Staggered fermions and chiral-symmetry breaking in transverse lattice regulated QED

Paul A. Griffin*
Department of Physics, University of Florida, Gainesville, Florida 32611
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Staggered fermions are constructed for the transverse lattice regularization scheme. The weak perturbation theory of transverse lattice noncompact QED is developed in the light-cone gauge, and we argue that for fixed lattice spacing this theory is ultraviolet finite, order by order in perturbation theory. However, by calculating the anomalous scaling dimension of the link fields, we find that the interaction Hamiltonian becomes nonrenormalizable for $g^2(a) > 4\pi$, where $g(a)$ is the bare (lattice) QED coupling constant. We conjecture that this is the critical point of the chiral-symmetry-breaking phase transition in QED. Nonperturbative chiral-symmetry breaking is then studied in the strong-coupling limit. The discrete remnant of chiral symmetry that remains on the lattice is spontaneously broken, and the ground state to lowest order in the strong-coupling expansion corresponds to the classical ground state of the two-dimensional spin-$\frac{1}{2}$ Heisenberg antiferromagnet.

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I. INTRODUCTION

Staggered fermions [1] for lattice gauge theory [2,3] have the desirable property of preserving a discrete remnant of chiral symmetry, and are therefore useful for studying nonperturbative chiral-symmetry breaking in gauge theories. Staggered fermions have been constructed for the four-dimensional (4D) Euclidean formulation of lattice gauge theory, and for the Hamiltonian formulation of lattice gauge theory, based on a three-dimensional spatial lattice and one continuum time variable. In Secs. II and III, we construct staggered fermions for the transverse lattice formalism of Bardeen *et al.* [4,5], which is based on a two-dimensional spatial lattice and two continuum space-time coordinates. Wilson fermions for the transverse lattice were constructed in Ref. [4].

The transverse lattice construction is a minimalist's nonperturbative regularization scheme for gauge fields [4]. After choosing an axial gauge and imposing the Gauss constraint, the degrees of freedom of the gauge field are reduced to two spatial components, and these can be regulated by mapping them to link fields on a two-dimensional lattice. The link fields are nonperturbative excitations of the gauge fields, and are scalars with respect to the two continuous space-time coordinates perpendicular to the lattice, so their ultraviolet (UV) behavior is softened.

The basic disadvantage of the transverse lattice construction is the breaking of $(3+1)$-dimensional Lorentz invariance down to $(1+1)$-dimensional Lorentz invariance plus discrete 2D lattice translations and rotations. This means, for example, that pure $(3+1)$-dimensional gauge theory has three bare coupling constants when regulated this way, as dictated by $1+1$ Lorentz invariance [5]. One assumes that the full $3+1$ Lorentz-invariant theory is recovered in the scaling region of the lattice theory for a line of tricritical points of the coupling constants. The tricritical points are determined by examining $3+1$ relativistic dispersion relations.

Weak coupling perturbation theory of transverse lattice noncompact QED (TLQED) is discussed in Sec. IV. After gauge fixing in light-cone gauge, the UV properties of the theory are studied. We argue that the usual diagrammatic UV divergences are cut off by the finite transverse lattice spacing. The transverse lattice construction converts a four-dimensional field theory into a two-dimensional field theory with a finite (for finite sites on the lattice) number of "flavors" which is then UV finite, diagram by diagram, for fixed lattice spacing.

In Sec. V we calculate the anomalous scaling dimension of the link fields on the lattice, and find that the interaction Hamiltonian becomes a nonrenormalizable interaction for $g^2(a) > 4\pi$, where $g(a)$ is the bare QED coupling constant. The anomalous scaling dimension is calculated by normal ordering the link fields and is nonperturbative because the link fields are exponentials of the gauge fields.

The relationship between this phase transition and the phase transition of the sine-Gordon model, the quenched ladder approximation of QED, and quenched noncompact lattice QED is discussed. Based on these analogies, we conjecture that this critical point corresponds to the nonperturbative chiral-symmetry-breaking phase transition in QED.

Recent interest in chiral symmetry breaking in QED was generated by Miransky [6] who used the ladder approximation of the Schwinger-Dyson equation to argue for the existence of a nontrivial UV renormalization-group fixed point of the QED coupling constant. This phenomenon is closely related to the collapse of the Dirac wave function in supercritical ($Z > 137$, for which $\alpha = Ze^2/4\pi > 1$) Coulomb fields [6]. The fixed point is

*Electronic address (Internet): pgriffin@ufhepa.phys.ufl.edu

the boundary of the chirally symmetric ladder QED phase and its strong coupling phase which has spontaneous chiral symmetry breaking [7]. That the strong coupling phase of QED breaks chiral symmetry spontaneously is understood analytically via the strong coupling expansion of lattice gauge theory [1,8,9], and via lattice gauge theory Monte Carlo simulations [10–12]. It is not clear, however, that lattice gauge theory data support the existence of a nontrivial UV fixed point for full QED. It may be the case that the renormalized charge of the continuum theory vanishes at the critical point [13].

In Sec. VI, we study the strong coupling limit of TLQED by calculating the energy shift of the infinite coupling vacuum states to lowest order in the inverse coupling $1/g$. We find that the discrete remnant of chiral symmetry on the transverse lattice is spontaneously broken and that the chiral condensate $\langle \bar{\psi}\psi \rangle$ is nonvanishing for the lowest energy state. We discuss our results further in Sec. VII.

II. STAGGERED FERMIONS FOR THE TRANSVERSE LATTICE

In this section, we construct staggered fermions for the transverse lattice, and in the process, introduce lots of notation that will be used in later sections.

The initial strategy is to write the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ in appropriate component form, and find a fermion equation on the transverse lattice which reproduces these equations in the continuum limit. We use the chiral representation of γ matrices,

$$\gamma^0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.1}$$

and define the fermion ψ components:

$$\psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \quad \varphi = \begin{bmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{bmatrix}, \quad \chi = \begin{bmatrix} \chi^{(1)} \\ \chi^{(2)} \end{bmatrix}. \tag{2.2}$$

In light-cone coordinates,

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3), \tag{2.3}$$

the component equations are

$$\begin{aligned} \sqrt{2}\partial_-\chi^{(1)} &= im\varphi^{(1)} + [\partial_1 - i\partial_2]\chi^{(2)}, \\ \sqrt{2}\partial_+\chi^{(2)} &= im\varphi^{(2)} + [\partial_1 + i\partial_2]\chi^{(1)}, \\ \sqrt{2}\partial_+\varphi^{(1)} &= im\chi^{(1)} - [\partial_1 - i\partial_2]\varphi^{(2)}, \\ \sqrt{2}\partial_-\varphi^{(2)} &= im\chi^{(2)} - [\partial_1 + i\partial_2]\varphi^{(1)}. \end{aligned} \tag{2.4}$$

Now consider a complex one-component fermion field on a discrete square lattice of points $\mathbf{x}_\perp = a(n_x, n_y)$, with lattice spacing a and basis vectors $\boldsymbol{\alpha} = (a,0)$ or $(0,a)$. We will also use the notation $\mathbf{x} = (a,0)$ and $\mathbf{y} = (0,a)$. In this section, the lattice is taken to be infinite. The fermion field ϕ is a continuous function of the light-cone coordinates $x^\pm$, and satisfies the equation

$$\partial_0\phi = P_3(\mathbf{x}_\perp)\partial_3\phi + P_1(\mathbf{x}_\perp)\Delta_1\phi + P_2(\mathbf{x}_\perp)\Delta_2\phi, \tag{2.5}$$

where P_1, P_2, and P_3 are unknowns to be determined by matching to the continuum equations (2.4) with zero mass, and Δ_α is the symmetric lattice derivative

$$\Delta_\alpha f(\mathbf{x}_\perp) = \frac{1}{2a}[f(\mathbf{x}_\perp + \boldsymbol{\alpha}) - f(\mathbf{x}_\perp - \boldsymbol{\alpha})]. \tag{2.6}$$

Since Eq. (2.5) is linear in lattice derivatives, there will be fermion doubling in each lattice direction [1]; i.e., one lattice fermion will correspond to four fermion components in the continuum. In lattice coordinate space, this means that different linear combinations of four adjacent sites will correspond to four different fields in the continuum. We will see how this explicitly occurs in momentum space at the end of this section.

To be more specific, introduce a lattice parity $P_L[\mathbf{x}_\perp] = (-1)^{n_x+n_y}$. If $P_L[\mathbf{x}_\perp]$ is $+1(-1)$, then $\mathbf{x}_\perp$ is an even (odd) site. For the moment, consider the fermion at even or odd sites to be different continuum fields, labeled ϕ_{even} and ϕ_{odd}. Making the ansatz $P_3 = P_L$, the equation of motion (2.5) becomes

$$\begin{aligned} \sqrt{2}\partial_-\phi_{\text{even}} &= P_1\Delta_1\phi_{\text{odd}} + P_2\Delta_2\phi_{\text{odd}}, \\ \sqrt{2}\partial_+\phi_{\text{odd}} &= P_1\Delta_1\phi_{\text{even}} + P_2\Delta_2\phi_{\text{even}}. \end{aligned} \tag{2.7}$$

If we select $P_1 = 1$ and $P_2 = -iP_L$, then Eq. (2.7) are just the massless continuum equations for the Dirac fermion components χ of Eq. (2.4). This is not the complete result, however, because we know that there should be four continuum components. The full result is obtained by breaking up the lattice further into sublattices graded by $(-1)^{n_y}$. The full result is that with the P_1, P_2, P_3 selected above:

$$\begin{aligned} \chi^{(1)} &= \tfrac{1}{2}[\phi(\mathbf{x}_\perp) + \phi(\mathbf{x}_\perp + \mathbf{s})], \quad \mathbf{x}_\perp \text{ even}, \\ \chi^{(2)} &= \tfrac{1}{2}[\phi(\mathbf{x}_\perp) + \phi(\mathbf{x}_\perp + \mathbf{s})], \quad \mathbf{x}_\perp \text{ odd}, \\ \varphi^{(1)} &= \tfrac{1}{2}[\phi(\mathbf{x}_\perp) - \phi(\mathbf{x}_\perp + \mathbf{s})](-1)^{n_x}, \quad \mathbf{x}_\perp \text{ odd}, \\ \varphi^{(2)} &= \tfrac{1}{2}[\phi(\mathbf{x}_\perp) - \phi(\mathbf{x}_\perp + \mathbf{s})](-1)^{n_x}, \quad \mathbf{x}_\perp \text{ even}, \end{aligned} \tag{2.8}$$

where $\mathbf{s} = a(1,1)$. One can easily check that these fields obey the massless version of Eqs. (2.4). Each continuum field is associated with the face of the lattice with center $\mathbf{x}_\perp + \frac{1}{2}\mathbf{s}$. Label each point on the lattice by $((-1)^{n_x}, (-1)^{n_y})$, so that there are four types of points with respect to this grading. Then $\chi^{(1)}, \varphi^{(2)}$ are associated with type A faces, and $\chi^{(2)}, \varphi^{(2)}$ are associated with type B faces, where the faces are labeled in Fig. 1.

