PHYSICAL REVIEW D VOLUME 47, NUMBER 9 1 MAY 1993

Third-order spin polarizabilities of the nucleon

Silvestre Ragusa
Departamento de Física e Ciência dos Materiais, Instituto de Física e Química de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13 560-970, São Carlos, São Paulo, Brazil
(Received 6 September 1991)

We study the structure-constant (polarizability) coefficients of the spin-dependent terms of the nucleon Compton scattering amplitude which are third order in the frequency of the incoming photon. It is shown that these spin-polarizability coefficients can be related to sums of products of electromagnetic transition moments of the nucleon, involving the electric and magnetic dipoles, the electric and magnetic quadrupoles, and the charge electric and magnetic dipole mean-square radii. Three sum rules involving products of the electric dipole transition moments emerge from the calculation.

PACS number(s): 13.60.Fz

I. INTRODUCTION

As is well known, to second order in the frequency of the incoming photon, the nucleon Compton-scattering amplitude is described in terms of nucleon static electromagnetic properties (charge, magnetic moment, and corresponding mean-square radii) and two additional parameters or structure constants that cannot be fixed by low-energy theorems. They represent the electric (α) and magnetic (β) polarizabilities of the nucleon [1–4]. Baldin [2] has shown that α can be expressed in terms of sums of products of dipole transition moments of the nucleon. A first contribution to the magnetic polarizability, involving products of magnetic dipole transition moments, was obtained by Petrun'kin [4], and afterwards we have obtained its complete expression, containing a second, diamagnetic, term involving moments of the current commutators [5]. The recent experimental and theoretical results have been reviewed by L'Vov and Petrun'kin [6(a)], and by Friar [6(b)].

In this paper we study the structure constants which appear at third order in the frequency of the incoming photon. Their interest lies in the fact that it is at this order that structure constants related to the spin of the nucleon first appear. Lin [7] has studied the third-order low-energy theorems in nucleon Compton scattering. To that order one has six structure-constant coefficients in the amplitude tensor T_{ij}, all of them connected to third-order spin terms of the amplitude. Because of the transversality condition, only four of these structure constants will be present in the amplitude $A=\varepsilon'^{i}T_{ij}\varepsilon^{j}$. We shall refer to them as spin polarizabilities of the nucleon.

The electric and magnetic dipole polarizabilities characterize the induced dipole moments of hadrons subjected to external electromagnetic fields and have a classical counterpart easy to grasp. Under the action of an electric field **E**, a classical system acquires an induced electric dipole moment due to a deformation of the charge distribution, which is proportional to the field, $\mathbf{p}=\alpha\mathbf{E}$, giving rise to an interaction energy proportional to $\mathbf{p}\cdot\mathbf{E}=\alpha E^2$. The quantity α, called the electric polarizability of the system, measures its response to deformation. In the case of an electromagnetic wave, we have $\mathbf{E}=-\partial\mathbf{A}/\partial t$, where the vector potential $\mathbf{A}=\sum_k \mathbf{A}_k$, in a plane-wave decomposition, is a sum of terms $\mathbf{A}_k=\boldsymbol{\varepsilon}A_{0k}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)+\text{c.c.}$ where $\boldsymbol{\varepsilon}$ is the polarization vector, **k** is the wave vector ($\boldsymbol{\varepsilon}\cdot\mathbf{k}=0$), and $\omega=|\mathbf{k}|$ is the frequency of the wave. The electric field will then be a sum of terms $\mathbf{E}_k=i\omega\boldsymbol{\varepsilon}A_{0k}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)+\text{c.c.}$ Therefore the crossed term of the interaction energy of the two waves, $\boldsymbol{\varepsilon},\mathbf{k}$ and $\boldsymbol{\varepsilon}',\mathbf{k}'$ will have the form $\alpha\omega\omega'\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'$, which is quadratic in the frequency. Similar considerations apply to the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$. Under its action the system acquires an induced magnetic dipole moment $\mathbf{p}_m=\beta\mathbf{B}$, giving rise to an interaction energy proportional to βB^2. The quantity β is called the magnetic polarizability of the system. In the case of our plane-wave decomposition, the crossed interaction energy term will be proportional to $\beta(\mathbf{k}'\times\boldsymbol{\varepsilon}')\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})$, which is also quadratic in the frequency. This term and the previous electric one are exactly those that appear to second order in the Compton amplitude where the unprimed (primed) variables refer to the incident (outgoing) photon. We shall be working in the Breit frame, where $\omega'=\omega$.

For effects that depend on the spin **S** of the system, the discussion is a little more involved. A possible form for the induced electric dipole vector carrying information about the spin **S** is $\mathbf{p}=\gamma\nabla(\mathbf{S}\cdot\mathbf{B})$. The quantity γ may be called a spin polarizability of the system. In the case of our plane-wave expansion, the crossed interaction energy term for a spin-$\frac{1}{2}$ system ($\mathbf{S}=\boldsymbol{\sigma}/2$) is then $\gamma\omega[\boldsymbol{\varepsilon}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\boldsymbol{\varepsilon}')-\boldsymbol{\varepsilon}'\cdot\mathbf{k}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})]$, where we have taken $\omega'=\omega$. This term, of order ω^3, has exactly the same form of one of the four third-order terms of the Compton amplitude that cannot be fixed by the low-energy theorems. This will be discussed in Sec. II, where all spin polarizabilities and their corresponding phenomenological interactions will be presented. Because of this correspondence, the four associated structure constants will be called spin polarizabilities of the nucleon. The experimental determination of these quantities probably will have to wait some time, since the main efforts in the near future will probably be directed to a better experimental determination of the dipole polarizabilities [6]. On the theoretical side a quark-model calculation may come to mind and for that purpose expressions of the spin polarizabilities in terms of the nucleon electromagnetic transition moments can be of help. This is the main motivation for this paper. It is shown here that these spin polarizabilities can be related to sums of products of electromagnetic transition moments of the nucleon involving the electric and magnetic dipoles, the electric and magnetic quadrupoles, and the charge electric dipole and magnetic dipole mean-square radii. We have obtained also three sum rules which relate the static electromagnetic properties of the nucleon to sums of products of electric dipole transition moments.

A calculation of these multiple transition moments to low-lying nucleon resonances could be attempted by using available experimental data allied with the recent determination of the nucleon transition amplitudes in the context of the relativized quark model [8] or of the nonrelativistic constituent quark model [9]. On the assumption of dominance of the low-lying resonances, we would then be able to estimate the spin polarizabilities. In turn, the three sum rules could be used as a test for the low-lying dominance hypothesis.

In Sec. II we give a general discussion of the scattering amplitude. Then, by using gauge invariance, we will show how the structure constants can be related to products of the electromagnetic current transition matrix elements. Section III is devoted to the expansion of the transition matrix elements to the order needed in our discussion. First, by means of a Lorentz transformation, we bring each intermediate excited state to rest and then expand the resulting nucleon transition moments in the nucleon variables. The relation of this expansion to the electromagnetic multipole transition moments is then established. The whole procedure is rather long but straightforward. In Sec. IV the expressions of the third-order spin polarizabilities are written down together with the three sum rules involving the electric dipole transition moments. The results are discussed in Sec. V.

II. SCATTERING AMPLITUDE

In the transverse gauge ($\varepsilon_0=\varepsilon_0'=0$, $\boldsymbol{\varepsilon}\cdot\mathbf{k}=\boldsymbol{\varepsilon}'\cdot\mathbf{k}'=0$), the scattering amplitude of light by the nucleon can be written as

$$A=\varepsilon'^i T_{ij}\varepsilon^j=\varepsilon'^i(U_{ij}+E_{ij})\varepsilon^j , \tag{2.1}$$

where ε'^i and ε^i are the polarization vectors of the incoming and outgoing photons with momenta $k^\mu=(\omega,\mathbf{k})$ and $k'^\mu=(\omega,\mathbf{k}')$, respectively. Following Low [10], we have separated out the contribution U_{ij} of the one-nucleon on-shell intermediate state. U_{ij} is called the unexcited part of T_{ij} and the rest, E_{ij}, is called the excited part. U_{ij} is the space-space component of

$$U_{\mu\nu}=V^2\left[\frac{\langle \mathbf{p}'|J_\mu|\mathbf{p}+\mathbf{k}\rangle\langle \mathbf{p}+\mathbf{k}|J_\nu|\mathbf{p}\rangle}{E(\mathbf{p}+\mathbf{k})-E-\omega}\frac{m}{E(\mathbf{p}+\mathbf{k})}+\frac{\langle \mathbf{p}'|J_\nu|\mathbf{p}-\mathbf{k}'\rangle\langle \mathbf{p}-\mathbf{k}'|J_\mu|\mathbf{p}\rangle}{E(\mathbf{p}-\mathbf{k}')-E+\omega'}\frac{m}{E(\mathbf{p}-\mathbf{k}')}\right] , \tag{2.2a}$$

