Equivalence and compositeness: Beyond $1/N_c$ in four-fermion theories

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The equivalence between four-fermion and Yukawa models under the conditions $Z_3=Z_4=0$ is rederived. These compositeness conditions are shown to be a simple self-consistent procedure for computing beyond the usual large-N_c or fermion bubble approximation. In the case of the two-flavor Nambu–Jona-Lasinio model the $1/N_c$ expansion is found to diverge for $N_c \le 24$. The correct mass relationship and chiral-symmetry-restoration temperature T_c are obtained as $m_\sigma^2=2m_q^2+m_\pi^2$ and $T_c=f_\pi$.

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The relationship between Nambu–Jona-Lasinio (NJL) type [1] four-fermion models and boson-fermion ("Yukawa") theories has often been a subject of investigation. The first systematic study was made by Lurié and Macfarlane [2], who used the Schwinger-Dyson equations to demonstrate the equivalence between four-fermion and Yukawa models provided one imposes the condition $Z_3=0$ in the latter. Subsequently, Eguchi [3] reformulated the equivalence using functional integrals, pointing to $Z_3=Z_4=0$ as compositeness conditions. Eguchi also employed renormalization-group arguments to conjecture on $Z_3=Z_4=0$ as eigenvalue conditions for finite radiatively generated couplings when the equivalent Yukawa theory possesses nontrivial ultraviolet fixed points.

More recently, there has been a revival of interest in the NJL-Yukawa connection with compositeness being imposed as a boundary condition on the renormalization-group equations [4,5]. These studies, which were directed to dynamical electroweak symmetry breaking, are notable in demonstrating numerically the failure of the customary large-N_c or fermion bubble approximation in predicting mass relationships. Lattice methods have also been brought to bear [6].

In this paper we will show the relevance of these ideas to low-energy hadronic physics where NJL-Yukawa models may be viewed as effective theories for QCD [7]. We formulate equivalence directly in terms of the compositeness conditions $Z_3=Z_4=0$ rather than as boundary conditions. This has the advantage of avoiding some subtleties in the renormalization-group approach (see below) and more readily extending to finite temperature. Further, we thereby easily obtain the relationships the underlying four-fermion model imposes on the equivalent Yukawa theory as analytic functions of N_c and so may address the convergence of the $1/N_c$ expansion.

We begin by recounting some features of the Gell-Mann–Lévy linear σ model [8]. The approximately chiral SU(2)-invariant Lagrangian density is

$$\mathcal{L}_\sigma=\bar\psi[i\partial\!\!\!/-g(\sigma+i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau})]\psi+\tfrac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma+\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi})-\frac{\mu^2}{2}(\sigma^2+\boldsymbol{\pi}^2)-\frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2+c\sigma\ , \tag{1}$$

where ψ is an isodoublet of u- and d-type quark fields with color multiplicity N_c and c is the small explicit symmetry-breaking term generating PCAC (partial conservation of axial vector current). For $\mu^2\ge 0$ and $c=0$, the model is manifestly renormalizable, the counterterm Lagrangian density being

$$\delta\mathcal{L}_\sigma=(Z_2-1)\bar\psi i\partial\!\!\!/\psi-(Z_1-1)g\bar\psi(\sigma+i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau})\psi+\frac{Z_3-1}{2}(\partial_\mu\sigma\partial^\mu\sigma+\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi})-\frac{\delta\mu^2}{2}(\sigma^2+\boldsymbol{\pi}^2)-(Z_4-1)\frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2\ . \tag{2}$$

A straightforward evaluation of the one-loop renormalization constants using an O(4) cutoff Λ and minimal subtraction at a scale M gives

$$\begin{aligned}
Z_1&=Z_2=1-\frac{g^2}{4\pi^2}\ln\left[\frac{\Lambda}{M}\right],\\
\delta\mu^2&=\frac{4g^2N_c-3\lambda}{8\pi^2}\Lambda^2+\frac{3\lambda}{4\pi^2}\mu^2\ln\left[\frac{\Lambda}{M}\right],\\
Z_3&=1-\frac{N_cg^2}{2\pi^2}\ln\left[\frac{\Lambda}{M}\right],\\
Z_4&=1+\left[3\lambda-\frac{2N_cg^4}{\lambda}\right]\frac{1}{2\pi^2}\ln\left[\frac{\Lambda}{M}\right].
\end{aligned}\tag{3}$$