The equation of motion of the massless theory, with one four component Dirac fermion in the continuum limit, is

$$\partial_0\phi - (-1)^{n_x+n_y}\partial_3\phi - \Delta_1\phi + i(-1)^{n_x+n_y}\Delta_2\phi = 0. \tag{2.9}$$

We can see more explicitly how the continuum components arise by doing a momentum space mode expansion of the fermion. A fermion mode in momentum space is given by

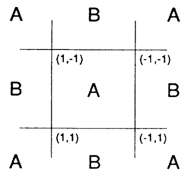

FIG. 1. Each point on the lattice is labeled by $((-1)^{n_x},(-1)^{n_y})$. The fields $\chi^{(1)},\varphi^{(2)}$ are associated with type A faces, and $\chi^{(2)},\varphi^{(1)}$ are associated with type B faces.

$$\tilde{\phi}(k^{\pm},l_\alpha)=\frac{1}{4\pi^2 a}\int d^2x \sum_{\mathbf{x}_\perp} a^2 e^{-ik^0x^0}e^{+ik^3x^3} \times e^{+il_\alpha n_\alpha}\phi(x^{\pm},n_\alpha) , \tag{2.10}$$

where $-\pi<l_\alpha\leq\pi$, and the signature of the spacetime metric is $(+,-,-,-)$.

In momentum space, the equation of motion is

$$ik^0\tilde{\phi}(l)+ik^3\tilde{\phi}(l+\pi)+i\left[\frac{\sin l_1}{a}\right]\tilde{\phi}(l) - \left[\frac{\sin l_2}{a}\right]\tilde{\phi}(l+\pi)=0 , \tag{2.11}$$

where $\tilde{\phi}(l+\pi)=\tilde{\phi}(l_1+\pi,l_2+\pi)$. In the continuum limit, as $a\to 0$, finite energy states are located about $l_\alpha\sim\epsilon$, or $l_\alpha\sim\pi-\epsilon$. There are clearly four continuum fermion components for one transverse lattice fermion. This is just the standard fermion doubling problem, which works to our advantage, in this case, because four continuum components are desired. Divide momentum space into four regions:

$$\begin{aligned} &l_\alpha=p_\alpha+\Pi_A \pmod{2\pi} ,\\ &-\tfrac{1}{2}\pi<p_\alpha\leq\tfrac{1}{2}\pi ,\\ &\Pi_A=\{(0,0),(\pi,0),(0,\pi),(\pi,\pi)\} , \end{aligned} \tag{2.12}$$

and let

$$\begin{aligned} &\tilde{\phi}(p+\Pi_1)=c_1(p)+c_4(p) ,\\ &\tilde{\phi}(p+\Pi_2)=c_3(p)+c_2(p) ,\\ &\tilde{\phi}(p+\Pi_3)=c_3(p)-c_2(p) ,\\ &\tilde{\phi}(p+\Pi_4)=c_1(p)-c_4(p) . \end{aligned} \tag{2.13}$$

With the "continuum" momentum variables

$$k^1=\frac{\sin(p_1)}{a} , \quad k^2=\frac{\sin(p_2)}{a} , \tag{2.14}$$

the massless Dirac equation (2.11) reduces, in the c_i basis, to $ik_\mu\gamma^\mu c=0$.

III. LOCAL GAUGE INVARIANCE, LATTICE SYMMETRIES, AND THE FERMION ACTION

In this section we introduce gauge fields to make the theory defined in the previous section locally gauge invariant at each site. This, however, fails because of the 2D gauge anomaly, and a second set of fermion fields has to be introduced, leading to a second fermion flavor in the continuum limit. Lattice symmetries and the form of the fermion action is discussed, with particular care given to analysis of allowed mass terms.

To promote $\delta_G\phi=i\Lambda\phi$ to a local gauge symmetry, introduce the 2D vector gauge fields A_i and 2D scalar fields A_α with transformation laws

$$\begin{aligned} &\delta_G A_i(\mathbf{x}_\perp,x^{\pm})=\partial_i\Lambda(\mathbf{x}_\perp,x^{\pm}) , \quad i=0,3 ,\\ &\delta_G A_\alpha(\mathbf{x}_\perp,x^{\pm})=\Delta_\alpha^{+}\Lambda(\mathbf{x}_\perp,x^{\pm}) , \quad \alpha=x,y . \end{aligned} \tag{3.1}$$

The forward lattice derivative

$$\Delta_\alpha^{+}f(\mathbf{x}_\perp)=\frac{1}{a}[f(\mathbf{x}_\perp+\boldsymbol{\alpha})-f(\mathbf{x}_\perp)] \tag{3.2}$$

obeys the integration by parts rule

$$\sum_{\mathbf{x}_\perp} f\Delta_\alpha^{+}g=-\sum_{\mathbf{x}_\perp}(\Delta_\alpha^{-}f)g ,$$

where

$$\Delta_\alpha^{-}f(\mathbf{x}_\perp)=\frac{1}{a}[f(\mathbf{x}_\perp)-f(\mathbf{x}_\perp-\boldsymbol{\alpha})] . \tag{3.3}$$

The Lagrangian for the gauge fields is

$$\mathcal{L}_{\text{gauge}}=\sum_{\mathbf{x}_\perp} a^2\left[\frac{1}{4g_1^2}(F_{ij})^2+\frac{2}{4g_2^2}(F_{i\alpha})^2+\frac{1}{4g_3^2}(F_{\alpha\beta})^2\right] , \tag{3.4}$$

where g_1, g_2, and g_3 will be fixed by requiring 3+1 Lorentz invariance in the continuum limit. The field strengths $F_{\mu\nu}$ are

$$\begin{aligned} &F_{ij}=\partial_i A_j-\partial_j A_i , \quad F_{\alpha\beta}=\Delta_\alpha^{+}A_\beta-\Delta_\beta^{+}A_\alpha ,\\ &F_{i\alpha}=\partial_i A_\alpha-\Delta_\alpha^{+}A_i . \end{aligned} \tag{3.5}$$

For the fermion fields, we dress derivatives via the usual minimal coupling procedure:

$$\begin{aligned} &\partial_i\phi\to D_i\phi=(\partial_i-iA_i)\phi ,\\ &\Delta_\alpha\phi\to D_\alpha\phi=\{\phi(\mathbf{x}_\perp+\boldsymbol{\alpha})e^{-iaA_\alpha(\mathbf{x}_\perp)} -\phi(\mathbf{x}_\perp-\boldsymbol{\alpha})e^{iaA_\alpha(\mathbf{x}_\perp-\alpha)}\}/2a . \end{aligned} \tag{3.6}$$

However, this construction does not yield a gauge-invariant theory. The 2D kinetic terms for the fermions are

$$\begin{aligned} \mathcal{L}&=i\sqrt{2}\phi^\dagger(\partial_- -iA_-)\phi+\cdots , \quad \mathbf{x}_\perp \text{ even}\\ &=i\sqrt{2}\phi^\dagger(\partial_+ -iA_+)\phi+\cdots , \quad \mathbf{x}_\perp \text{ odd} . \end{aligned} \tag{3.7}$$