where $|\mathbf{p}+\mathbf{k}\rangle$ and $|\mathbf{p}-\mathbf{k}'\rangle$ are one-nucleon on-shell intermediate states with energies $E(\mathbf{p}+\mathbf{k})$ and $E(\mathbf{p}-\mathbf{k}')$, respectively, and an invariant density has been used. A summation over the spins of these intermediate states is implied. The nucleon matrix element for the electromagnetic current operator J_μ is given by

$$\langle \mathbf{p}_2|J_\mu|\mathbf{p}_1\rangle=\frac{1}{V}e\bar{u}(\mathbf{p}_2)\left[F_1(q^2)\gamma_\mu+\frac{i}{2m}F_2(q^2)\sigma_{\mu\nu}q^\nu\right]u(\mathbf{p}_1) , \tag{2.2b}$$

with $q=p_2-p_1$ and the normalization $\bar{u}u=1$. V designates the normalization volume and $\mathbf{p}$ ($\mathbf{p}'$) is the momentum of the initial (final) nucleon with energy E (E') and mass m. The Breit frame, where

$$\mathbf{p}'=-\mathbf{p}=\frac{\mathbf{k}-\mathbf{k}'}{2},\quad E'=E,\quad \omega'=\omega , \tag{2.3}$$

and $E(\mathbf{p}+\mathbf{k})=E(\mathbf{p}-\mathbf{k}')$, is the frame where the requirement of T invariance achieves its simplest form as used by Pais [11] to find the minimal basis $B_{ij}^{(N)}$ in which E_{ij} is to be expanded. To order ω^3 we have

$$\begin{aligned}E_{ij}&=\sum_N a_N(\mathbf{k},\mathbf{k}')B_{ij}^{(N)}\\&=[a_1(0)+a_{1,1}\mathbf{k}\cdot\mathbf{k}'+a_{1,2}\omega^2]\delta_{ij}+i\omega(a_{2,1}+a_{2,2}\mathbf{k}\cdot\mathbf{k}'+\gamma_1\omega^2)\varepsilon_{ijm}\sigma^m+\beta(-k_ik_j'+\mathbf{k}\cdot\mathbf{k}'\delta_{ij})\\&\quad+a_4(0)k_i'k_j+a_5(0)(k_i'k_j'+k_ik_j)+i\omega\gamma_2[\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})\delta_{ij}-\mathbf{k}\cdot\mathbf{k}'\epsilon_{ijm}\sigma^m]\\&\quad+i\omega\gamma_3[k_i(\boldsymbol{\sigma}\times\mathbf{k})_j-k_j'(\boldsymbol{\sigma}\times\mathbf{k}')_i]+i\omega\gamma_4[k_i(\boldsymbol{\sigma}\times\mathbf{k}')_j-k_j'(\boldsymbol{\sigma}\times\mathbf{k})_i-2\mathbf{k}\cdot\mathbf{k}'\varepsilon_{ijm}\sigma^m]\\&\quad+i\omega a_{9,1}[k_i'(\boldsymbol{\sigma}\times\mathbf{k}')_j-k_j(\boldsymbol{\sigma}\times\mathbf{k})_i]+i\omega a_{10,1}[k_i'(\boldsymbol{\sigma}\times\mathbf{k})_j-k_j(\boldsymbol{\sigma}\times\mathbf{k}')_i] ,\end{aligned} \tag{2.4}$$

where we have used the indication $a_{2,3}=\gamma_1$ for what we shall call the first third-order spin polarizability of the nucleon, $a_3(0)=\beta$ for the magnetic dipole polarizability, and $a_{6,1}-a_{8,1}=\gamma_2-\gamma_4$ for the other three spin polarizabilities, so-called for reasons to be discussed below. The expansion of the coefficients is in accordance with the even-crossing-symmetry property of E_{ij}, that is, invariance under the transformation $i\leftrightarrow j$, $\mathbf{k}\leftrightarrow-\mathbf{k}'$,

$\omega \leftrightarrow -\omega' = -\omega$. For convenience, we have used parity- and time-reversal-invariant basis elements $B_{ij}^{(N)}$ for $N=7-10$, of which the corresponding basis elements $E_{ij}^{(N)}$ of Pais, there numbered $N=8-11$, are linear combinations. With the basis elements that we have chosen, the contraction with k'^i automatically gives final terms which are independent of each other, as they are written in Eq. (2.11). The coefficients a_1, $a_{1,1}$, $a_{2,1}$, $a_{2,2}$, a_4, and a_5 are the ones that obey low-energy theorems [10–13]. We quote their values, here calculated in the Breit frame:

$$a_1(0) = -\frac{e^2}{m} , \tag{2.5a}$$

$$a_{2,1} = \frac{2\mu - 1}{2m^2} e^2 , \tag{2.5b}$$

$$a_{1,1} = \frac{e^2}{4m^3} , \tag{2.5c}$$

$$a_4(0) = 0 , \tag{2.5d}$$

$$a_5(0) = \frac{\mu(1-\mu)e^2}{4m^3} , \tag{2.5e}$$

and

$$a_{2,2} = \frac{1-2\mu}{8m^4} e^2 , \tag{2.5f}$$

where μ is the magnetic moment of the nucleon in units of $e/2m$. Equation (2.5a) is the frequency zeroth-order low-energy theorem first derived by Thirring [12] and Eq. (2.5b) gives Low's first-order result, associated with the term of order ω in Eq. (2.4). Equation (2.5c) gives the second-order result derived by Singh [13], here calculated in the Breit frame (in the laboratory frame it would give zero), and Eq. (2.5f) gives the third-order result of Lin [7], associated with the first term of order ω^3 in Eq. (2.4). Because of the transversality condition, a_4 and a_5 will actually not be present in the amplitude, and so they do not give interesting low-energy theorems. However, as will become clear later, one needs the value of a_5 to disentangle the known part of the second-order coefficient $a_{1,2}$, whose unknown part α represents the electric polarizability of the system. We have, in the Breit frame,

$$a_{1,2} = \alpha + \frac{e^2}{m} \left[2F_1' + \frac{2\mu^2 - 1}{4m^3} \right] , \tag{2.5g}$$

where $F_1' = [dF_1(t)/dt]_{t=0}$.

We shall be concerned with the third-order unknown coefficients $\gamma_1 - \gamma_4$, $a_{9,1}$, and $a_{10,1}$. Actually, again by the transversality condition, the last two coefficients will not be present in the scattering amplitude. However, we shall need the expression for $a_{9,1}$ to obtain one of the sum rules. From Eq. (2.4) we have

$$\begin{aligned} \varepsilon'^i E_{ij} \varepsilon^j = & (a_1 + a_{1,1}\mathbf{k}\cdot\mathbf{k}' + a_{1,2}\omega^2)\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon} \\ & + i\omega(a_{2,1} + a_{2,2}\mathbf{k}\cdot\mathbf{k}' + \gamma_1\omega^2)\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}'\times\boldsymbol{\varepsilon}) \\ & + \beta(\mathbf{k}'\times\boldsymbol{\varepsilon}')\cdot(\mathbf{k}\times\boldsymbol{\varepsilon}) \\ & + i\omega\varepsilon'\cdot(\gamma_2 B^{(6)} + \gamma_3 B^{(7)} + \gamma_4 B^{(8)})\cdot\varepsilon , \end{aligned} \tag{2.6}$$

where a dyadic notation has been used in the last three terms. Note that we have the two unknown second-order terms $a_{1,2}\omega^2\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}$ and $\beta(\mathbf{k}'\times\boldsymbol{\varepsilon}')\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})$, which correspond to the first two phenomenological interaction terms mentioned in the Introduction. This is what motivated the names electric and magnetic polarizabilities given, respectively, to $a_{1,2}$ (or, as has become common, to its unknown part α) and β. Next, we have the four other unknown terms of third order, $\gamma_1 - \gamma_4$, which correspond to the four classical phenomenological interactions mentioned in Sec. I. One can see that γ_3 corresponds to the one discussed there $\gamma_3\omega\varepsilon'\cdot B^{(4)}\cdot\varepsilon \sim \gamma_3 \mathbf{E}\cdot\nabla(\mathbf{S}\cdot\mathbf{B})$, and the other three are related to an equal number of independent phenomenological interactions according to the following scheme:

$$\gamma_1\omega^3\varepsilon'\cdot B^{(2)}\cdot\varepsilon \sim \mathbf{E}\cdot[\mathbf{S}\times(\nabla\times\mathbf{B})] ,$$

$$\gamma_2\omega\varepsilon'\cdot(2B^{(6)} - B^{(8)})\cdot\varepsilon \sim \mathbf{B}\cdot\nabla(\mathbf{S}\cdot\mathbf{E}) ,$$

and

$$\gamma_4\omega\varepsilon'\cdot B^{(8)}\cdot\varepsilon \sim \gamma_4 \mathbf{B}\cdot[\nabla\times(\nabla\times\mathbf{E})] .$$

This correspondence is the motivation for the name spin polarizabilities given to $\gamma_1 - \gamma_4$.

