in our results, due to our inability to fix μ at an exact value.

V. RESULTS AND DISCUSSION

In order to obtain numerical results, we use the values indicated in Sec. III for the parameters of the quark model. For the quark potential, we use $a=0.67$, $b=0.18$ GeV^2, and $c=-0.84$ GeV, and for the light constituent mass $m_q=0.33$ GeV. With these values one determines variationally the meson wave-function sizes $\beta_M=0.42$ GeV and $\beta_\rho=0.31$ GeV. One also finds $E^{(2s)}-E^{(1s)}=0.83$ GeV [Eq. (25)], which is the splitting between the ground state and first radial excitation in a $Q\bar{q}$ system. There are no experimental data to compare this result; however, in the same model one finds the splitting between the ρ and its first radial excitation to be 0.66 GeV, which is remarkably close to the experimental value of 681 ± 8 MeV [12]. In addition, we take $\Delta m_B=46$ MeV [12] for the B^*-B mass difference as input to determine the "magnetic" correction [Eq. (24)].

For the heavy-quark masses we use $m_b=5.0$ GeV and $m_c=1.5$ GeV as central values. Uncertainties are considered by varying m_c (the most significant source of error among the two masses) up to 1.8 GeV [13].

To define α_s we use $\Lambda_{\rm QCD}$ in the range 100–250 MeV, with a fiducial value of 150 MeV. Finally, we take the low-energy scale μ at 350 ± 100 MeV. We take these values for μ because, although naively μ should be the mass of the light quark entering the weak current (m_q in our model), in our problem such a quark is lighter than the Bohr momentum of the bound state ($\sim\beta_\rho$ in our model), and clearly the scaling behavior used in the determination of the coefficients $c_i(\mu)$ is not valid below that point. We thus use μ in a range near β, namely, $\beta_\rho\lesssim\mu\lesssim\beta_M$. We take this error as inherent to our model, and further improvements would require the treatment of hadronization more systematically within the theory of QCD.

The numerical results are shown in Tables I and II. Table I displays the dependence on $\Lambda_{\rm QCD}$ for μ fixed at 350 MeV. From this table one notices that the inclusion of short-distance QCD tends to decrease the dominant

TABLE I. The four $1/m_Q$ corrections to the ratio $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ [cf. Eq. (35)], the *leading log* correction, and the resulting value of the ratio as a function of α_s (parametrized in terms of $\Lambda_{\rm QCD}$), for fixed $\mu=350$ MeV.

$\Lambda_{\rm QCD}$ (MeV)	R_{J_1}	R_{J_2}	$R_{\rm kin}$	$R_{\rm mag}$	$\frac{c_0^{[B]}(\mu)}{c_0^{[D]}(\mu)}$	$f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$
No QCD	0.066	0.	−0.022	0.043		1.09
100	0.054	−0.014	−0.022	0.035	1.09	1.14
150	0.050	−0.018	−0.022	0.033	1.10	1.15
250	0.039	−0.030	−0.022	0.026	1.12	1.14

TABLE II. The four $1/m_Q$ corrections to the ratio $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ [cf. Eq. (35)], the *leading log* correction, and the resulting value of the ratio as a function of μ, for fixed $\Lambda_{\rm QCD}=150$ MeV.

μ (MeV)	R_{J_1}	R_{J_2}	$R_{\rm kin}$	$R_{\rm mag}$	$\frac{c_0^{[B]}(\mu)}{c_0^{[D]}(\mu)}$	$f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$
300	0.047	−0.022	−0.022	0.030	1.10	1.14
350	0.050	−0.018	−0.022	0.033	1.10	1.15
400	0.053	−0.015	−0.022	0.034	1.10	1.16

$1/m_Q$ corrections (R_{J_1} and $R_{\rm mag}$), inducing a partial *cancellation* of the overall $1/m_Q$ correction, and at the same time generating a correction $c_0^{[B]}(\mu)/c_0^{[D]}(\mu)$ to the leading term that, as a net effect, *increases* the ratio $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$. This result shows that in these heavy-to-light processes the logarithmic corrections are as important as, or more than, $1/m_Q$ corrections, as the running occurs over a large range of scales.