There is only a single left- or right-handed fermion for each local U(1) gauge symmetry, and therefore, the local U(1) symmetries are anomalous. The anomaly breaks the U(1)×U(1) symmetry of pairs of sites (say $\mathbf{x}_\perp$ and $\mathbf{x}_\perp+P_L[\mathbf{x}_\perp]\mathbf{y}$) to the diagonal U(1). In principle, one can construct transverse lattice QED with the remaining U(1) symmetry [14]. However, in practice, it will be easier to add a second flavor of lattice fermions to cancel the anomalies and preserve the full set of U(1) symmetries. The fermion action takes the form

$$\mathcal{L}_F = i\sum_{f=1}^{2}\sum_{\mathbf{x}_\perp} a^2\phi_f^\dagger\{[D_0+(-1)^{n_x+n_y+f}D_3]\phi_f - \kappa[D_x+i(-1)^{n_x+n_y+f}D_y]\phi_f\}\ , \tag{3.8}$$

where κ is a hopping parameter that will be fixed by requiring 3+1 Lorentz invariance. While $\kappa=1$ in the classical continuum limit, it will receive quantum corrections and in fact will have to be renormalized With two flavors on the lattice, there will be two Dirac fermions in the continuum limit. Their components φ and χ are constructed from different flavors of the lattice fermions, i.e., ϕ_1 and ϕ_2 contribute to each of the two continuum Dirac fermions. The components of the continuum fermions

$$\Psi_j = \begin{bmatrix}\varphi_j\\ \chi_j\end{bmatrix}\ ,\quad j=1,2\ , \tag{3.9}$$

are

$$\begin{aligned}
\chi_1^{(1)} &= \tfrac{1}{2}[\varphi_1(\mathbf{x}_\perp)+\phi_1(\mathbf{x}_\perp+\mathbf{s})]\ ,\quad \mathbf{x}_\perp \text{ even}\ ,\\
\chi_1^{(2)} &= \tfrac{1}{2}[\phi_1(\mathbf{x}_\perp)+\phi_1(\mathbf{x}_\perp+\mathbf{s})]\ ,\quad \mathbf{x}_\perp \text{ odd}\ ,\\
\varphi_1^{(1)} &= \tfrac{1}{2}[\phi_2(\mathbf{x}_\perp)-\phi_2(\mathbf{x}_\perp+\mathbf{s})](-1)^{n_x}\ ,\quad \mathbf{x}_\perp \text{ even}\ ,\\
\varphi_1^{(2)} &= \tfrac{1}{2}[\phi_2(\mathbf{x}_\perp)-\phi_2(\mathbf{x}_\perp+\mathbf{s})](-1)^{n_x}\ ,\quad \mathbf{x}_\perp \text{ odd}\ ,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\chi_2^{(1)} &= \tfrac{1}{2}[\phi_2(\mathbf{x}_\perp)+\phi_2(\mathbf{x}_\perp+\mathbf{s})]\ ,\quad \mathbf{x}_\perp \text{ odd}\ ,\\
\chi_2^{(2)} &= \tfrac{1}{2}[\phi_2(\mathbf{x}_\perp)+\phi_2(\mathbf{x}_\perp+\mathbf{s})]\ ,\quad \mathbf{x}_\perp \text{ even}\ ,\\
\varphi_2^{(1)} &= \tfrac{1}{2}[\phi_1(\mathbf{x}_\perp)-\phi_1(\mathbf{x}_\perp+\mathbf{s})](-1)^{n_x}\ ,\quad \mathbf{x}_\perp \text{ odd}\ ,\\
\varphi_2^{(2)} &= \tfrac{1}{2}[\phi_1(\mathbf{x}_\perp)-\phi_1(\mathbf{x}_\perp+\mathbf{s})](-1)^{n_x}\ ,\quad \mathbf{x}_\perp \text{ even}\ .
\end{aligned} \tag{3.11}$$

The mass term $\sum_{\mathbf{x}_\perp} a^2(m/\sqrt{2})\sum_j \bar{\Psi}_j\Psi_j$ is found, by explicit evaluation, to be

$$\mathcal{L}_m = \sum_{\mathbf{x}_\perp} a^2\left\{\frac{m}{\sqrt{2}}(-1)^{n_x}[\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1]\right\}\ , \tag{3.12}$$

and it clearly preserves the U(1) symmetries for all the sites.

We now discuss the other symmetries of the massless lattice action.

(i) Continuous two-dimensional Lorentz invariance:

$$\delta\phi_f(x) = \lambda(-1)^{n_x+n_y+f}\phi_f(x')\ . \tag{3.13}$$

In terms of the continuum fermion Ψ defined in Eq. (3.9), the transformation is

$$\delta\Psi = \lambda\gamma^0\gamma^3\Psi(x')\ . \tag{3.14}$$

as expected.

(ii) Two-dimensional parity:

$$\begin{aligned}
\phi_1(x^3) &\to (-1)^{n_y}\phi_2(-x^3)\ ,\\
\phi_2(x^3) &\to (-1)^{n_y}\phi_1(-x^3)\ .
\end{aligned} \tag{3.15}$$

In terms of the continuum fields,

$$\Psi(x^3) \to \sigma^3\gamma^5\gamma^3\Psi(-x^3)\ , \tag{3.16}$$

where the σ^i are Pauli matrices that act in flavor space.

(iii) y-coordinate shift invariance:

$$\begin{aligned}
\phi_1(\mathbf{x}_\perp) &\to \phi_2(\mathbf{x}_\perp+\mathbf{y})\ ,\\
\phi_2(\mathbf{x}_\perp) &\to \phi_1(\mathbf{x}_\perp+\mathbf{y})\ .
\end{aligned} \tag{3.17}$$

This is a flavor change in terms of Ψ:

$$\Psi(\mathbf{x}_\perp) \to \sigma^1\Psi(\mathbf{x}_\perp+\mathbf{y})\ . \tag{3.18}$$

Note that two shifts generate a simple translation.

(iv) x-coordinate shift invariance:

$$\begin{aligned}
\phi_1(\mathbf{x}_\perp) &\to \phi_2(\mathbf{x}_\perp+\mathbf{x})\ ,\\
\phi_2(\mathbf{x}_\perp) &\to \phi_1(\mathbf{x}_\perp+\mathbf{x})\ .
\end{aligned} \tag{3.19}$$

This is an axial flavor change in terms of Ψ:

$$\Psi(\mathbf{x}_\perp) \to \sigma^1\gamma^5\Psi(\mathbf{x}_\perp+\mathbf{x})\ . \tag{3.20}$$

Again, two shifts generate a simple translation. The combination of x and y shifts generates $\Psi\to\gamma^5\Psi$, which is a discrete chiral symmetry. This takes $\chi_j\to+\chi_j$ and $\varphi_j\to-\varphi_j$, and corresponds to a discrete Z_2 subgroup of the 4D anomalous axial U(1) symmetry. The mass term (3.12) explicitly breaks both the discrete chiral symmetry (generated by $\mathbf{x}_\perp\to\mathbf{x}_\perp+\mathbf{s}$), and the axial vector flavor symmetry.

(v) y-axis reversal:

$$\phi_f(n_y) \to (-1)^{n_y}\phi_f(-n_y)\ . \tag{3.21}$$

In terms of Ψ,

$$\Psi(n_y) \to \sigma^2\gamma^5\gamma^2\Psi(-n_y)\ . \tag{3.22}$$

(vi) x-axis reversal:

$$\phi_f(n_x) \to (-1)^{n_x}\phi_f(-n_x)\ . \tag{3.23}$$

In terms of Ψ,

$$\Psi(n_x) \to -\sigma^1\gamma^5\gamma^1\Psi(-n_x)\ . \tag{3.24}$$

(vii) x-y rotational invariance:

$$\begin{aligned}
\phi_f(n_x,n_y) &\to \phi_f(-n_y,n_x)\ ,\quad \mathbf{x}_\perp \text{ even}\ ,\\
\phi_f(n_x,n_y) &\to i(-1)^{f+1}\phi_f(-n_y,n_x)\ ,\quad \mathbf{x}_\perp \text{ odd}\ ,
\end{aligned} \tag{3.25}$$

Writing this in terms of Ψ is not particularly illuminating. This symmetry is why we need only one arbitrary parameter κ in the Lagrangian, instead of one for the D_x term and one for the D_y term. This can be interpreted as a spin transformation; these transformations are typically applied to staggered fermion systems to diagonalize γ matrices in the fermion action (see Refs. [15], [16], and [17] for further discussion on this point).

The 2D gauge theory for each site also has an anomalous chiral transformation:

$$\delta\phi_f(\mathbf{x}_\perp)=i\lambda(-1)^{f+1}\phi_f(\mathbf{x}_\perp) \ . \tag{3.26}$$

This symmetry is broken at one loop in perturbation theory by the usual anomaly. It corresponds in the 4D continuum limit to a broken axial-vector flavor symmetry, under which the continuum components transform as

$$\delta\Psi=-i\lambda\sigma_3\gamma_5\Psi \ . \tag{3.27}$$

There are no continuous global flavor symmetries for this model. This is in contrast with the naive (Wilson) and staggered (Susskind) formulations of QED on 4D Euclidean lattices [15,18]. The action for a single 4D naive massless fermion on the 4D lattice has U(4) vector and axial-vector flavor symmetries, which is a subgroup of the full U(16) flavor symmetries of the 16 continuum Dirac fermions of this model. The minimal staggered massless fermion action on the 4D lattice has U(1) vector and axial-vector flavor symmetries on the lattice, which is a subgroup of the U(4) flavor symmetries of the 4 continuum Dirac fermions for this model. The transverse lattice model constructed in this section has no continuous flavor symmetries, and has only two continuum Dirac fermions in the continuum limit. For the fermions on the 4D lattices, the axial-vector flavor symmetries which exist in the lattice action are spontaneously broken in the strong coupling limit. The nonvanishing of the order operator $\overline{\Psi}\Psi$, which signals the breaking of the axial-vector flavor symmetries of the lattice models, is confirmed, by Monte Carlo simulations, for the scaling region of the theory [19,20]. This order operator breaks all of the continuum axial flavor symmetries, and one expects the full multiplet of Goldstone bosons associated with the full set of broken axial symmetries in the scaling regime. In Sec. VI, we will show that the discrete chiral symmetry of the transverse lattice model is spontaneously broken in the strong coupling limit by the nonvanishing vacuum expectation value of $\overline{\Psi}\cdot\Psi$.