To get information about the third-order coefficients, we consider one of the gauge conditions associated with each photon, $k'^\mu T_{\mu\nu} = T_{\nu\lambda}k^\lambda = 0$. From here we obtain, with $T = U + E$,

$$k'^i E_{ij} + \omega' E_{0j} = -k'^i U_{ij} - \omega' U_{0j} . \tag{2.7}$$

The time-space component E_{0j} satisfies the relation [14]

$$E_{0j} = -k'^i \Gamma_{ij} , \tag{2.8}$$

where, with an invariant density for the intermediate states,

$$\Gamma_{ij} = V^2 \sum_n \left[\frac{\langle \mathbf{p}'|J_i|n,\mathbf{p}+\mathbf{k}\rangle\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle}{[E_n(\mathbf{p}+\mathbf{k}) - E'][E_n(\mathbf{p}+\mathbf{k}) - E - \omega]} \frac{M_n}{E_n(\mathbf{p}+\mathbf{k})} - \frac{\langle \mathbf{p}'|J_j|n,\mathbf{p}-\mathbf{k}'\rangle\langle n,\mathbf{p}-\mathbf{k}'|J_i|\mathbf{p}\rangle}{[E_n(\mathbf{p}-\mathbf{k}') - E][E_n(\mathbf{p}-\mathbf{k}') - E + \omega']} \frac{M_n}{E_n(\mathbf{p}-\mathbf{k}')} \right] , \tag{2.9}$$

which is odd under crossing in the Breit frame, $\mathbf{p}' = -\mathbf{p}$, since here $E' = E$, $\omega' = \omega$, and $E_n(\mathbf{p}+\mathbf{k}) = E_n(\mathbf{p}-\mathbf{k}')$. Expanding Γ_{ij} in the basis elements $B_{ij}^{(N)}$ defined in Eq. (2.4) and using (2.8), we shall have the general form of E_{0j}. As we need E_{ij} to order ω^3, E_{0j} will have to be considered to this same order, and Eq. (2.8) tells us that it is enough to consider Γ_{ij} to order ω^2 only. As Γ_{ij} is odd under crossing, we then have, to order ω^2,

$$\Gamma_{ij}=\sum_N b_N(\mathbf{k},\mathbf{k}')B_{ij}^N$$
$$=b_{1,1}\omega\delta_{ij}+i[b_2(0)+b_{2,1}\mathbf{k}\cdot\mathbf{k}'+b_{2,2}\omega^2]\varepsilon_{ijm}\sigma^m+\sum_{N=6}^{10}b_N(0)B_{ij}^{(N)}\ . \quad (2.10)$$

The basis elements $B_{ij}^{(3-5)}$ are absent because their coefficients should start with a factor ω, giving terms of order ω^3. Substituting Eq. (2.10) into (2.8), we shall have the general form of E_{0j}. Taking it into Eq. (2.7) and making use of Eq. (2.4), we obtain

$$-k_j'[a_1+a_{1,1}\mathbf{k}\cdot\mathbf{k}'+(a_{1,2}+a_5-b_{1,1})\omega^2]-k_j[a_4\omega^2+a_5\mathbf{k}\cdot\mathbf{k}']$$
$$-i\omega(\boldsymbol{\sigma}\times\mathbf{k}')_j[a_{2,1}-b_2+(a_{2,2}-\gamma_2-\gamma_4-b_{2,1}+b_6+b_8)\mathbf{k}\cdot\mathbf{k}'+(\gamma_1+a_{9,1}-b_{2,2}-b_9)\omega^2]$$
$$-i\omega(\boldsymbol{\sigma}\times\mathbf{k})_j[(\gamma_3-b_7)\mathbf{k}'\cdot\mathbf{k}+(a_{10,1}-b_{10})\omega^2]$$
$$-i\omega\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})k_j'[\gamma_2+\gamma_4-b_6-b_8]-i\omega\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})k_j(a_{9,1}-b_9)$$
$$=V^2\frac{m}{E(\mathbf{p}+\mathbf{k})}[\langle\mathbf{p}'|J_0|\mathbf{p}+\mathbf{k}\rangle\langle\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle-\langle\mathbf{p}'|J_j|\mathbf{p}-\mathbf{k}'\rangle\langle\mathbf{p}-\mathbf{k}'|J_0|\rangle]\ , \quad (2.11)$$

where use has been made of the equation of continuity to rewrite the right-hand side of Eq. (2.7) in a convenient way. Calculating the right-hand side of Eq. (2.11), we shall obtain the values of all coefficients in its left-hand side. Without performing any calculation, it is easy to see that $a_{2,1}-b_2$ and all the next coefficients that are associated with terms of even order in ω will be equal to zero. This is because the right-hand side of Eq. (2.11) can produce only terms of odd order in ω because of the fact that the nucleon matrix element of J_0 can produce only even terms in ω and of J_j only odd. We then have from the last three even terms on the left-hand side of Eq. (2.11) the relations

$$\gamma_2+\gamma_4=b_6+b_8\ , \quad (2.12)$$

$$\gamma_3=b_7\ , \quad (2.13)$$

$$a_{9,1}=b_9,\quad a_{10,1}=b_{10}\ , \quad (2.14)$$

and, from the one before,

$$a_{2,1}=b_2,\quad a_{2,2}=b_{2,1}\ , \quad (2.15)$$

and

$$\gamma_1=b_{2,2}\ . \quad (2.16)$$

The results of (2.12), (2.13), and (2.16) pave the way for the derivation of the multipole expressions of the spin coefficients. By expanding the current transition matrix elements in Eq. (2.9) in terms of multipole moments, we shall have Γ_{ij} in terms of them. Then, from Eq. (2.10), we shall have all the b's in terms of the transition moments and, by making use of Eqs. (2.12), (2.13), and (2.16), we shall have the spin polarizabilities in terms of them. The method cannot give γ_2 and γ_4 separately. We shall come back to this point later. Performing the actual calculation of the right-hand side of Eq. (2.11), we obtain, together with Eqs. (2.12)–(2.16), the low-energy results quoted in Eqs. (2.5a) and (2.5c)–(2.5e) and the value of the coefficient of $k_j'\omega^2$ in the left-hand side of (2.11),

$$a_{1,2}+a_5-b_{1,1}=(e^2/m)[2F_1'+(\mu^2+\mu-1)/4m^2]\ .$$

From here we get the result in Eq. (2.5g) by making use of the value of a_5 and of the relation $b_{1,1}=\alpha$, which will be shown below, in Eq. (2.21).

Next, we will show that three sum rules will appear when we make use of the second independent gauge condition $T_{0\lambda}k^\lambda=0$. This gives us the relation

$$E_{00}\omega+E_{0j}k^j=-U_{00}\omega-U_{0j}k^j\ , \quad (2.17)$$

where [10], with an invariant density for the excited intermediate states,

$$E_{00}=V^2\sum_n\left[\frac{\langle\mathbf{p}'|J_0|n,\mathbf{p}+\mathbf{k}\rangle\langle n,\mathbf{p}+\mathbf{k}|J_0|\mathbf{p}'\rangle}{E_n(\mathbf{p}+\mathbf{k})-E-\omega}+\text{c.t.}\right]\ . \quad (2.18)$$

Here c.t. stands for the crossed term. The general form of E_{00} is provided by Singh's lemma [13] according to which $E_{00}=k'^ik^j\Lambda_{ij}$, where Λ_{ij} is even under crossing. Being so, its expansion in terms of the basis elements defined in Eq. (2.4) is

$$\Lambda_{ij}=\alpha\delta_{ij}+ic_{2,1}\omega\varepsilon_{ijm}\sigma^m+O(\omega^2)\ ,$$

which gives, to order ω^3,

$$E_{00}=k'^ik^j(\alpha\delta_{ij}+i\omega c_{2,1}\varepsilon_{ijm}\sigma^m)\ . \quad (2.19)$$

Using this result and Eq. (2.8) in Eq. (2.17), we obtain

$$\omega(\alpha-b_{1,1})\mathbf{k}'\cdot\mathbf{k}-i[b_2(0)+b_{2,1}\mathbf{k}\cdot\mathbf{k}'+(b_{22}-c_{2,1}+2b_9)\omega^2]\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})$$
$$=V^2\frac{m}{E(\mathbf{p}+\mathbf{k})}[\langle\mathbf{p}'|J_0|\mathbf{p}+\mathbf{k}\rangle\langle\mathbf{p}+\mathbf{k}|J_0|\mathbf{p}\rangle-\text{c.t.}]\ , \quad (2.20)$$

where use has been made again of the equation of continuity to rewrite the right-hand side in a convenient form. Calculating the right-hand side of this equation, we obtain

$$b_{1,1}=\alpha\ , \quad (2.21)$$

$$b_2(0)=\frac{2\mu-1}{2m^2}e^2 , \tag{2.22}$$

$$b_{2,1}=\frac{1-2\mu}{8m^4}e^2 , \tag{2.23}$$

and the relation

$$b_{2,2}-c_{2,1}+2b_9=-\frac{e^2}{m^2}\left[\mu F_1'+F_2'+\frac{2\mu^2-2\mu+1}{8m^2}\right] , \tag{2.24}$$

where $F_a'=[dF_a(t)/dt]_{t=0}$.