The coupling constants obey the renormalization-group equations (to one-loop order)

$$M\frac{dg}{dM}=\beta_g(g,\lambda)=\frac{N_c}{4\pi^2}g^3\ , \qquad (4)$$

$$M\frac{d\lambda}{dM}=\beta_\lambda(g,\lambda)=\frac{1}{2\pi^2}(3\lambda^2+2N_cg^2\lambda-2N_cg^4)\ , \qquad (5)$$

or, in terms of $X=\lambda/g^2$,

$$M\frac{dX}{dM}=\beta_X(g,X)=\frac{g^2}{2\pi^2}[3X^2+N_cX-2N_c]\ . \qquad (4')$$

When $\mu^2<0$ and/or $c>0$, the $SU(2)_L\otimes SU(2)_R$ symmetry of $\mathcal{L}_\sigma$ is spontaneously broken down to $SU(2)_V$ through the vacuum expectation value $\langle\sigma\rangle=v$ obeying the (tree) tadpole equation $v[\mu^2+\lambda v^2]=c$. Shifting the fields by $(\sigma,\boldsymbol{\pi})\to(v+\sigma,\boldsymbol{\pi})$ in $\mathcal{L}_\sigma$, the fermions acquire a mass $m_q=gv$ ($=gf_\pi$ by the Goldberger-Treiman relation) and the σ field a mass $m_\sigma^2=2Xm_q^2+m_\pi^2$, while the pions are pseudo Nambu-Goldstone bosons: $m_\pi^2=c/v$. As is well known [9], if the same shift is performed in $\delta\mathcal{L}_\sigma$, the counterterms of (2) and (3) are all that is required to cancel the divergences in the broken symmetry phase also.

Consider now imposing the conditions $Z_3=Z_4=0$; then,

$$\mathcal{L}_\sigma+\delta\mathcal{L}_\sigma=Z_2\bar\psi i\partial\!\!\!/\psi-Z_1g\bar\psi(\sigma+i\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau})\psi-\frac{\mu^2+\delta\mu^2}{2}(\sigma^2+\boldsymbol{\pi}^2)+c\sigma\ . \qquad (6)$$

Solving the Euler-Lagrange field equations, one finds the meson fields as $q\bar q$ composites:

$$(\sigma,\boldsymbol{\pi})=\frac{(c-Z_1g\bar\psi\psi,-Z_1g\bar\psi i\gamma_5\boldsymbol{\tau}\psi)}{\mu^2+\delta\mu^2}\ . \qquad (7)$$

Substitution of Eq. (7) into Eq. (6), or equivalently integrating σ and $\overline{\boldsymbol{\pi}}$ out of the generating functional $W[\bar\eta,\eta]$ for fermion Green's functions, yields

$$\mathcal{L}_\sigma+\delta\mathcal{L}_\sigma=\psi(i\partial\!\!\!/-m)\psi+\frac{G}{2}[(\bar\psi\psi)^2+(\bar\psi i\gamma_5\boldsymbol{\tau}\psi)^2]\ , \qquad (8)$$

where

$$G=(Z_1)^2\frac{g^2}{\mu^2+\delta\mu^2}\ ,\quad m=\frac{Gc}{Z_1g}\ , \qquad (9)$$

which is nothing but the Lagrangian density $\mathcal{L}_{NJL}$ of the NJL model. Clearly, the normal (broken) symmetry phases of the NJL model correspond to $G\le G_c$ ($G>G_c$), with G_c obtained by setting $\mu^2=0$ in (9). Note that (7) and (9) yield the current-algebra relation