With the symmetries listed above, we will now see which "mass terms" of the form

$$\overline{\Psi}FG\Psi \tag{3.28}$$

can occur in the Lagrangian, where F is a 2×2 matrix in flavor space, and G is a 4×4 matrix with spinor indices. Terms of this form could in principle be generated as contact terms when evaluating the fermion self-energy at one loop (for discussion of this in the context of 4D lattice QCD, see Ref. [21]). The 2D Lorentz invariance places a strong constraint on the form of G, namely, $[\gamma^0\gamma^3,G]=0$. There are only eight linearly independent G matrices that satisfy the commutator: 1, γ^5, γ^α, $\gamma^5\gamma^\alpha$, $i\gamma^1\gamma^2$, and $i\gamma^0\gamma^3$, where $\alpha=1,2$. The y-coordinate shift invariance requires that $[\sigma^1,F]=0$: $F=1$ or σ^1. For $F=1$, applying the symmetries (ii), (v), and (vi) above leaves just one G, $G=1$. This is the mass term $\overline{\Psi}\Psi$ given by Eq. (3.12). For $F=\sigma^3$, applying the same symmetries leaves $G=\gamma^5\gamma^1$, so there is a second possible term, $\overline{\Psi}i\sigma^1\gamma^5\gamma^1\Psi$. In terms of the lattice fermions, it is given by

$$\mathcal{L}'=m'\sum_{\mathbf{x}_\perp}a^2[\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1] \ . \tag{3.29}$$

However, this is not rotationally invariant, unlike Eq. (3.12). Under symmetry (vii),

$$\mathcal{L}'\to m'\sum_{\mathbf{x}_\perp}a^2(-1)^{n_x+n_y}[\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1] \ . \tag{3.30}$$

In particular if a y-coordinate shift is then implemented on Eq. (3.30), and another rotation is performed, one gets $-\mathcal{L}'$. So $m'=0$, by a combination of symmetries (iii) and (vii), and the lattice symmetries are sufficient to guarantee that no additional counterterms, other than the flavor symmetric mass term, will be required at one loop. In addition, if the bare mass is set to zero, the renormalized mass in weak perturbation theory will also vanish, by the symmetry (iv). However, this symmetry will be spontaneously broken in the intermediate and strong coupling regimes, to which we now turn.

IV. ULTRAVIOLET FINITENESS OF PERTURBATION THEORY

In this section, the weak coupling perturbation expansion will be developed for the transverse lattice theory with Lagrangians given by Eqs. (3.4) and (3.8). It will be argued that the transverse lattice regulates all the UV divergences for each diagram in perturbation theory. We will use this formalism in the next section to calculate the nonperturbative scaling dimension of the interaction Hamiltonian.

Axial gauges minimize the mixing of longitudinal and transverse degrees of freedom and are therefore particularly useful in the context of the transverse lattice construction. Spacelike axial gauges are problematic for weak coupling because of difficulties implementing Gauss's law [22], so the light-cone gauge $A_-=0$ will be used. In light-cone gauge, if the field theory is quantized on the null plane $x^+=0$, then A_+ is a constrained field. So in this section, we use the light-cone quantization scheme—the light-cone gauge with the null-plane Cauchy surface.

Only half of the fermion fields $\phi^{(f)}$ satisfy dynamical equations on the null plane. With the definitions

$$\begin{aligned}\chi&=\phi_f \ , \quad (-1)^{n_x+n_y+f}=-1 \ ,\\ \psi&=\phi_f \ , \quad (-1)^{n_x+n_y+f}=+1 \ ,\end{aligned} \tag{4.1}$$

one finds that only the ψ are dynamical fields. The constraint equations in the light-cone gauge for the fields A_+ and χ are

$$\partial_-^2A_+=J_-=\left[\frac{g_1}{g_2}\right]^2\partial_-\Delta_\alpha^-A_\alpha-g_1^2\sqrt{2}\psi^\dagger\psi \tag{4.2}$$

and

$$\partial_-\chi=\frac{1}{\sqrt{2}}[D_1-iD_2]\psi \ . \tag{4.3}$$

Using $\frac{1}{2}|x^--y^-|=1/\partial_-^2$, which satisfies $\partial_-^2\frac{1}{2}|x^--y^-|=\delta(x^--y^-)$, the constraint equation (4.2) is integrated:

$$A_+=\tfrac{1}{2}\int dy^-|x^- - y^-|J_- + Fx^- + G \ . \tag{4.4}$$

The constant $G(x^+,\mathbf{x}_\perp)$ is set to zero as a gauge fixing constraint; it fixes x^+-dependent (and x^--independent) infinitesimal gauge transformations. The $F(x^+,\mathbf{x}_\perp)x^-$ term corresponds to the θ angle of the Schwinger model [23]. In the continuum limit, the physical 3+1 Lorentz-covariant vacuum should correspond to $F=0$, so it can be set to zero identically.

To remove the coupling constant dependence from the canonical commutation relations, we let

$$A_\alpha \to g_2 A_\alpha \ . \tag{4.5}$$

The current J_- and covariant derivative D_α must be changed accordingly. With this field redefinition, the light-cone momentum $P^+=(P^0+P^3)/\sqrt{2}$ and the light-cone energy $P^-=(P^0-P^3)/\sqrt{2}$ are

$$\begin{aligned} P^+ &= \int d\mathcal{M}\{\partial_- A_\alpha \partial_- A_\alpha + i\sqrt{2}\psi^\dagger\psi\} \ , \\ P^- &= \int d\mathcal{M}\left\{ \frac{g_2^2}{2g_3^2}F_{12}F_{12} - \frac{1}{g_1^2}J_- \frac{1}{\partial_-^2}J_- \right. \\ &\quad \left. + \frac{i\kappa^2}{\sqrt{2}}(\delta_{\alpha\beta}-i\epsilon_{\alpha\beta})\psi^\dagger D_\alpha \frac{1}{\partial_-}[D_\beta\psi] \right\} , \end{aligned} \tag{4.6}$$

where the constraint equations have been applied, and the measure is

$$d\mathcal{M} = \sum_{\mathbf{x}_\perp} a^2 dx^- \ . \tag{4.7}$$

The light-cone Hamiltonian P^- can be divided into free and interacting parts:

$$\begin{aligned} P_0^- = \int d\mathcal{M} \Big\{ & c_1 A_\alpha \Delta_\beta^- \Delta_\beta^+ A_\alpha + c_2(\Delta_\alpha A_\alpha)^2 \\ & + \frac{i}{\sqrt{2}}\psi^\dagger \Delta_\alpha \Delta_\alpha \frac{1}{\partial_-}\psi \Big\} \end{aligned} \tag{4.8}$$

where

$$c_1 = -\frac{1}{2}\left[\frac{g_2}{g_3}\right]^2 , \quad c_2 = \frac{1}{2}\left[\frac{g_1^2}{g_2^4} - \frac{1}{g_3^2}\right] , \tag{4.9}$$

and

$$\begin{aligned} P_{\text{int}}^- = \int d\mathcal{M} \Bigg\{ & -\sqrt{2}\left[\frac{g_1}{g_2}\right]^2 \Delta_\alpha^- A_\alpha \frac{1}{\partial_-}[\psi^\dagger\psi] - g_1^2 \psi^\dagger\psi \frac{1}{\partial_-^2}[\psi^\dagger\psi] \\ & + \frac{i\kappa^2}{\sqrt{2}}\psi^\dagger \left[(\delta_{\alpha\beta}-i\epsilon_{\alpha\beta})D_\alpha \frac{1}{\partial_-}[D_\beta\psi] - \frac{1}{\kappa^2}\Delta_\alpha \frac{1}{\partial_-}[\Delta_\alpha\psi] \right] \Bigg\} . \end{aligned} \tag{4.10}$$

We define the beta functions $\beta_{tl}(\gamma)=a\partial\gamma/\partial a$ for each coupling constant $\gamma=g_1,g_2,g_3,\kappa$. The coupling constants g_1 and g_3 are fixed with respect to g_2 for each value of lattice spacing a by requiring that the photons obeys a covariant dispersion relation. The appropriate renormalization scheme for covariant dispersion relations is

$$c_1 = -\tfrac{1}{2} \ , \quad c_2 = 0 \ . \tag{4.11}$$

At the tree level, this implies $g_1=g_2=g_3$. This relation will receive corrections in perturbation theory; the beta functions

$$\begin{aligned} \beta_{tl}(g_1) &= \beta_{tl}(g_2) + O(g_2) \ , \\ \beta_{tl}(g_3) &= \beta_{tl}(g_2) + O(g_2) \ , \end{aligned} \tag{4.12}$$

are determined by the renormalization conditions Eqs. (4.11).

The quantum theory is defined on a square, doubly periodic, transverse lattice with N^2 sites, N even. A real scalar field $\sigma(\mathbf{x}_\perp)=(\mathbf{x}_\perp+N\boldsymbol{\alpha})$ has the mode expansion

$$\sigma(\mathbf{x}_\perp) = \sum_l \omega^{l\cdot n}\tilde{\sigma}(l) + \text{c.c.} \ , \tag{4.13}$$

where ω is the phase factor $e^{2\pi i/N}$, the l_α are integer momenta which take values from $-N/2$ to $N/2-1$, and the inner product $l\cdot n$ is shorthand for $\sum_\alpha l_\alpha n_\alpha$. With these definitions, the mode expansions for the fields are

$$\begin{aligned} A_\alpha &= \frac{1}{\sqrt{4\pi}Na}\sum_l \int_0^\infty \frac{d\eta}{\eta}\{e^{-i\eta x^-}\omega^{l\cdot n}a_\alpha(l,\eta) + e^{+i\eta x^-}\omega^{-l\cdot n}a_\alpha^\dagger(l,\eta)\} \ , \\ \psi &= \frac{1}{\sqrt{2\pi}Na}\sum_l \int_0^\infty \frac{d\eta}{\sqrt{\eta}}\{e^{-i\eta x^-}\omega^{l\cdot n}b(l,\eta) + e^{+i\eta x^-}\omega^{-l\cdot n}d^\dagger(l,\eta)\} \ , \\ \psi^\dagger &= \frac{1}{\sqrt{2\pi}Na}\sum_l \int_0^\infty \frac{d\eta}{\sqrt{\eta}}\{e^{-i\eta x^-}\omega^{l\cdot n}d(l,\eta) + e^{+i\eta x^-}\omega^{-l\cdot n}b^\dagger(l,\eta)\} \ . \end{aligned} \tag{4.14}$$