The result in Eq. (2.21) could be anticipated by noting that, as the nucleon matrix element of J_0 can produce terms of even order in ω, the first term of Eq. (2.20) has to be zero. Substitution of Eqs. (2.22) and (2.23) into Eq. (2.15) will give us the two remaining low-energy results quoted in Eqs. (2.5b) and (2.5f). In the course of the calculation, we have then recovered all the low-energy theorems to order ω^3. By having $b_2(0)$, $b_{2,1}$, $b_{2,2}$, $c_{2,1}$, and $b_{9,1}$ in terms of the multipole transition moments, Eqs. (2.22)–(2.24) will give us three sum rules involving these quantities. The expression for $c_{2,1}$ can be obtained very easily by just pushing Baldin's calculation of E_{00} in Eq. (2.18) to the next order. Following Baldin, we expand the transition matrix of J_0, here to order ω^2:

$$V\langle \mathbf{p}'|J_0|n,\mathbf{p}+\mathbf{k}\rangle=ik'^i\langle \mathbf{0}|d_i|n,\mathbf{0}\rangle+O(\omega^2,\bar{Q}_{ij},m_i) , \tag{2.25}$$

where d_i is the electric dipole moment operator taken between the nucleon and the excited state at rest (indicated by $|\mathbf{0}\rangle$ and $|n,\mathbf{0}\rangle$, respectively) and the last term, which is of order ω^2, contains the quadrupole moment, charge mean-square radius, and magnetic dipole transition moments. A complete discussion of this term will be given in Sec. III [Eq. (3.56)]. Here we just need the fact that both $\bar{Q}_{ij}$ and m_i have opposite parity to d_i. Being so, there will be no cross products in the numerators of Eq. (2.18), the first one being given by

$$V^2\langle \mathbf{p}'|J_0|n,\mathbf{p}+\mathbf{k}\rangle\langle n,\mathbf{p}+\mathbf{k}|\mathbf{p}\rangle =k'^ik^j\langle \mathbf{0}|d_i|n,\mathbf{0}\rangle\langle n,\mathbf{0}|d_j|\mathbf{0}\rangle+O(\omega^4) . \tag{2.26}$$

Expanding the denominators of (2.18), we will have a term of order ω^2 symmetric in i and j, and another of order ω^3 antisymmetric in i and j, with exactly the same structure of Eq. (2.19), with

$$\alpha\delta_{ij}=\sum_n\frac{\langle \mathbf{0}|d_i|n,\mathbf{0}\rangle\langle \mathbf{0}|d_j|n,\mathbf{0}\rangle+(i,j)}{M_n-m} , \tag{2.27}$$

which is the result of Baldin, and

$$ic_{2,1}\varepsilon_{ijm}\sigma^m=\sum_n\frac{\langle \mathbf{0}|d_i|n,\mathbf{0}\rangle\langle n,\mathbf{0}|d_j|\mathbf{0}\rangle-(i,j)}{(M_n-m)^2} , \tag{2.28}$$

where (i,j) stands for the preceding term with i and j interchanged. Now that we have the expression for $c_{2,1}$, we only need those of $b_{2,2}$ and b_9 to get the sum rule associated with Eq. (2.24).

III. EXPANSION OF THE CURRENT MATRIX ELEMENTS IN MULTIPOLES

In this section we shall expand the current matrix elements in electromagnetic multipoles to the same order ω^2, which we need for Γ_{ij}. We start by relating the matrix elements present in Eq. (2.9) to those in which the intermediate state n is at rest. Call L the Lorentz transformation with velocity

$$\mathbf{v}=\frac{\mathbf{p}+\mathbf{k}}{E_n(\mathbf{p}+\mathbf{k})}=\frac{\mathbf{k}'+\mathbf{k}}{2E_n[(\mathbf{k}'+\mathbf{k})/2]} , \tag{3.1}$$

which brings the intermediate state n at rest in the Breit frame. Then, calling U the state unitary transformation corresponding to L, we have $U|n,\mathbf{p}+\mathbf{k}\rangle=|n,\mathbf{0}\rangle$ and, for the matrix element of the current, having the nucleon state with momentum $\mathbf{p}$ and z component of the spin λ [15],

$$\langle n,\mathbf{p}+\mathbf{k}|J_\mu|\mathbf{p},\lambda\rangle=\sum_\sigma\langle n,\mathbf{0}|UJ_\mu U^{-1}|\mathbf{q},\sigma\rangle D_{\sigma\lambda}(R) , \tag{3.2}$$

where $\mathbf{q}$ is the transformed of $\mathbf{p}$ by L,

$$\mathbf{q}=\mathbf{p}+\mathbf{v}\left[\frac{\mathbf{p}\cdot\mathbf{v}}{v^2}(\gamma-1)-\gamma E(\mathbf{p})\right] =\frac{\mathbf{k}'-\mathbf{k}}{2}-\frac{\gamma E}{E_n}\frac{\mathbf{k}'+\mathbf{k}}{2} , \tag{3.3}$$

where $\gamma=(1-v^2)^{-1/2}$ and we have used the relation $\mathbf{p}\cdot\mathbf{v}=0$. $D(R)=\exp(-i\Theta\mathbf{n}\cdot\sigma/2)$ is the rotation matrix corresponding to the rotation of the nucleon momentum $\mathbf{p}$, with $\Theta\mathbf{n}$ given by $\mathbf{p}\times\mathbf{p}_R=p^2\sin\Theta\mathbf{n}$, where

$$\mathbf{p}_R=[E(\mathbf{p})+m\gamma][E(\mathbf{q})+m]^{-1}\mathbf{q}+\gamma m\mathbf{v}$$

is the rotated vector. To order ω^2 we have

$$D(R)=1-\frac{i\sigma\cdot(\mathbf{v}\times\mathbf{p})}{4m}=1+\frac{i\sigma\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n} . \tag{3.4}$$

Hence, to order ω^2,

$$\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle=\langle n,\mathbf{0}|J_j+v_j(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{J}+J_0)|\mathbf{q}\rangle \times\left[1+\frac{i\sigma\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}\right] . \tag{3.5}$$

Likewise,

$$\langle \mathbf{p}'|J_i|n,\mathbf{p}+\mathbf{k}\rangle=\left[1+\frac{i\sigma\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}\right] \times\langle \mathbf{q}'|J_i+v_i(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{J}+J_0)|n,\mathbf{0}\rangle , \tag{3.6}$$

where $\mathbf{q}$ is the transformed of $\mathbf{p}'=-\mathbf{p}$ by L,

$$\mathbf{q}'=-\frac{\mathbf{k}'-\mathbf{k}}{2}-\frac{\gamma E}{E_n}\frac{\mathbf{k}'+\mathbf{k}}{2} . \tag{3.7}$$

Equation (3.6) follows from (3.5) when the Hermiticity of the current is used together with the fact that we go from $\mathbf{p}$ to $\mathbf{p}'$ by exchanging $\mathbf{k}$ and $\mathbf{k}'$, which will take us from $\mathbf{q}$ to $\mathbf{q}'$. To simplify the notation, an index n has not been attached to $\mathbf{v}$, γ, $\mathbf{q}$, and $\mathbf{q}'$ to show their dependence on the state n.

We shall perform now the multipole expansion of $\langle n,\mathbf{0}|J_j|\mathbf{q}\rangle$ and $\langle n,\mathbf{0}|J_0|\mathbf{q}\rangle$. We shall start with the last one. According to Eq. (3.5), we shall need the expansion of $\langle n,\mathbf{0}|J_0|\mathbf{q}\rangle$ only to first order, but we shall extend it to third order to show how the various electric transition moments that we are interested in can be calculated from the charge density matrix element. As

$$J_0(\mathbf{r})=\exp(-i\mathbf{P}\cdot\mathbf{r})J_0(\mathbf{0})\exp(i\mathbf{P}\cdot\mathbf{r}) ,$$

where $\mathbf{P}$ is the momentum operator, we have the identity used by Baldin,

$$\langle n,\mathbf{0}|J_0(\mathbf{0})|\mathbf{q}\rangle=\frac{1}{V}\left\langle n,\mathbf{0}\left|\int J_0(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}dV\right|\mathbf{q}\right\rangle . \tag{3.8}$$