Table II, on the other hand, displays the μ dependence, for $\Lambda_{\rm QCD}$ fixed at 150 MeV. As μ is one of the most uncertain parameters of the model, it is fortunate that it affects only some of the $1/m_Q$ terms, but not the leading QCD correction $c_0^{[B]}/c_0^{[D]}$, and consequently the resulting ratio $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ is not as sensitive to μ as it would be otherwise. Indeed, the form factors $f_{A_1}^{[B]}(1)$ and $f_{A_1}^{[D]}(1)$ are separately much more sensitive to μ than their ratio. Nevertheless, we found that, in addition to m_c, μ is the parameter that causes the largest uncertainty on $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$. For instance, a (rather large) variation on β_ρ, the momentum spread in the ρ meson, between 260 and 360 MeV induces an error of only $\sim 0.6\%$ on $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$. We thus take the error in $\mu=350\pm 50$ MeV as representative of the overall accuracy of this model.

Still the most significant uncertainty arises from the value of m_c. Taking the value of $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ with the error induced by μ as indicated above, we find

$$\text{for } m_c-1.3 \text{ GeV}, \quad f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)\approx 1.18\pm 0.01 \ ,$$
$$\text{for } m_c=1.5 \text{ GeV}, \quad f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)\approx 1.15\pm 0.01 \ , \quad (40)$$
$$\text{for } m_c=1.8 \text{ GeV}, \quad f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)\approx 1.11\pm 0.01 \ ,$$

that is, the 20% uncertainty on m_c shown above induces an uncertainty of the same magnitude on the $1/m_Q$ corrections to $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$, but this translates into only an $\sim 3\%$ error on the value of $f_{A_1}^{[B]}(1)/f_{A_1}^{[D]}(1)$ itself, a rather remarkable result.

From Eq. (6) and the values (40), one finds the ratio of differential rates at $y=1$. For example, taking the full range of values of Eq. (40) dominated by the uncertainty on m_c indicated, we get

$$\left.\frac{d\Gamma(B\to\rho e\nu)/dy}{d\Gamma(D\to\rho e\nu)/dy}\right|_{y\to 1}=\left|\frac{V_{ub}}{V_{cd}}\right|^2(22\pm 1)\ . \quad (41)$$

Over all we have considered ratios of B and D decaying either both into charged ρ's or both into neutral ρ's; otherwise, one must include an extra factor of $\frac{1}{2}$ in the rate into a neutral ρ because of isospin symmetry.

Finally, we should estimate the fraction of phase space that must be measured in order to experimentally determine the ratio of the B and D differential rates at $y=1$. Since these rates vanish at $y=1$, actual measurements are done at $y>1$, from which one must extrapolate to $y\to 1$. Also, since all four form factors affect the rate as y increases from unity, y should be close enough to unity as to ensure that the main contribution to the rate is due to the form factor f_{A_1} alone.

Given the full expression of the differential rate in Eq. (4), and assuming that all form factors are of the same size and shape, one can see that by taking $y\lesssim 1.1$ the terms in Eq. (4) due to f_{A_1} are at least 10 times larger than those due to other form factors. In $D\to\rho e\nu$, more than one-third of the total rate originates from the region $y\le 1.1$, while in $B\to\rho e\nu$, the fraction of the rate for $y\le 1.1$ is about $0.1\to 1$ % of the total rate (the uncertainty is due to our ignorance on the shape of the form factors as a function of y). Therefore, the main experimental limitation arises from the B decay measurements. Nevertheless, assuming a branching fraction $B(B\to\rho e\nu)\sim 10^{-4}$, a sample of the order of 10^8 B's should provide a few tens of events with $y\le 1.1$, in principle, a task achievable in a B factory.

ACKNOWLEDGMENTS

One of us (C.D.) is grateful to R. D. Peccei, H. R. Quinn, M. A. Shifman, B. Hill, and M. B. Wise for discussions, and to N. Isgur for useful suggestions. C.D. also acknowledges the hospitality of the ITP at U.C. Santa Barbara, where part of this work was done. This work was supported in part by NSF Grant No. PHY89-04035; the U.S. Department of Energy, Contract No. DE-AT03-88ER 40384 Mod A006-Task C; Fundación Andes, Chile, Contract No. C-11666/7; and FONDECYT, Chile, Project No. 92-0806.

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