The canonical (anti)commutation relations for the creation and annihilation operators are

$$[a_\alpha(l,\eta),a^\dagger_\beta(l',\eta')]=\delta_{l,l'}\delta_{\alpha\beta}\eta\delta(\eta-\eta') ,$$
$$\{b(l,\eta),b^\dagger(l',\eta')\}=\delta_{l,l'}\delta_{\alpha\beta}\eta\delta(\eta-\eta') , \tag{4.15}$$
$$\{d(l,\eta),d^\dagger(l',\eta')\}=\delta_{l,l'}\delta_{\alpha\beta}\eta\delta(\eta-\eta') .$$

The modes a,b,d annihilate the light-cone vacuum, and the normal-ordered expressions for the fermion charge $Q_F=\sqrt{2}\int\psi^\dagger\psi$, momentum P^+, and free Hamiltonian P_0^- are

$$Q_F=\sum_l\int\frac{d\eta}{\eta}\{b^\dagger(l,\eta)b(l,\eta)-d^\dagger(l,\eta)d(l,\eta)\} ,$$
$$P^+=\sum_l\int d\eta\{a^\dagger_\alpha(l,\eta)a_\alpha(l,\eta)+b^\dagger(l,\eta)b(l,\eta)+d^\dagger(l,\eta)d(l,\eta)\} , \tag{4.16}$$
$$P_0^-=\sum_l\int\frac{d\eta}{\eta^2}\{\tfrac{1}{2}k_\perp^2(\Delta^+\Delta^-;l)a^\dagger_\alpha(l,\eta)a_\alpha(l,\eta)+\tfrac{1}{2}k_\perp^2(\Delta^2;l)[b^\dagger(l,\eta)b(l,\eta)+d^\dagger(l,\eta)d(l,\eta)]\} ,$$

where

$$k_\perp^2(\Delta^+\Delta^-;l)=\sum_\alpha\left[\frac{2\sin(\pi l_\alpha/N)}{a}\right]^2 ,$$
$$k_\perp^2(\Delta^2;l)=\sum_\alpha\left[\frac{\sin(2\pi l_\alpha/N}{a}\right]^2 . \tag{4.17}$$

The photon states $a^\dagger(l,\eta)|0\rangle$ satisfy the free-field equation

$$[P^+P_0^- -\tfrac{1}{2}k_\perp^2(\Delta^+\Delta^-)]a^\dagger|0\rangle=0 . \tag{4.18}$$

As the lattice size becomes large, $k_\perp^2(\Delta^+\Delta^-)\rightarrow k_1^2+k_2^2$, where $k_\alpha=2\pi l_\alpha/Na$. Hence Eq. (4.18) is the 3+1 Lorentz-covariant free photon dispersion relation for finite lattice size. A similar relation holds for the fermion states.

We now argue that light-cone perturbation theory [24–26] is finite, diagram by diagram. The S matrix is $\langle f|T\exp(-i\int dx^+P^{-I}_{\rm int})|i\rangle$, where T denotes time ordering with respect to x^+, and $P^{-I}_{\rm int}$ is the normal-ordered interaction light-cone Hamiltonian (4.10) in the interaction picture. Diagrammatic perturbation theory is generated by expanding the time-ordered exponential and inserting complete sets of intermediate states. In general, the S matrix will have an overall energy conservation factor $-2\pi i\delta(P^-_{0,f}-P^-_{0,i})$, and each intermediate state will have the factor $1/(P^-_{0,f}-P_0^-+i\epsilon)$. Matrix elements of the interaction Hamiltonian with intermediate or final states will always include the factor $\delta(\sum_f\eta_f-\sum_i\eta_i)$, where the n_f are outgoing momenta and the η_i are incoming momenta, because all vertices conserve light-cone momentum η. The light-cone momentum is bounded from below by zero for all states.

One delicate aspect of light-cone perturbation theory is the limit $\eta\rightarrow 0$ in intermediate loops. Certain connected one-loop diagrams are ill defined for zero η in continuum QED and QCD (see Refs. [27] and [28]), and need to be regularized. The regulator can be removed when calculating gauge invariant combinations of one-loop diagrams; i.e., the $\eta=0$ region does not contribute to gauge invariant processes at one loop. These divergences are particular to canonical Hamiltonian perturbation theory and do not correspond to the UV divergences of covariant perturbation theory. Also, they are not infrared (IR) divergences since the parity operator P, where $P\psi(x^-,x^+)P^{-1}=\psi(x^+,x^-)$, acts on the modes as $Pb^\dagger(l,\eta)P^{-1}\propto b^\dagger(l,k_\perp^2(l)/2\eta)$, and interchanges the large and small η regions. Two popular regularization schemes for the $\eta=0$ region are a sharp η cutoff [27,28] and the discrete light-cone approach [29–31]. However, these cutoffs may not be good regulators to higher order in perturbation theory because the $\eta=0$ region can contribute to connected diagrams in light-cone field theory [32]. One signature of this problem would be the loss of gauge or Lorentz symmetries; counterterms would have to be added to restore the symmetries order by order in perturbation theory.

The regular UV divergences of QED arise from integration over the transverse momentum $k_\perp$ of the fermions and the gauge fields in the $1/(P^-_{0,f}-P^-+i\epsilon)$ terms of the S matrix [33]. These divergences are explicitly cut off by the transverse lattice construction, since the perpendicular momentum is bounded by $8/a^2$. There also are IR divergences for the gauge fields and massless fermions that arise when summing over $k_\perp=0$ in the denominators. These correspond to the IR divergences of covariant perturbation theory, and are regulated by introducing small mass terms: $k_\perp^2\rightarrow k_\perp^2+\mu^2$.

The last source of perturbative UV divergences is the continuum 2D field theory. Divergent tadpoles of the perturbation theory are eliminated by normal ordering the light-cone Hamiltonian. The nonlocal operator $1/\partial_-$ in the interaction Hamiltonian Eq. (4.10) softens the UV structure of the vertices, as opposed to derivative interactions, which can violate UV finiteness [34]. For instance, the four-fermion term in Eq. (4.10) is scaling dimension zero (verses two) because of the nonlocal $1/\partial_-^2$ factor. And by further power counting arguments, the interaction light-cone Hamiltonian is UV finite, diagram by diagram.

In principle, the 2D fermions ψ in the light-cone Hamiltonian can be bosonized. With the bosonization relations $\psi=:\exp(i\sqrt{2\pi}\Phi):$ and $\psi^\dagger=:\exp(-i\sqrt{4\pi}\Phi):$, where

Φ is a canonical boson, the light-cone Hamiltonian of TLQED is mapped to a bosonic light-cone Hamiltonian with nonderivative interactions. It is well known that a bosonic theory in two dimensions with no derivative interactions is UV finite, diagram by diagram [34]. It is also possible to bosonize the 2D covariant Lagrangian. Then the bosonization dictionary which translates between fermions and bosons will be more complicated [35].

V. THE NONPERTURBATIVE ULTRAVIOLET DIVERGENCE AT $g_2^2=4\pi$

While the transverse lattice theory of QED is UV finite diagram by diagram, it can happen that an infinite number of diagrams conspire to generate a new UV divergence. This phenomenon occurs in the 2D sine-Gordon model [34,36–38]. The basic signature of this phenomenon in the sine-Gordon model is that the anomalous scaling dimension of the interaction $(\alpha/\beta^2)\cos(\beta\phi)$ is greater than two for $\beta^2>8\pi$, and the interaction becomes nonrenormalizable. For this region of coupling, the energy density is unbounded from below [34], and the connected Green's functions diverge order by order in α, starting at order α^2 [36,37].

For TLQED we will now calculate the leading anomalous scaling dimension of the interaction light-cone Hamiltonian (4.10). It is obtained by considering the parts of Eq. (4.10) that contain noninteracting products of link fields. The prototypical term of this type is

$$\frac{\kappa_0^2}{a_3^2}\psi^\dagger\epsilon_{\alpha\beta}D_\alpha\frac{1}{\partial_-}[D_\beta\psi] \ , \qquad (5.1)$$

where κ_0^2 is the bare coupling and a_3 is the cutoff of the 2D continuum theory. This is a bare expression, since it depends upon a_3, and it needs to be renormalized with respect to an arbitrary mass scale. We will calculate the divergent tadpole contributions and renormalize this term. In Eq. (5.1), the factor $1/a_3^2$ accounts for the naive scaling dimension of this interaction, which is $\frac{1}{2}+\frac{1}{2}-1=0$, where each $\frac{1}{2}$ comes from the fermions and -1 comes from $1/\partial_-$. Since the fermion fields $\psi^\dagger$ and ψ in Eq. (5.1) occur at different lattice sites and therefore anticommute, and the two link fields commute because of $\epsilon_{\alpha,\beta}$, we only have to normal order each link field to obtain the tadpole contributions. Consider the exponential

$$e^{iag_2A_\alpha}=:e^{iag_2A_\alpha}:\exp\{+\tfrac{1}{2}(g_2a)^2[A_\alpha^+,A_\alpha^-]\} \ , \qquad (5.2)$$

where $A^+(A^-)$ includes only raising (lowering) operators in the fields mode expansion (4.14). After applying the commutation relations, we get

$$[A_\alpha^+,A_\alpha^-]=\frac{1}{4\pi(Na)^2}\sum_l\int_{\delta_l^+}^{\Lambda_l^+}\frac{d\eta}{\eta} \ . \qquad (5.3)$$