Expanding the exponential, we have, to third-order,

$$\int J_0(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}dV=Q+iq^id_i-\tfrac{1}{2}q^iq^a\bar{q}_{ia}-\frac{i}{6}q^iq^aq^b\bar{o}_{iab} , \tag{3.9}$$

where Q is the charge operator,

$$d_i=\int J_0x_idV \tag{3.10}$$

is the electric dipole operator, and

$$\bar{q}_{ia}=\int J_0x_ix_adV=q_{ia}+\tfrac{1}{3}\rho\delta_{ia} \tag{3.11}$$

contains the (traceless) electric quadrupole moment

$$q_{ia}=\int J_0(x_ix_a-\tfrac{1}{3}r^2\delta_{ia})dV \tag{3.12}$$

($q^a{}_a=0$) and the charge mean radius

$$\rho=\int J_0r^2dV . \tag{3.13}$$

Also,

$$\bar{o}_{iab}=\int J_0x_ix_ax_bdV=o_{iab}+\tfrac{1}{5}(s_i\delta_{ab}+\text{c.p.}) , \tag{3.14}$$

where c.p. stands for cyclic permutation of the indices of the previous term, contains the completely symmetric (traceless) octopole moment

$$o_{iab}=\int J_0[x_ix_ax_b-\tfrac{1}{5}r^2(x_i\delta_{ab}+\text{c.p.})] \tag{3.15}$$

and the electric dipole mean-square radius

$$s_i=\int J_0r^2x_idV . \tag{3.16}$$

Substituting Eq. (3.9) into (3.8) and noting that the charge operator is diagonal in the baryon states, we obtain (in the sequel we shall drop the comma between n and $\mathbf{0}$ in the n excited rest-state vector)

$$V\langle n\mathbf{0}|J_0|\mathbf{q}\rangle=iq^i\langle n\mathbf{0}|d_i|\mathbf{0}\rangle-\tfrac{1}{2}q^iq^a\langle n\mathbf{0}|\bar{q}_{ia}|\mathbf{q}\rangle-\frac{i}{6}q^iq^aq^b\langle n\mathbf{0}|\bar{o}_{iab}|\mathbf{q}\rangle . \tag{3.17}$$

Next, we expand the various matrix elements in powers of the nucleon momentum $\mathbf{q}$. Introducing the notation

$$\left[\frac{\partial}{\partial q^i}\langle n\mathbf{0}|X|\mathbf{q}\rangle\right]_{\mathbf{q}=0}=(\langle n\mathbf{0}|X|\mathbf{q}\rangle_{,i})_0 , \tag{3.18}$$

we obtain, to third order

$$V\langle n\mathbf{0}|J_0|\mathbf{q}\rangle=iq^i\langle n\mathbf{0}|d_i|\mathbf{0}\rangle-\tfrac{1}{2}q^iq^a\langle n\mathbf{0}|\bar{Q}_{ia}|\mathbf{0}\rangle-\frac{i}{6}q^iq^aq^b\langle n\mathbf{0}|\bar{O}_{iab}|\mathbf{0}\rangle , \tag{3.19}$$

where the second matrix element on the right-hand side is defined by

$$\langle n\mathbf{0}|\bar{Q}_{ia}|\mathbf{0}\rangle=\langle n\mathbf{0}|\bar{q}_{ia}|\mathbf{0}\rangle-i[(\langle n\mathbf{0}|d_i|\mathbf{0}\rangle,a)_0+(i,a)] , \tag{3.20}$$

which is symmetric in i and a, and the third by

$$\langle n\mathbf{0}|\bar{O}_{iab}|\mathbf{0}\rangle=\langle n\mathbf{0}|\bar{o}_{iab}|\mathbf{0}\rangle-i[(\langle n\mathbf{0}|\bar{q}_{ia}|\mathbf{q}\rangle_{,b})_0+\text{c.p.}]-[(\langle n\mathbf{0}|d_i|\mathbf{q}\rangle_{,ab})_0+\text{c.p.}] , \tag{3.21}$$

which is completely symmetric in i, a, and b. In Eq. (3.20) we have a contribution of the moving dipole to the electric multipoles contained in $\bar{q}_{ia}$, and in (3.21) we have contributions of the moving electric dipole d_i and of those in $\bar{q}_{ia}$ to the multipoles contained in $\bar{o}_{iab}$. We introduce the (traceless) generalized electric quadrupole transition moment

$$\langle n\mathbf{0}|Q_{ia}|\mathbf{0}\rangle=\langle n\mathbf{0}|\bar{Q}_{ia}|\mathbf{0}\rangle+\tfrac{1}{3}\delta_{ia}\langle n\mathbf{0}|\bar{Q}_i{}^i|\mathbf{0}\rangle , \tag{3.22}$$

and the generalized charge mean-square radius

$$\langle n\mathbf{0}|R|\mathbf{0}\rangle=-\langle n\mathbf{0}|\bar{Q}_i{}^i|\mathbf{0}\rangle , \tag{3.23}$$

with which

$$\langle n\mathbf{0}|\bar{Q}_{ia}|\mathbf{0}\rangle=\langle n\mathbf{0}|Q_{ia}|\mathbf{0}\rangle+\tfrac{1}{3}\delta_{ia}\langle n\mathbf{0}|R|\mathbf{0}\rangle . \tag{3.24}$$

We also write

$$\langle n\mathbf{0}|\bar{O}_{iab}|\bar{0}\rangle=\langle n\mathbf{0}|O_{iab}|\mathbf{0}\rangle+\tfrac{1}{5}[\langle n\mathbf{0}|S_j|\mathbf{0}\rangle\delta_{ab}+\text{c.p.}] , \tag{3.25}$$

where

$$\langle n\mathbf{0}|O_{iab}|\mathbf{0}\rangle=\langle n\mathbf{0}|\bar{O}_{iab}|\mathbf{0}\rangle+\tfrac{1}{5}[\langle n\mathbf{0}|\bar{O}^s{}_{si}|\mathbf{0}\rangle\delta_{ab}+\text{c.p.}] \tag{3.26}$$

is the (traceless) generalized electric octopole transition moment and

$$\langle n\mathbf{0}|S_i|\mathbf{0}\rangle=-\langle n\mathbf{0}|\bar{O}^a{}_{ai}|\mathbf{0}\rangle \tag{3.27}$$

is the generalized electric dipole mean-square-radius transition moment. From (3.19) we have the relations

$$V(\langle n\mathbf{0}|J_0|\mathbf{q}\rangle_{,i})_0=i\langle n\mathbf{0}|d_i|\mathbf{0}\rangle , \tag{3.28}$$

$$V(\langle n\mathbf{0}|J_0|\mathbf{q}\rangle_{,ia})_0=-\langle n\mathbf{0}|\bar{Q}_{ia}|\mathbf{0}\rangle . \tag{3.29}$$

and

$$V(\langle n0|J_0|\mathbf{q}\rangle_{,iab})_0 = -i\langle n0|\overline{O}_{iab}|0\rangle \ , \tag{3.30}$$

by means of which we can calculate all the previous multipole transition moments in terms of the charge density transition matrix.

We come now to the multipole decomposition of the current transition matrix. Here it is simpler to start from the following identity due to Foldy [16], which is a consequence of the equation of continuity,

$$V\langle n0|J_j|\mathbf{q}\rangle = i[E_n(0)-E(\mathbf{q})]\langle n0|D_j(-\mathbf{q})|\mathbf{q}\rangle - i\varepsilon_j{}^{ar}q_a\langle n0|M_r(-\mathbf{q})|\mathbf{q}\rangle \ , \tag{3.31}$$

where

$$D_j(-\mathbf{q}) = \int_0^1 ds \int J_0(\mathbf{r})x_j e^{-is\mathbf{q}\cdot\mathbf{r}}dV \tag{3.32}$$

and

$$M_j(-q) = \int_0^1 s\, ds \int (\mathbf{r}\times\mathbf{J})_j e^{-is\mathbf{q}\cdot\mathbf{r}}dV \ . \tag{3.33}$$

Expanding the exponential in Eq. (3.32) and recalling Eqs. (3.10), (3.11), and (3.14), we obtain

$$D_j(-\mathbf{q}) = d_j + \frac{i}{2}q^a\overline{q}_{ja} - \tfrac{1}{6}q^a q^b \overline{o}_{jab} \ . \tag{3.34}$$

We note that Eq. (3.32) gives the relation

$$iq^j D_j(-\mathbf{q}) = \int J_0(\mathbf{r})\exp(-i\mathbf{q}\cdot\mathbf{r})dV - Q \ ,$$

and, as the charge Q is diagonal in the baryon states,

$$iq^j\langle n0|D_j(-\mathbf{q})|\mathbf{q}\rangle = V\langle n0|J_0|\mathbf{q}\rangle \ . \tag{3.35}$$

This relation together with (3.34) is equivalent to Eq. (3.17). It also shows that the equation of continuity is obtained again when we contract Eq. (3.31) with q^j. Expanding the exponential in Eq. (3.33), we obtain

$$M_j(-\mathbf{q}) = \mu_j + \frac{i}{3}q^a\lambda_{ja} \ , \tag{3.36}$$

where

$$\boldsymbol{\mu} = \tfrac{1}{2}\int \mathbf{r}\times\mathbf{J}\, dV \tag{3.37}$$

is the magnetic moment operator and

$$\lambda_{ja} = \int (\mathbf{r}\times\mathbf{J})_j x_a dV = h_{ja} + \varepsilon_{jam} i^m \tag{3.38}$$

contains the (already traceless) magnetic quadrupole moment symmetric operator

$$h_{ja} = \tfrac{1}{2}(\lambda_{ja} + \lambda_{aj}) \ , \tag{3.39}$$

and the magnetic dipole mean-square radius i_m is given by

$$\varepsilon_{jam} i^m = \tfrac{1}{2}(\lambda_{ja} - \lambda_{aj}) \tag{3.40}$$

or

$$\mathbf{i} = \tfrac{1}{2}\int (\mathbf{r}\times\mathbf{J})\times\mathbf{r}\, dV \ . \tag{3.41}$$