$$m\langle\bar\psi\psi\rangle=-f_\pi^2m_\pi^2[1+O(m_\pi^2)]\ .$$

Implementing the $Z_3=0$ condition,

$$g^{-2}(M)=\frac{N_c}{2\pi^2}\ln\left[\frac{\Lambda}{M}\right]\ , \qquad (10)$$

and so

$$Z_1=Z_2=1-\frac{1}{2N_c}\ . \qquad (11)$$

As in Ref. [2] and contrary to Ref. [3], we stress that $Z_3=0$ has nothing to do with divergences in the infinite cutoff limit. Indeed, the renormalized coupling constant vanishes for $\Lambda\to\infty$; however, this is merely the triviality disease of nonasymptotically free-field theories, to which class the linear σ model belongs. More to the point, Z_1 and Z_2 being cutoff and scale independent, (10) determines the running of the Yukawa coupling constant, in which case Λ is a parameter playing the role of a dimensionally transmuted coupling constant.

Using (10), the $Z_4=0$ condition reduces to the quadratic equation

$$3X^2+N_cX-2N_c=0\ , \qquad (12)$$

with positive root

$$X=X_+(N_c)=\frac{N_c}{6}\left[\sqrt{1+24/N_c}-1\right]\ . \qquad (10')$$

Note that X is also scale independent. While $X(N_c\to\infty)\to2$ and $m_\sigma\to2m_q+O(1/N_c,m_\pi^2)$, the usual asymptotic value is approached only for $N_c\gg24$. In fact, $N_c=3$ lies *outside* the radius of convergence of $X_+(N_c)$. For the real world, $X_+(3)=1$, and so $m_\sigma=(2m_q^2+m_\pi^2)^{1/2}$ and the σ meson is a true resonance, lying well below the $q\bar q$ threshold.

The compositeness conditions can be given another interpretation: With $Z_3=Z_4=0$, $\mathcal{L}_\sigma$ corresponds to starting with $\mathcal{L}_{NJL}$, introducing the composite fields, and then adding and subtracting kinetic and self-coupling terms for them to be determined self-consistently. Hence the $Z_3=Z_4=0$ conditions may be viewed as a Hartree-Fock-type procedure, extending the methods of NJL as well as Eguchi and Sugawara [10] beyond fermion bubbles. It is the self-interactions of the composite mesons which are responsible for the σ's binding energy.

In order to estimate the constituent quarks and σ-meson masses, we use quark-counting rules and g_A to relate the meson-quark coupling to the pion-nucleon coupling constant: $g_\pi=gN_cg_A$. Taking the accurate experimental values $g_\pi=13.4\pm0.1$ [11], $g_A=1.2573\pm0.0028$, $f_\pi=92.5\pm0.2$ MeV, and $m_\pi=138$ MeV [12], we find $g=3.55\pm0.04$, $m_q=328\pm4$ MeV, and $m_\sigma=484\pm5$ MeV.

The renormalization-group approach is fundamentally different. By rescaling the meson fields in (1) as $(\sigma,\boldsymbol{\pi})\to(\sigma,\boldsymbol{\pi})/g(M)$ and defining

$$\tilde Z(M)=g^{-2}(M)\ ,\qquad \tilde\lambda(M)=\lambda(M)g^{-4}(M)=\tilde Z(M)X(M)\ , \qquad (13)$$

one observes that $\tilde{\mathcal{L}}_\sigma$ becomes a four-fermion theory at a scale Λ if $g(M\to\Lambda)\to\infty$ with $X(M\to\Lambda)$ finite and positive. Then, by the multiplicative nature of the renormalization group, the four-fermion model is identified with the coupling-constant trajectories of the Yukawa theory subject to these boundary conditions [4]. Indeed, integrating (4) thus also yields (10). Qualitatively similar results for Abelian and discrete chiral models have been obtained within the context of the renormalization group by Hasenfratz *et al.* [6] and Zinn-Justin [13].