The small η regulator δ_l^+ and the large η regulator Λ_l^+ are related by x_3 parity, as discussed in the preceding section and in Ref. [32]. The relationship is

$$\delta_l^+=\frac{k_\perp^2(\Delta^+\Delta^-;l)}{2\Lambda_l^+} \ . \qquad (5.4)$$

In terms of a fixed x_3 momentum cutoff $\Lambda\approx 1/a_3$, the large η cutoff Λ_l^+ is given by the 2D relativistically correct expression

$$\Lambda_l^+=\frac{\Lambda+\sqrt{\Lambda^2+k_\perp^2(l)}}{\sqrt{2}} \ . \qquad (5.5)$$

Here, $k_\perp$ plays the role of a mass for each 2D theory. In the limit $\Lambda>>k_\perp(l)$,

$$\int_{\delta_l^+}^{\Lambda_l^+}\frac{d\eta}{\eta}=\ln\left[\frac{4\Lambda^2}{k_\perp^2}\right] \qquad (5.6)$$

and

$$e^{iag_2A_\alpha}=:e^{iag_2A_\alpha}:\left\{\frac{\left[\prod_l k_\perp^2(l)\right]^{1/N^2}}{4\Lambda^2}\right\}^{g_2^2/8\pi} \ . \qquad (5.7)$$

In Eqs. (5.6) and (5.7), the IR divergence at $l=0$ is regulated by adding a small mass: $k_\perp(l=0)\rightarrow\mu^2$. We see that the exponentials have the anomalous scaling dimension $g_2^2/4\pi$; i.e., they scale as $\Lambda^{-g_2^2/4\pi}$, where Λ is the UV momentum cutoff. The interaction term (5.1) is multiplicatively renormalized by defining a renormalized coupling $\kappa(m)$ as

$$\kappa_0^2=Z_\kappa\kappa^2(m) \ ,$$
$$Z_\kappa=\tfrac{1}{4}(2ma_3)^{2-g_2^2/2\pi}\prod_l[k_\perp(l)]^{g_2^2/4\pi N^2} \ , \qquad (5.8)$$

where m is an arbitrary mass scale. The renormalized interaction term is then

$$m^{2-g_2^2/2\pi}\kappa^2(m)\psi^\dagger\epsilon_{\alpha\beta}D_\alpha\frac{1}{\partial_-}[D_\beta\psi] \ . \qquad (5.9)$$

For $g_2^2<4\pi$, the interaction term has dimension less than two. For this region of coupling constant, the UV finiteness of each diagram in the theory is sufficient to guarantee finiteness of the full theory. For $g_2^2=4\pi$ the interaction term (5.1) is a marginal operator, and the theory will be well defined if the renormalization of g_2 with respect to the 2D continuum theory is allowed. This is the situation for the sine-Gordon model as its critical point [6,38]. For $g_2^2>4\pi$ the theory is nonrenormalizable, the hopping parameter κ has negative scaling dimension, and the operator product of the interaction Hamiltonian with itself is too singular to allow consistent perturbation theory about the free-field vacuum.

Therefore, for TLQED, we find the somewhat surprising result that the weak perturbation theory is valid only for $\alpha(a)=g_2^2/4\pi<1$, independent of a. The coupling $g_2^2(a)$ is the bare coupling and in the scaling regime of full TLQED it may be quite far from the renormalized QED coupling constant $g_{\rm ren}$. Only for very weak coupling is $g_2(a)\approx g_{\rm ren}$ in the full theory. However, recall that in the quenched approximation of lattice QED [10], chiral symmetry is spontaneously broken beyond a certain critical value $\alpha\sim 1$. Similarly, the analytic calculations in the ladder approximation of quenched QED also exhibit a critical coupling which corresponds to the chiral-symmetry-breaking phase transition [6,7]. We therefore make the conjecture that $g_2^2(a)=4\pi$ is in general the chiral-symmetry-breaking critical point in TLQED, and that specifically, in the quenched approximation of TLQED, for which $g_2(a)=g_{\rm ren}$, chiral symmetry is broken for $\alpha>1$. This is discussed further in Sec. VII.

VI. STRONG COUPLING LIMIT

Does TLQED realize spontaneous chiral symmetry breaking in the strong coupling regime? This means that a nonvanishing chiral condensate $\langle\overline{\Psi}\cdot\Psi\rangle$ must appear, or equivalently, in terms of the lattice fermions,

$$\sum_{\mathbf{x}_\perp}(-1)^{n_x}\langle\mathrm{vac}|\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1|\mathrm{vac}\rangle\neq 0\ , \tag{6.1}$$

where $|\mathrm{vac}\rangle$ is the full interacting vacuum state. Such a nonvanishing condensate would signal the spontaneous breaking of the discrete chiral symmetry of the lattice theory. Since it is a discrete symmetry in the strong coupling region, there will be no accompanying Goldstone boson in this region, and Coleman's theorem [39], prohibiting spontaneous breaking of continuous internal symmetries in two dimensions without anomalies or a Higgs mechanism, will not be violated. The discrete chiral symmetry of the lattice model corresponds to the 4D anomalous U(1) chiral symmetry, and we would not expect Goldstone bosons for this broken symmetry in the scaling regime of the transverse lattice model. However, nonvanishing of the condensate Eq. (6.1) in the scaling regime would also signal the breaking of the nonanomalous continuum U(2) axial flavor symmetries and we would expect their accompanying Goldstone bosons in the scaling regime.

We will now show that spontaneous chiral symmetry breaking does occur in TLQED in the infinite coupling $g_i\to\infty$ limit, where $i=1,2,3$ (here we assume $g_1\sim g_2\sim g_3$) by calculating the energy difference between various vacuum configurations defined below to lowest order in $1/g$. As we will see, this calculation is complicated by the fact that the field theory of rigid rotators is fraught with divergences. In the end however, the vacuum energy density shift will be a finite quantity.

Unlike the previous weak coupling analysis, it is convenient to perform the analysis in the $A_3=0$ gauge, and with equal time quantization. The Hamiltonian density is then

$$\mathcal{H}=\frac{1}{2}\left[\frac{g_2}{a}\right]^2E_\alpha^2+A_0[\Delta_\alpha^-E_\alpha+a^2j_F]+H_F+O\left[\frac{1}{g^2}\right]\ , \tag{6.2}$$

where E_α is the electric field and momentum conjugate to A_α, and $j_F=\sum_f\phi^{\dagger(f)}\phi^{(f)}$ is the fermion current. Gauss's law

$$\mathcal{G}(\mathbf{x}_\perp)=\Delta_\alpha^-E_\alpha+a^2j_F(\mathbf{x}_\perp)=0 \tag{6.3}$$

is obtained by integrating out A_0, and is treated in the quantum theory as the weak constraint $\langle\mathcal{G}_\alpha\rangle=0$ for all physical correlation functions. To leading order in g, the vacuum must satisfy

$$E_\alpha(\mathbf{x}_\perp)|0\rangle=0\ ,\quad\forall\mathbf{x}_\perp\ . \tag{6.4}$$

The system will be quantized with respect to this "free-field" vacuum. The condition that all modes of canonical momentum annihilate the vacuum is reminiscent of the rigid rotator in quantum mechanics.

To regulate the IR behavior of the system, introduce periodic boundary conditions in the continuous spatial direction $z=x_3$:

$$-L\le z\le L\ . \tag{6.5}$$

The mode expansions for the second quantized fields are

$$E_\alpha=\frac{1}{2L}\left\{\sum_{n=1}^{\infty}[E_\alpha^ne^{+i\pi nz/L}+E_\alpha^{*n}e^{-i\pi nz/L}]+E_\alpha^0\right\}\ ,$$
$$A_\alpha=\sum_{n=1}^{\infty}[A_\alpha^ne^{+i\pi nz/L}+A_\alpha^{*n}e^{-i\pi nz/L}]+A_\alpha^0\ , \tag{6.6}$$

where E_α^{*n} (A_α^{*n}) are the complex conjugates of E_α^n (A_α^n), and *not* Hermitian conjugates in the sense of raising and lowering operators of the harmonic oscillator. The canonical commutation relations in terms of the modes are

$$[A_\alpha^{*n}(\mathbf{x}_\perp),E_\beta^m(\mathbf{y}_\perp)]=i\delta_{\alpha\beta}\delta^{n+m}\delta_{\mathbf{x}_\perp,\mathbf{y}_\perp}\ ,$$
$$[A_\alpha^n(\mathbf{x}_\perp),E_\beta^{*m}(\mathbf{y}_\perp)]=i\delta_{\alpha\beta}\delta^{n+m}\delta_{\mathbf{x}_\perp,\mathbf{y}_\perp}\ . \tag{6.7}$$

The free Hamiltonian and momentum for the gauge fields are given by

$$H_{\rm gauge}^0=\frac{1}{2L}\left[\frac{g_2}{a}\right]^2\sum_{\mathbf{x}_\perp,\alpha}\left[\sum_{n=1}^{\infty}E_\alpha^nE_\alpha^{*n}+\tfrac{1}{2}(E_\alpha^0)^2\right]\ ,$$
$$P_{\rm gauge}=\sum_{\mathbf{x}_\perp,\alpha,n}\frac{in\pi}{L}[A_\alpha^{*n}E_\alpha^n-A_\alpha^nE_\alpha^{*n}]\ . \tag{6.8}$$

All E_α^n and E_α^{*n} are lowering operators and annihilate the free-field vacuum, and the modes E_α^n, A_α^{*n} (E_α^{*n}, A_α^n) are eigenstates of momentum P with eigenvalues n ($-n$).