Substituting Eqs. (3.34) and (3.36) in (3.31), we obtain, to second order in $\mathbf{q}$,

$$V\langle n0|J_j|\mathbf{q}\rangle = i\left[M_n - m - \frac{q^2}{2m}\right]\left\langle n0\left| d_j + \frac{i}{2}q^a\overline{q}_{ja} - \tfrac{1}{6}q^a q^b\overline{o}_{jab}\right|\mathbf{q}\right\rangle - i\varepsilon_j{}^{ar}q_a\left\langle n0\left|\mu_r + \frac{i}{3}q^b\lambda_{rb}\right|\mathbf{q}\right\rangle \ . \tag{3.42}$$

Now we expand the various elements in powers of $\mathbf{q}$. Recalling Eqs. (3.20) and (3.21), we obtain

$$V\langle n0|J_j|\mathbf{q}\rangle = i\left[M_n - m - \frac{q^2}{2m}\right]\langle n0|d_j|0\rangle - (M_n - m)[\tfrac{1}{2}q^a\langle n0|\overline{Q}_{ja}|0\rangle + \frac{i}{6}q^a q^b\langle n0|\overline{O}_{jab}|0\rangle] - i\varepsilon_j{}^{ar}q_a\left[\langle n0|m_r|0\rangle + \frac{i}{3}q^b\langle n0|L_{rb}|0\rangle\right] , \tag{3.43}$$

where we have introduced the new transition matrix elements

$$\langle n0|m_r|0\rangle = \langle n0|\mu_r|0\rangle - \tfrac{1}{2}(M_n - m)\varepsilon_r{}^{ab}(\langle n0|d_a|\mathbf{q}\rangle_{,b})_0 \ , \tag{3.44}$$

which is the generalized magnetic dipole transition moment, which contains a contribution from the moving electric dipole moment, and

$$\langle n0|L_{rb}|0\rangle = \langle n0|\lambda_{rb}|0\rangle - 3i(\langle n0|\mu_r|\mathbf{q}\rangle_{,b})_0 - i\delta_{rb}(\langle n0|\mu^s|\mathbf{q}\rangle_{,s})_0 - (M_n - m)\varepsilon_r{}^{sa}[i(\langle n0|d_s|\mathbf{q}\rangle_{,ab})_0 - \tfrac{1}{2}(\langle n0|\overline{q}_{sb}|\mathbf{q}\rangle_{,a})_0] \ , \tag{3.45}$$

which contains contributions of the moving magnetic dipole moment and of moving electric multipoles to the magnetic multipoles contained in λ_{rb}. We have added the term proportional to δ_{rb} in the right-hand side of (3.45), which does not contribute to the last term of Eq. (3.43), to make $\langle n0|L_{rb}|0\rangle$ traceless. The symmetric and antisymmetric parts of Eq. (3.45) gives, respectively, the generalized magnetic quadrupole transition moment

$$\langle n0|H_{rb}|0\rangle = \tfrac{1}{2}[\langle n0|L_{rb}|0\rangle + (r,b)] \tag{3.46}$$

and the generalized magnetic dipole mean-square-radius transition moment

$$\langle n0|I_i|0\rangle = -\tfrac{1}{2}\varepsilon_i{}^{ab}\langle n0|L_{ab}|0\rangle \ . \tag{3.47}$$

We have

$$\langle n0|L_{rb}|0\rangle = \langle n0|H_{rb}|0\rangle + \varepsilon_{rb}{}^{a}\langle n0|I_a|0\rangle \ . \tag{3.48}$$

From (3.43) we see that the generalized magnetic dipole transition moment can be calculated from the current matrix element according to

$$\langle n0|m_j|0\rangle = -\frac{i}{2}V\varepsilon_j{}^{ab}(\langle n0|J_a|\mathbf{q}\rangle_{,b})_o \ . \tag{3.49}$$

We also have

$$\langle n0|L_{ij}|0\rangle = -V\varepsilon_i{}^{rs}(\langle n0|J_r|\mathbf{q}\rangle_{,sj})_o - \frac{i}{m}\varepsilon_{ij}{}^{s}\langle n0|d_s|0\rangle \ , \tag{3.50}$$

from which we can calculate both (3.46) and (3.47) from the current matrix element. The transition electric moments can be calculated either from (3.28)–(3.30) or, using (3.43), from

$$V\langle n0|J_j|0\rangle = i(M_n-m)\langle n0|d_j|0\rangle \ , \tag{3.51}$$

$$V[(\langle n0|J_j|\mathbf{q}\rangle_{,i})_0+(j,i)] = -(M_n-m)\langle n0|\bar{Q}_{ij}|0\rangle \ , \tag{3.52}$$

and

$$V\left\{\left[(\langle n0|J_j|\mathbf{q}\rangle_{,ab})_0+\frac{i}{m}\langle n0|d_j|0\rangle\delta_{ab}\right]+\text{c.p.}\right\} = -i(M_n-m)\langle n0|\bar{O}_{jab}|0\rangle \ . \tag{3.53}$$

Using Eqs. (3.17) and (3.43) in Eq. (3.5), we obtain our final desired complete multipole expansion to order ω^2,

$$\begin{aligned} V\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle = {} & i\langle n,0|d_j|0\rangle\left[M_n-m-\frac{q^2}{2m}+\frac{i\sigma\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}(M_n-m)\right]-i\langle n,0|d_a|0\rangle[\tfrac{1}{2}v^a(M_n-m)-q^a]v_j \\ & -(M_n-m)\left[\tfrac{1}{2}q^a\langle n,0|\bar{Q}_{ja}|0\rangle+\frac{i}{6}q^aq^b\langle n,0|\bar{O}_{jab}|0\rangle\right] \\ & -i\varepsilon_j{}^{ar}q_a\left[\langle n,0|m_r|0\rangle+\frac{i}{3}q^b\langle n,0|L_{rb}|0\rangle\right] \ . \end{aligned} \tag{3.54}$$

Also, from the Hermiticity of the current operator and from the fact that we go from $\mathbf{p}$ to $\mathbf{p}'$ by exchanging $\mathbf{k}$ and $\mathbf{k}'$, which will also take us from $\mathbf{q}$ to $\mathbf{q}'$,

$$\begin{aligned} V\langle\mathbf{p}'|J_i|n,\mathbf{p}+\mathbf{k}\rangle = {} & -i\left[M_n-m-\frac{q'^2}{2m}+\frac{i\sigma\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}(M_n-m)\right]\langle 0|d_i|n,0\rangle+iv_i[\tfrac{1}{2}v^a(M_n-m)-q'^a]\langle 0|d_a|n,0\rangle \\ & -(M_n-m)\left[\tfrac{1}{2}q'^a\langle 0|\bar{Q}_{ia}|n,0\rangle-\frac{i}{6}q'^aq'^b\langle 0|\bar{O}_{iab}|n,0\rangle\right] \\ & +i\varepsilon_i{}^{ar}q'_a\left[\langle 0|m_r|n,0\rangle-\frac{i}{3}q'^b\langle 0|L_{rb}|n,0\rangle\right] \ . \end{aligned} \tag{3.55}$$

We can easily justify Eq. (2.25) by making use of the fact that the matrix element of J_o can be obtained by the equation of continuity,

$$\langle n,\mathbf{p}+\mathbf{k}|J_0|\mathbf{p}\rangle = -\frac{1}{E_n(\mathbf{p}+\mathbf{k})-E}k^j\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle \ . \tag{3.56}$$

Using (3.54), we immediately get (2.25).