There is, however, a subtlety: As discussed by Bando *et al.* [5], the consistent way to formulate compositeness as a boundary condition is in terms of the Wilson renormalization group applied to the cutoff theory. Discrepancies arise from the continuum ($\Lambda \to \infty$) treatment unless (a) one adopts a mass-dependent renormalization prescription to suppress fermion self-energy and vertex contributions to the β functions or (b) they accidently cancel as in our example. A further complication is that to formulate compositeness boundary conditions at nonzero temperature T requires the finite-T renormalization group [14].

Conversely, in our formulation we already have directly from $Z_3 = Z_4 = 0$ the maximal information that can be extracted from the renormalization-group boundary conditions. In addition, the extension to finite T is immediate, particularly in the real-time formalism [15]: Vacuum expectation values $\langle \cdot \rangle$ are replaced by thermal averages $\langle\langle \cdot \rangle\rangle$ and (fermion) Green's functions by temperature Green's functions generated from $W[\bar{\eta},\eta;T]$. Temperature does not modify the ultraviolet behavior of the theory, and so the counterterms of (2) and (3) remain sufficient to renormalize the linear σ model at finite T. Then, under the compositeness condition $Z_3 = Z_4 = 0$, we have the identity $W_\sigma[\bar{\eta},\eta;T] = W_{\rm NJL}[\bar{\eta},\eta;T]$. Hence the fermion-temperature Green's functions coincide.

Now, the thermodynamic quantities, pressure P and energy density ϵ, obtain from

$$\langle\langle (T^{\mu\nu} + \delta T^{\mu\nu}) \rangle\rangle = (\epsilon + P) U^\mu U^\nu - P g^{\mu\nu} , \tag{14}$$

where U^μ is the four-velocity and $(\delta)T^{\mu\nu}$ is the stress-energy tensor derived from $(\delta)\mathcal{L}$. It follows that, under the compositeness conditions,

$$\langle\langle (T_\sigma^{\mu\nu} + \delta T_\sigma^{\mu\nu}) \rangle\rangle = \langle\langle T_{\rm NJL}^{\mu\nu} \rangle\rangle ,$$

so also the thermodynamics of the two models are equivalent when $Z_3 = Z_4 = 0$.

Here we apply equivalence and compositeness to the chiral-symmetry-restoration temperature T_c and the T dependence of the light-quark condensate [16,17]. Restricting our attention to the chiral limit ($m_\pi \to 0$), in the high-temperature mean-field approximation the tadpole equation for an arbitrary number N_f of flavors in (1) reads

$$v[(3+N_f^2-1)\lambda T^2/12 + N_f N_c g^2 T^2/12 + \lambda(v^2 - f_\pi^2)] = 0 , \tag{15}$$

where $v = v(T)$ and $v(0) = f_\pi$. The first two terms in (15) represent the σ and π loops, while the third is the fermion-loop contribution [17]. Keeping only the fermion bubble part, with $N_f = 2$, $N_c = 3$, so that $\lambda = 2g^2$, one finds $T_c = 2 f_\pi$ as the value at which (15) ceases to have nontrivial solutions $v \neq 0$. In the real world where $N_c = 3$ and the composite mesons contribute, $\lambda = g^2$ for $N_f = 2$, and one has instead $T_c = f_\pi$.

In the chiral limit, from (7) and the T independence of the renormalization constants,

$$\langle\langle \bar{\psi}\psi \rangle\rangle = \langle \bar{\psi}\psi \rangle \left[\frac{v(T)}{f_\pi} \right] , \tag{16}$$

and so the quark condensate melts as T approaches T_c. What we observe is that at *low* temperature $T \ll m_\sigma, m_q$, only the composite pion loop contributes to the mean-field tadpole equation,

$$v\lambda[(N_f^2 - 1)T^2/12 + v^2 - f_\pi^2] = 0 , \quad T \ll m_\sigma, m_q , \tag{17}$$

leading to ($N_f = 2$)

$$\langle\langle \bar{\psi}\psi \rangle\rangle = \langle \bar{\psi}\psi \rangle \left[1 - \frac{T^2}{8 f_\pi} \right] , \quad T \ll m_\sigma, m_q . \tag{18}$$

This result has also been obtained in the linear σ model by Contreras and Loewe [18] and is a general consequence of chiral symmetry [19]. It is only through the composite meson self-interactions that the NJL model also satisfies the low-temperature theorem.