Certain operators in the Hilbert space are exponentials of the modes A_α^n with charge n. For instance, the momentum zero mode state $e^{inaA_\alpha^0}|0\rangle$ has the energy eigenvalue $(g_2n)^2/4L$. Each state in the Hilbert space corresponds to a wave function in a first quantized theory where the dimensionless quantity aA_α^n plays the role of a coordinate. This relation can be used to calculate correlation functions. The correlation function of states is nonvanishing only if the total charge in the exponentials of the wave functions vanish.

More precisely, for the zero-mode expectation values, the two-point correlator is

$$\langle e^{-imaA_\alpha^0}e^{inaA_\beta^0}\rangle\equiv\mathcal{N}^{-1}\delta_{\alpha\beta}\int_{-\infty}^{\infty}dx\,e^{i(n-m)x}$$
$$=\delta_{\alpha\beta}\mathcal{N}^{-1}\delta(n-m)\ . \tag{6.9}$$

Note that for the compact U(1) theory, the integration region for x would be $[-\pi,\pi]$, n,m would be integers, and the correlator Eq. (6.9) would be $\delta_{n,m}$. In the noncompact case at hand, the result is a normalized Dirac delta function, which is ill defined for arbitrary real n,m; only for "integer" n,m do the noncompact and compact results coincide [40]. In this section, we will evaluate such correlators with noninteger arguments, and regulate the result by defining the cutoff delta function

$$\delta_\Lambda(x)=\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}dk\, e^{ikx} . \qquad (6.10)$$

The normalization is given by $\mathcal{N}^{-1}=\delta_\Lambda(0)$.

This is not the only expression which needs to be regulated in the theory. Consider the exponential of the field $e^{iaA_\alpha(z)}$ acting on the vacuum. This expression appears in the interaction Hamiltonian; it represents a link field carrying flux from $\mathbf{x}_\perp$ to $\mathbf{x}_\perp+\alpha$ and has the energy eigenvalue

$$H^0_{\text{gauge}}e^{iaA_\alpha(z)}|0\rangle=\frac{g_2^2}{4L}\left[1+2\sum_{n=1}^{\infty}\right]e^{iaA_\alpha(z)}|0\rangle =\frac{g_2^2}{2}\delta(0)e^{iaA_\alpha(z)}|0\rangle . \qquad (6.11)$$

The energy is infinite because the exponential is a product of an infinite number of states. The exponential receives contributions from all of the "standing waves" A_α^n and A_α^{*n} in the box. To regulate this UV divergence, introduce a cutoff in the number of modes counted in the delta function:

$$\delta^L_{\Lambda'}(z)=\frac{1}{2L}\sum_{j=-\Lambda'}^{\Lambda'}e^{i\pi jz/L} . \qquad (6.12)$$

Then the energy of the exponential is $(g_2^2/2)\delta^L_{\Lambda'}(0)$. The energy is proportional to the number of links and to the square of the flux carried by each link.

The next to leading order contribution to the Hamiltonian comes from the fermions and their interactions with the gauge field. We adopt the equal time anticommutation relations for the fermions:

$$\{\phi_f^\dagger(z,\mathbf{x}_\perp),\phi_g(z',\mathbf{y}_\perp)\}=\frac{1}{a^2}\delta(z-z')\delta_{\mathbf{x}_\perp,\mathbf{y}_\perp}\delta_{fg} . \qquad (6.13)$$

The free-field Hamiltonian density for the fermions is

$$\mathcal{H}_F^0=-i\sum_{\mathbf{x}_\perp,f}a^2(-1)^{n_x+n_y+f}\phi_f^\dagger\partial_z\phi_f . \qquad (6.14)$$

Because of the minus signs in this expression, the mode expansion for the fermions is

$$\phi_f=\frac{1}{\sqrt{2La^2}}\left[\sum_{n>0}(b_n^{\dagger(f)}e^{-i\pi nz/L}+d_n^{(f)}e^{i\pi nz/L})+b_0^{(f)}\right] , \quad (-1)^{n_x+n_y+f}=+1 ,$$
$$(6.15)$$
$$\phi_f=\frac{1}{\sqrt{2La^2}}\left[\sum_{n>0}(b_n^{(f)}e^{-i\pi nz/L}+d_n^{\dagger(f)}e^{i\pi nz/L})+b_0^{\dagger(f)}\right] , \quad (-1)^{n_x+n_y+f}=-1 ,$$

where, for each site $\mathbf{x}_\perp$ and flavor f,

$$\{b_n,b_m^\dagger\}=\delta_{n,m} , \quad \{d_n,d_m^\dagger\}=\delta_{n,m} , \quad \{b_0,b_0^\dagger\}=1 . \qquad (6.16)$$

The normal ordered free-field Hamiltonian is given by

$$H_F^0=\sum_{\mathbf{x}_\perp,f,n}\frac{n\pi}{L}(b_n^{\dagger(f)}b_n^{(f)}+d_n^{\dagger(f)}d_n^{(f)}) \qquad (6.17)$$

and

$$\langle 0|b_n^\dagger=\langle 0|d_n^\dagger=b_n|0\rangle=d_n|0\rangle=0 , \quad n>0 , \qquad (6.18)$$

for each fermion flavor. The zero modes appear in the charge operator

$$Q(\mathbf{x}_\perp)=\int dz\, j_F=\tfrac{1}{2}[b_0^{\dagger(f)},b_0^{(f)}]+\cdots , \qquad (6.19)$$

and in the mass operator

$$M(\mathbf{x}_\perp)=a^2\int dz[\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1] =b_0^{\dagger(1)}b_0^{(2)}+b_0^{\dagger(2)}b_0^{(1)}+\cdots . \qquad (6.20)$$

The chiral condensate order parameter is proportional to $\sum_{\mathbf{x}_\perp}(-1)^{n_x}M(\mathbf{x}_\perp)$.

The vacuum states of the full theory to order $O(g^0)$ will be a direct product of the gauge field vacuum $|0\rangle$ and the highest weight states for the fermion zero modes. To discuss chiral symmetry breaking in the zero-mode sector of the fermion theory, we diagonalize the charge Q and mass operator M simultaneously, via the Bogolubov transformation

$$b_0^{(1)}=\frac{1}{\sqrt{2}}(a_0+ic_0) , \quad b_0^{\dagger(1)}=\frac{1}{\sqrt{2}}(a_0^\dagger-ic_0^\dagger) ,$$
$$(6.21)$$
$$b_0^{(2)}=\frac{1}{\sqrt{2}}(a_0-ic_0) , \quad b_0^{\dagger(2)}=\frac{1}{\sqrt{2}}(a_0^\dagger+ic_0^\dagger) ,$$

where $\{a_0^\dagger,a_0\}=\{c_0^\dagger,c_0\}=1$. Then the mass and charge operators in the zero-mode sector for each site are

$$M=\tfrac{1}{2}[a_0^\dagger,a_0]-\tfrac{1}{2}[c_0^\dagger,c_0] , \quad Q=\tfrac{1}{2}[a_0^\dagger,a_0]=\tfrac{1}{2}[c_0^\dagger,c_0] . \qquad (6.22)$$

The operators $a_0^\dagger,a_0$ and $c_0^\dagger,c_0$ act on two level systems. The a operators are raising and lowering operators for the states $|\uparrow\rangle_a$ and $|\downarrow\rangle_a$,

$$a_0^\dagger|\downarrow\rangle_a=|\uparrow\rangle_a , \quad a_0^\dagger|\uparrow\rangle_a=0 ,$$
$$(6.23)$$
$$a_0|\uparrow\rangle_a=|\downarrow\rangle_a , \quad a_0|\downarrow\rangle_a=0 .$$

The vacuum states in the fermion sector are direct products of the two level states in the a and c systems:

$$|+\rangle=|\uparrow\rangle_a|\downarrow\rangle_c , \quad |-\rangle=|\downarrow\rangle_a|\uparrow\rangle_c . \qquad (6.24)$$

They satisfy $Q|\pm\rangle=0$ (Gauss's law) and $M|\pm\rangle=\pm|\pm\rangle$. The vacuum for each site on the lattice is therefore doubly degenerate at $O(g^0)$. Note that fermion zero-mode expectation values vanish: $\langle b_0^{(f)}\rangle=0$ and $\langle b_0^{\dagger(f)}\rangle=0$. The nonvanishing two-point functions are

$$\langle b_0^{\dagger(f)} b_0^{(f')} \rangle = +\langle b_0^{(f)} b_0^{\dagger(f')} \rangle = \tfrac{1}{2}\ , \quad f=f'\ ,$$
$$\langle b_0^{\dagger(f)} b_0^{(f')} \rangle = -\langle b_0^{(f)} b_0^{\dagger(f')} \rangle = \tfrac{1}{2}M\ , \quad f\neq f'\ . \qquad (6.25)$$

We now show that the degeneracy of the vacuum state is broken in perturbation theory by the interaction Hamiltonian

$$H_{\rm int} = i\kappa \sum_{\mathbf{x}_\perp, f} a^2 \int dz\, \phi_f^\dagger [D_x + i(-1)^{n_x+n_y+f} D_y]\phi_f\ , \qquad (6.26)$$

which is a gauge invariant operator since $[\mathcal{G}(\mathbf{x}_\perp), H_{\rm int}] = 0$. In the context of the 4D transverse lattice theory, the constant κ is dimensionless, since the fermions are dimension $\frac{3}{2}$ and the lattice derivatives go like $1/a$ and are dimension 1. When power counting for the continuous 2D theory however, the fermions are dimension $\frac{1}{2}$ and the derivative is dimension 0. Therefore κ is dimension 1 in the context of the 2D field theory: $\kappa \sim a/a_3$, where a_3 is the UV cutoff of the 2D theory at each site. To regulate the energy of states, we have introduced a cutoff in the number of modes, $\delta_{\Lambda'}^L(0)$. The UV cutoff a_3 is given by $a_3 \sim 1/\delta_{\Lambda'}^L(0)$, so that

$$\kappa = \kappa' a \delta_{\Lambda'}^L(0)\ , \qquad (6.27)$$

where κ' is a scale independent constant.