IV. MULTIPOLE EXPANSION OF Γ_{ij}

Now we substitute Eqs. (3.54) and (3.55) in Eq. (2.9) and pay attention to the fact that there are no cross products between matrix elements of multipoles of opposite parities. Also, by (3.1), (3.3), and (3.7), the interchange $\mathbf{k}\leftrightarrow-\mathbf{k}'$ leads to the transformations

$$\mathbf{v}\rightarrow-\mathbf{v}, \quad \mathbf{q}\leftrightarrow-\mathbf{q}' \ . \tag{4.1}$$

Finally, from Eqs. (3.3) and (3.7), we have $q'^2=q^2$. We then obtain

$$\Gamma_{ij}=\sum_n\left[1+\frac{\omega^2}{(M_n-m)^2}-\frac{\omega^2+\mathbf{k}\cdot\mathbf{k}'}{2}\frac{3M_n+m}{2(M_n-m)M_n^2}\right][\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle-(i,j)]$$
$$+\omega\sum_n\left[\frac{\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle+(i,j)}{M_n-m}\right]-\sum_n\left\{v_j\left[\tfrac{1}{2}v^a-\frac{q^a}{M_n-m}\right][\langle 0|d_i|n0\rangle\langle n0|d_a|0\rangle-(i,a)]-\text{c.t.}\right\}$$
$$+i\sum_n\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle+(i,j)]+i\sum_n[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle+(i,j)]\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}$$
$$-\tfrac{1}{6}\sum_n\{q^aq^b[\langle 0|d_i|n0\rangle\langle n0|\bar{O}_{jab}|0\rangle-\langle 0|\bar{O}_{jab}|n0\rangle\langle n0|d_i|0\rangle]-\text{c.t.}\}$$
$$+\tfrac{1}{4}\sum_n\{q'^aq^b[\langle 0|\bar{Q}_{ia}|n0\rangle\langle n0|\bar{Q}_{jb}|0\rangle-\langle 0|\bar{Q}_{jb}|n0\rangle\langle n0|\bar{Q}_{ia}|0\rangle]\}$$
$$+\frac{i}{2}\sum_n\left\{\varepsilon_j{}^{br}q'^aq_b\frac{\langle 0|\bar{Q}_{ia}|n0\rangle\langle n0|m_r|0\rangle+\langle 0|m_r|n0\rangle\langle n0|\bar{Q}_{ia}|0\rangle}{M_n-m}-\text{c.t.}\right\}$$
$$-\frac{i}{3}\sum_n\left\{\varepsilon_j{}^{ar}q_aq^b\frac{\langle 0|d_i|n_0\rangle\langle n0|L_{rb}|0\rangle+\langle 0|L_{rb}|n0\rangle\langle n0|d_i|0\rangle}{M_n-m}-\text{c.t.}\right\}$$
$$+\sum_n\varepsilon_i{}^{ar}\varepsilon_j{}^{bs}q'_aq_b\left[\frac{\langle 0|m_r|n0\rangle\langle n0|m_s|0\rangle-(r,s)}{(M_n-m)^2}\right],\qquad(4.2)$$

where c.t. stands for the crossed term $(i\leftrightarrow j,\mathbf{q}\leftrightarrow-\mathbf{q}',\mathbf{v}\leftrightarrow-\mathbf{v})$.

Now, from Eqs. (3.3) and (3.7), we have the following relations to order ω^2:

$$4q_aq_b=k_ak_b\left[1+\frac{m}{M_n}\right]^2-(k'_ak_b+k_ak'_b)\left[1-\frac{m^2}{M_n^2}\right]+k'_ak'_b\left[1-\frac{m}{M_n}\right]^2,\qquad(4.3a)$$

$$4q'_aq_b=k'_ak_b\left[1+\frac{m}{M_n}\right]^2-(k'_ak'_b+k_ak_b)\left[1-\frac{m^2}{M_n^2}\right]+k_ak'_b\left[1-\frac{m}{M_n}\right]^2,\qquad(4.3b)$$

and

$$q^2=q'^2=\tfrac{1}{2}\omega^2\left[1+\frac{m^2}{M_n^2}\right]-\tfrac{1}{2}\mathbf{k}\cdot\mathbf{k}'\left[1-\frac{m^2}{M_n^2}\right].\qquad(4.3c)$$

Next, we note that on reducing $\langle n0|\bar{O}_{jab}|0\rangle$ according to (3.25) the octopole can give no contribution since $\langle 0|d_i|n0\rangle\langle n0|O_{jab}|0\rangle=0$. In fact, as d_i has spin parity $J^P=1^-$ and O_{jab} is a 3^- object, the state n in the first matrix element can be nonzero only if it is a $\frac{1}{2}^-$ or $\frac{3}{2}^-$ state and in the second it can be only a $\frac{5}{2}^-$ or $\frac{7}{2}^-$ state. Also, on reducing $\langle 0|\bar{Q}_{ia}|n0\rangle$ according to (3.24), the mean-square radius can give no contribution to the term of Eq. (4.2) containing products of $\bar{Q}$'s. This is due to the fact that in the first place

$$\langle 0|R|n0\rangle\langle n0|Q_{jb}|0\rangle=0$$

because, for the first matrix element to be nonzero, n has to be a $\frac{1}{2}^+$ state (R has $J^P=0^+$) and for the second it has to be a $\frac{3}{2}^+$ or $\frac{5}{2}^+$ state (Q_{ia} has $J^P=2^+$). Second, the product

$$\langle 0|R|n0\rangle\langle n0|R|0\rangle\delta_{ia}\delta_{jb}$$

will be canceled by the crossed term. Together with (2.29) and (2.30), we introduce the following quantities which will be all present in the right-hand side of Eq. (4.2) when we make use of Eqs. (4.3a)–(4.3c):

$$\sum_n[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle-(i,j)]=iA_1\varepsilon_{ijm}\sigma^m,\qquad(4.4)$$

$$\tfrac{1}{8}\sum_n\frac{3M_n+m}{(M_n-m)M_n^2}[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle-(i,j)]=iA_2\varepsilon_{ijm}\sigma^m,\qquad(4.5)$$

$$\tfrac{1}{8}\sum_n\frac{1}{M_n^2}[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle-(i,j)]=iA_3\varepsilon_{ijm}\sigma^m.\qquad(4.6)$$

Note that the right-hand side of these relations is dictated by the fact that their left-hand sides are quantities antisymmetric in i,j between nucleon states at rest. Also,

$$\tfrac{1}{4}\sum_n \frac{1}{mM_n}[\langle 0|d_i|n0\rangle\langle n0|d_j|0\rangle+(i,j)]=A_4\delta_{ij} \tag{4.7}$$

is symmetric in i,j. Next, we have various sets of relations whose writing will be simplified if we call f_{Nn} ($N=1,2,3$) the three M_n-dependent coefficients in (4.3a) and (4.3b),

$$f_{1n}=\left[1+\frac{m}{M_n}\right]^2,\quad f_{2n}=\left[1-\frac{m^2}{M_n^2}\right],\quad f_{3n}=\left[1-\frac{m}{M_n}\right]^2. \tag{4.8}$$

The first set of relations is, with a convenient numerical factor,

$$\tfrac{1}{60}\sum_n f_{Nn}[\langle 0|d_i|n0\rangle\langle n0|S_j|0\rangle-\langle 0|S_j|n0\rangle\langle n0|d_i|0\rangle]=iB_N\varepsilon_{ijr}\sigma^r, \tag{4.9}$$

with $N=1,2,3$. A $B_N'\delta_{ij}$ contribution is excluded by the fact that under time reversal the left-hand side of (4.9) changes sign (since $\langle 0|d_i|n0\rangle$ goes into $\langle n0|d_i|0\rangle$ and similarly for S_i), as does σ_r on the right-hand side. Next, we define

$$\tfrac{1}{16}\sum_n f_{Nn}[\langle 0|Q_{ia}|n0\rangle\langle n0|Q_{jb}|0\rangle-\langle 0|Q_{jb}|n0\rangle\langle n0|Q_{ia}|0\rangle]=iC_N[\delta_{ij}\varepsilon_{abr}+\delta_{ab}\varepsilon_{ijr}+\delta_{ib}\varepsilon_{ajr}+\delta_{ja}\varepsilon_{ibr}]\sigma^r, \tag{4.10}$$

which is symmetric and traceless in (i,a) and (j,b) as are the Q's. A term

$$C_N'[\delta_{ij}\delta_{ab}+\delta_{ib}\delta_{aj}-\tfrac{2}{3}\delta_{ia}\delta_{jb}]$$

is excluded by T invariance. Next comes the set

$$\tfrac{1}{24}\sum_n f_{Nn}\frac{\langle 0|R|n0\rangle\langle n0|m_r|0\rangle-\langle 0|m_r|n0\rangle\langle n0|R|0\rangle}{M_n-m}=D_N\sigma_r \tag{4.11}$$

and the set

$$\tfrac{1}{12}\sum_n f_{Nn}\frac{\langle 0|Q_{ia}|n0\rangle\langle n0|m_r|0\rangle+\langle 0|m_r|n0\rangle\langle n0|Q_{ia}|0\rangle}{M_n-m}=E_N(\delta_{ir}\sigma_a+\delta_{ar}\sigma_i-\tfrac{2}{3}\delta_{ia}\sigma_r), \tag{4.12}$$

which is symmetric and traceless in i and a, as is Q_{ia}.