Of course, in the real world chiral symmetry is only approximate and only approximately restored, as signaled by a minimum of $m_\sigma(T)$ near T_c; $m_\pi(T)$ is an increasing function of T, while the condensate never melts, but merely fades away as $T \to \infty$ [16,18].

Finally, as deconfinement takes place at or near T_c, one should expect G to have significant temperature variations there. Hence the NJL and linear σ models can only be trusted in the low-temperature regime.

In conclusion, we have shown the equivalence of the NJL model to a linear σ model, both at zero and at nonzero temperature, when the compositeness conditions $Z_3 = Z_4 = 0$ are imposed. Equivalence and compositeness comprise a simple self-consistent scheme for computing beyond the customary large-N_c or fermion bubble approximation in four-fermion theories. We have also demonstrated that $N_c = 3$ lies outside the radius of convergence of the $1/N_c$ expansion. Finally, much of the important physics, such as the σ meson binding energy and the finite-temperature behavior of the quark condensate, lies precisely in the "$1/N_c$ suppressed" composite meson self-interaction.

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

[2] D. Lurié and A. J. Macfarlane, Phys. Rev. **136**, B816 (1964).

[3] T. Eguchi, Phys. Rev. D **14**, 2755 (1976); **17**, 611 (1978).

[4] W. A. Bardeen, C. T. Hill, and M. Lindner, Phys. Rev. D **41**, 1647 (1990).

[5] M. Bando, T. Kugo, N. Maekawa, N. Sasakura, Y. Watabiki, and K. Suehiro, Phys. Lett. B **246**, 466 (1990).

[6] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, and Y. Shen, Nucl. Phys. **B365**, 79 (1991).

[7] Wei-Min Zhang and L. Wilets, Phys. Rev. C **45**, 1900

(1992); V. Bernard, A. A. Osipov, and Ulf-G. Meissner, Phys. Lett. B **285**, 119 (1992); V. Dmitrasinovic, R. H. Lemmer, and R. Tegen, *ibid.* **284**, 201 (1992); H. Weigel, R. Alkofer, and H. Reinhardt, *ibid.* **284**, 296 (1992), and references therein.
[8] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[9] B. W. Lee, Nucl. Phys. **B9**, 649 (1969); J. L. Gervais and B. W. Lee, *ibid.* **B12**, 627 (1969); K. Symanzik, Commun. Math. Phys. **16**, 48 (1970).
[10] T. Eguchi and H. Sugawara, Phys. Rev. D **10**, 4257 (1974).
[11] O. Dumbrajs *et al.*, Nucl. Phys. **B216**, 277 (1983).
[12] Particle Data Group, K. Hikasa *et al.*, Phys. Rev. D **45**, S1 (1992).
[13] J. Zinn-Justin, Nucl. Phys. **B367**, 105 (1991).
[14] H. Matsumoto, Y. Nakano, and H. Umezawa, Phys. Rev. D **29**, 1116 (1984); K. Funakubo and M. Sakamoto, Phys. Lett. B **186**, 205 (1987); O. J. P. Eboli and G. C. Marquez, Phys. Lett. **162B**, 189 (1985).
[15] A. J. Niemi and G. W. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984); Nucl. Phys. **B230**, 181 (1984).
[16] T. Hatsuda and T. Kunihiro, Phys. Rev. Lett. **55**, 158 (1985); Phys. Lett. B **185**, 304 (1987).
[17] J. Cleymans, A. Kocic, and M. D. Scadron, Phys. Rev. D **39**, 323 (1989). Note, however, their Eq. (9) errs in the sign and coefficient of the fermion bubble term.
[18] C. Contreras and M. Loewe, Int. J. Mod. Phys. A **5**, 2297 (1990).
[19] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).