Then we define

$$\tfrac{1}{2}\sum_n f_{Nn}\frac{0|m_i|n0\rangle\langle n0|m_j|0\rangle-(i,j)}{(M_n-m)^2}=iF_N\varepsilon_{ijm}\sigma^r. \tag{4.13}$$

Finally, on reducing $\langle n0|L_{rb}|0\rangle$ according to (3.48), will appear the set

$$\tfrac{1}{12}\sum_n f_{Nn}\frac{\langle 0|d_i|0\rangle\langle n0|I_j|0\rangle+\langle 0|I_j|n0\rangle\langle n0|d_i|0\rangle}{M_n-m}=G_N\varepsilon_{ijm}\sigma^m \tag{4.14}$$

(a contribution $G_N'\delta_{ij}$ is excluded since under time reversal the left-hand side changes sign because $\langle n0|I_j|0\rangle$ goes into $-\langle 0|I_j|n0\rangle$, with a minus sign not present for d_i) and the set

$$\tfrac{1}{12}\sum_n f_{Nn}\frac{\langle 0|d_i|n0\rangle\langle n0|H_{rb}|0\rangle+\langle 0|H_{rb}|n0\rangle\langle n0|d_i|0\rangle}{M_n-m}=H_N(\delta_{ir}\sigma_b+\delta_{ib}\sigma_r-\tfrac{2}{3}\delta_{rb}\sigma_i), \tag{4.15}$$

which is symmetric and traceless in r and b, as is H_{rb}.

Also, we shall make use of the following identities (just start by expanding $\boldsymbol{\sigma}\cdot\{[(\mathbf{p}\times\mathbf{q})\times\mathbf{a}]\times\mathbf{b}\}$ in two different ways):

$$\sigma_i(\mathbf{k}'\times\mathbf{k})_j+\sigma_j(\mathbf{k}'\times\mathbf{k})_i=B_{ij}^{(10)}-B_{ij}^{(8)}+2B_{ij}^{(6)}, \tag{4.16a}$$

$$\varepsilon_{ijm}(k^m\boldsymbol{\sigma}\cdot\mathbf{k}'+k'^m\boldsymbol{\sigma}\cdot\mathbf{k})=-B_{ij}^{(8)}-B_{ij}^{(10)}, \tag{4.16b}$$

and

$$\varepsilon_{ijm}(k^m\boldsymbol{\sigma}\cdot\mathbf{k}+k'^m\boldsymbol{\sigma}\cdot\mathbf{k}')=-B_{ij}^{(7)}-B_{ij}^{(9)}+2\omega^2B^{(2)}. \tag{4.16c}$$

After a rather long but straightforward calculation, the final result that follows from (4.2) is that Γ_{ij} will have the form (2.10) with the following values for the b's:

$$b_{1,1}=\alpha, \tag{4.17}$$

$$b_2(0)=A_1, \tag{4.18}$$

$$b_{2,1}=A_4+4B_2+4C_1-2D_3+2E_3-2G_2-2H_2, \tag{4.19}$$

$$b_{2,2}=c_{2,1}-2A_2-B_1-B_3-2C_2-6E_2-2F_2+2G_1+2G_3-2H_1-2H_3=\gamma_1, \tag{4.20}$$

$$b_6 = A_4 + C_1 - C_3 - 3E_1 + 3E_3 + F_1 - F_3 \ , \quad (4.21)$$

$$b_7 = -A_3 - B_3 - C_2 + D_2 + 2E_2 + F_2 - G_3 + 2H_1 + H_3 = \gamma_3 \ , \quad (4.22)$$

$$b_8 = A_2 + B_2 + C_1 - D_3 - 2E_3 - F_1 + G_2 - 3H_2 \ , \quad (4.23)$$

$$b_9 = A_2 - B_1 - C_2 + D_2 + 2E_2 + F_2 - G_1 + H_1 + 2H_3 = a_{9,1} \ , \quad (4.24)$$

$$b_{10} = -A_3 + B_2 + C_3 - D_1 - 2E_1 - F_3 + G_2 - 3H_2 = a_{10,1} \ , \quad (4.25)$$

where use has been made of Eqs. (2.13) and (2.16) for the last steps in Eqs. (4.22) and (4.20), respectively. From (2.12) we find that $\gamma_2 + \gamma_4$ will be given by the sum of the right-hand sides of (4.21) and (4.23).

Equations (4.20) and (4.22) and the sum of (4.21) and (4.23) give us the relation between the spin polarizabilities and the multipole transition moments. The method cannot give γ_2 and γ_4 separately. This could be settled if we analyze E_{ij} directly. We intend to come back to this point in the future. The result (4.17) was already obtained in Eq. (2.21). The next three relations, Eqs. (4.18)–(4.20), will give rise to the three sum rules when we compare with the results obtained in Eqs. (2.22)–(2.24), respectively, with the help of (4.24). The comparison of the first pair of equations gives

$$\frac{2\mu - 1}{2m^2} e^2 = A_1 \quad (4.26)$$

or, from (4.4),

$$i \frac{2\mu - 1}{2m^2} e^2 \epsilon_{ijm} \sigma^m = \sum_n [\langle 0|d_i|n0\rangle \langle n0|d_j|0\rangle - (i,j)] \ . \quad (4.27)$$

From the second pair, we get

$$\frac{1-2\mu}{8m^4} e^2 = A_4 + 4B_2 + 4C_1 - 2D_3 + 2E_3 - 2G_2 - 2H_2 \ , \quad (4.28)$$

and from the third pair in conjunction with Eq. (4.24),

$$\frac{e^2}{m^2} \left[\mu F_1' + F_2' + \frac{2\mu^2 - 2\mu + 1}{8m^2} \right] = 3B_1 + B_3 + 4C_2 - 2D_2 + 2E_2 - 2G_3 - 2H_3 \ . \quad (4.29)$$

V. DISCUSSION

We have studied the terms of the excited-state part of the nucleon Compton amplitude that are of third-order in the frequency of the incoming photon. Their interest comes from the fact that it is at this order that effects depending on the spin of the target first appear. These quantities cannot be expressed in terms of the electromagnetic static properties of the nucleon, as given by the low-energy theorems. There are four of them present in the scattering amplitude scalar. We have shown that there is a correspondence of these four terms with phenomenological interactions describing the influence of electromagnetic fields in a medium with spin in terms of polarizability effects, and this correspondence justified the name of spin polarizabilities given to them.

Following the approach that has been used for the second-order electromagnetic dipole polarizabilities [2,3], we have tried to express the third-order spin polarizability in terms of multipole nucleon transition moments. Using gauge invariance, we have obtained a closed expression for two of them (γ_1 and γ_3) and for the sum of the other two ($\gamma_2 + \gamma_4$) in terms of products of electromagnetic transition moments of the nucleon involving the electric and magnetic dipole, quadrupole, and dipole mean-square radii. Three sum rules arise from the calculation involving products of electric dipole transition moments. The method cannot give separate expressions for two of the spin polarizabilities γ_2 and γ_4. For that purpose one would have to analyze the excited-state contribution E_{ij} directly. Work in this direction is in progress. These closed expressions can be of help for an estimate of two of the spin polarizabilities and for the sum of the other two, on the assumption of dominance of the low-lying nucleon resonances, using available experimental data allied with the recent determination of the nucleon transition amplitudes in the context of the relativized quark model [7] or of the nonrelativistic constituent quark model [8]. The three sum rules could be used to test the low-lying nucleon resonance dominance hypothesis.

[1] A. Klein, Phys. Rev. **99**, 998 (1955).
[2] A. M. Baldin, Nucl. Phys. **18**, 310 (1960).
[3] V. A. Petrun'kin, Zh. Eksp. Teor. Fiz. **40**, 1148 (1961) [Sov. Phys. JETP **13**, 808 (1961)].
[4] S. R. Choudhury and D. Z. Freedman, Phys. Rev. **168**, 1739 (1968).
[5] S. Ragusa, Phys. Rev. D **11**, 1536 (1975).
[6] (a) A. I. L'Vov and V. A. Petrun'kin, in *Proceedings of the Workshop on Perspectives on Photon Interactions with Hadrons and Nuclei,* Lecture Notes in Physics Vol. 365 (Springer-Verlag, Berlin, 1990).
[6] (b) J. L. Friar, in *Electron-Nucleus Scattering,* Proceedings of the Workshop, Marciana Marina, Italy, 1988, edited by A. Fabrocini *et al.* (World Scientific, Singapore, 1989).
[7] K. Y. Lin, Nuovo Cimento A **2**, 695 (1971); Phys. Rev. D **24**, 1014 (1981).
[8] M. Warns, H. Schröder, W. Pfeil, and H. Rollnik, Z. Phys. C **45**, 613 (1990); **45**, 627 (1990).
[9] N. Isgur and G. Karl, Phys. Rev. D **18**, 4187 (1978); **19**, 2653 (1979); **23**, 817 (1981); N. Isgur, G. Karl, and R. Koniuk, *ibid.* **25**, 2394 (1982).
[10] F. F. Low, Phys. Rev. **96**, 1428 (1954).
[11] A. Pais, Nuovo Cimento A **53**, 433 (1968).
[12] W. Thirring, Philos. Mag. **41**, 1193 (1950).
[13] V. Singh, Phys. Rev. Lett. **19**, 730 (1967); Phys. Rev. **165**, 1532 (1968).
[14] S. Ragusa, Phys. Rev. D **29**, 2636 (1984).
[15] S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), Chap. 4.
[16] L. L. Foldy, Phys. Rev. **92**, 178 (1953).