The first-order shift $\langle H_{\rm int} \rangle$ in the vacuum energy vanishes because the expectation value of a single link field vanishes. The second-order shift is given by

$$W_2 = \sum_n{}' \frac{\langle 0 | H_{\rm int} | n \rangle \langle n | H_{\rm int} | 0 \rangle}{0 - W_n}\ , \qquad (6.28)$$

where $W_n = (g_2^2/4L)\delta_{\Lambda'}^L(0) + W_{n,F}$ is the energy eigenvalue of link states $|n\rangle$, and $W_{n,F}$ is the fermion sector contribution. We will calculate the shift in the vacuum energy due to the assignment of the fermion vacuum to the zero-mode states $|\pm\rangle$ at each site on the lattice, which will be denoted as δW_2.

To calculate the second-order energy shift of the vacuum, we need the correlation function

$$\int dz\, f(z) \int dz' g(z') \langle e^{-iqaA_\alpha(z)} e^{+iq'aA_\beta(z')} \rangle\ . \qquad (6.29)$$

This correlator occurs when summing over intermediate states in Eq. (6.28). Integrating out the zero modes A_α^0 in the exponentials yields the factor $\delta_{\alpha\beta}\delta_\Lambda(q-q')/\delta_\Lambda(0)$. From the next lowest mode $i(A_\alpha^1 - A_\alpha^{*1})$, there is the factor

$$\delta_\Lambda(2\sin(z\pi/L) - 2\sin(z'\pi/L))/\delta_\Lambda(0)$$
$$= \frac{L}{2\pi}[\delta_\Lambda(z-z') + \Theta(z')\delta_\Lambda(z+z'-L) + \Theta(-z')\delta_\Lambda(z+z'+L)]/\delta_\Lambda(0)\ . \qquad (6.30)$$

The only term on the right-hand side (RHS) of Eq. (6.30) that contributes to the correlator is $\delta_\Lambda(z-z')$. The other two terms will lead to vanishing contributions because there is no overlap with these delta functions and the delta functions that appear when integrating out the cosine terms; for instance, integrating out $(A_\alpha^1 + A_\alpha^{*1})$ yields a term $\delta_\Lambda(2\cos(z\pi/L) - 2\cos(z'\pi/L))$ which has no overlap with the second two terms in Eq. (6.30). In the presence of the first term of Eq. (6.30), all the other modes in the correlator contribute factors of unity. Therefore, the correlator Eq. (6.29) is given by

$$\delta_{\alpha\beta} \frac{\delta_\Lambda(q-q')}{\delta_\Lambda^2(0)} \int dz\, f(z) g(z) \qquad (6.31)$$

The parameter Λ is an ultraviolet regulator. If the z direction were discretized, then the $\sin(nz\pi/L)$ and $\cos(mz\pi/L)$ terms which appear as arguments in the delta functions of the correlation function (6.29) would take on discrete values (n,m would be bounded by $\sim[2\pi/a_3]$, where a_3 is the lattice spacing in the discretized z direction). For integer charges q,q', which is all that we will have to consider in this section, all of the correlation functions would then be normalizable. The discrete version of the normalized delta function $\delta_\Lambda(z)/\delta_\Lambda(0)$ would be $(a_3)^{-1}\delta_{z_i}/(a_3)^{-1}$. Hence $\delta_\Lambda(0) \approx 1/a_3$.

While Eq. (6.28) is a complicated sum over four-point correlation functions of the fermion modes, the only terms which contribute to the shift in a vacuum energy δW_2 are products of four fermion zero modes. The nontrivial part of this observation is that a typical two-zero mode contribution $\langle b_0^{(f)} b_n^{(f)} b_n^{\dagger(f')} b_0^{\dagger(f')} \rangle$ is proportional to $\langle b_0^{(f)} b_0^{\dagger(f')} \rangle \delta^{ff'}$, and this by the first of Eqs. (6.25) is independent of the choice $|+\rangle$ or $|-\rangle$ for the vacuum state at that site. Using the link field two-point correlation function (6.29) given by Eq. (6.31) and the fermion zero-mode two-point correlators Eqs. (6.25), the shift in the energy density is

$$\delta w_2 = \frac{\kappa'^2}{16\pi g_2^2} \left[\frac{1}{(La)^2} \right] \times \sum_{\mathbf{x}_L} M(\mathbf{x}_\perp)[M(\mathbf{x}_\perp + \mathbf{x}) - M(\mathbf{x}_\perp + \mathbf{y})]\ . \qquad (6.32)$$

This is minimized for $M(\mathbf{x}_\perp)M(\mathbf{x}_\perp + \mathbf{x}) = -1$ and $M(\mathbf{x}_\perp)M(\mathbf{x}_\perp + \mathbf{y}) = +1$. There are two fermion vacuum configurations, related by an overall sign change, that obey these conditions and the symmetry of these ground states is made clear by Fig. 2. Both configurations break the discrete U(1) axial chiral symmetry since the order parameter $\sum_{\mathbf{x}_\perp}(-1)^{n_x} M(\mathbf{x}_\perp)$ is nonvanishing for these vacuum configurations. If the order parameter is nonvanishing in the scaling regime, then the full set of nonanomalous continuum axial flavor symmetries will be broken.

There is a simple way of approaching the continuum limit of this leading-order result in the strong coupling regime such that Eq. (6.32) remains finite; i.e., let the longitudinal IR regulator $L \to \infty$ and the transverse UV regular $a \to 0$ such that La remains finite. So although the energy and correlators of link fields require UV regulators, the shift in the vacuum energy density is finite in the continuum limit. We briefly list the sources of the regulated divergences that contribute to Eq. (6.32). The product of four zero modes contributes $1/L^2a^4$; the energy in

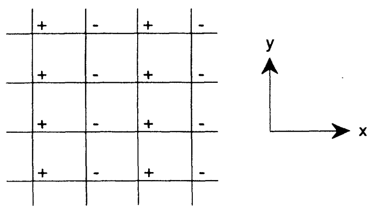

FIG. 2. The plus and minus signs for each site refer to the fermion zero mode states $|+\rangle$ and $|-\rangle$. Up to an overall change in sign, this configuration minimizes the order $1/g^2$ correction to the vacuum energy density in the strong coupling limit.

the denominator of (6.28) contributes $1/\delta^L_{\Lambda'}(0)\sim a_3$; from the integral over intermediate states we get $1/\delta_\Lambda(0)\sim a_3$; from the κ^2 coupling constant there is a factor of a^2/a_3^2; examination of Eq. (6.26) shows that $H^2_{\rm int}$ contributes a factor of a^2, and we multiply by $1/a^2$ to make (6.28) into a density. The result is the net factor of $1/(La)^2$.

To interpret this result further, consider the spin transformation $b_0^{(2)}\to\alpha b_0^{(2)}$, where $\alpha(\mathbf{x}_\perp)=(-1)^{n_y}$. Following the analysis of Semenoff [9], define the vector

$$\psi=\begin{bmatrix} b_0^{(1)} \\ b_0^{(2)} \end{bmatrix}, \tag{6.33}$$

and the currents $S_j=\psi^\dagger\sigma_j\psi$ where σ_j are Pauli matrices. Then the Hamiltonian density in the zero-mode sector that has expectation value given by Eq. (6.32) can be written as

$$\delta e_2=\frac{\kappa'^2}{16\pi g_2^2}\left[\frac{1}{(La)^2}\right]\sum_{\mathbf{x}_\perp}\mathbf{S}(\mathbf{x}_\perp)\cdot[\mathbf{S}(\mathbf{x}_\perp+\mathbf{x})+\mathbf{S}(\mathbf{x}_\perp+\mathbf{y})]+\text{const}\ . \tag{6.34}$$

This is the Hamiltonian density for the quantum spin-$\frac{1}{2}$ Heisenberg antiferromagnet, and the configuration given by Fig. 2 is just the classical ground state of the system [41]. It has Néel order, i.e., the expectation value of Eq. (6.32) is nonvanishing and the global flavor SU(2) of the Hamiltonian (6.34) is spontaneously broken. We can consider Eq. (6.34) to be Hamiltonian in the fermion sector to leading order in the strong coupling expansion.

To study chiral symmetry breaking to higher order in the strong coupling expansion, we need to treat the quantum fluctuations of the spin-$\frac{1}{2}$ Heisenberg antiferromagnet in the zero-mode sector, and include the effect of nonzero modes on the vacuum state. There is no exact solution of the ground state of the quantum $d=2$ quantum spin-$\frac{1}{2}$ Heisenberg antiferromagnet [42], and no proof that Néel order persists in the full quantum theory. However, numerical simulations indicate that this may be the case [43]. For a similar analysis of regular Hamiltonian lattice gauge theory the situation is better, because Néel order has been proven to exist in the three dimensions [9,42].

VII. DISCUSSION

The transverse lattice regulation of QED that has been studied in this paper is a "minimal" way of regulating the diagrammatic divergences of the perturbation theory, and it exhibits a phase transition at a critical value of the lattice QED coupling constant, and chiral symmetry breaking in the strong coupling regime.[1]

In Sec. V, we took advantage of the UV finiteness of each diagram in weak perturbation theory to find a nonperturbative UV divergence at $g_2^2(a)=4\pi$. The transverse lattice regulates the usual UV divergences of four dimensional QED, but the "finite" two-dimensional field theories for each site conspire to generate a nonrenormalizable interaction. The signature of the nonrenormalizability is the anomalous scaling dimension of the interaction Hamiltonian. If the dimension of any part of the interaction Hamiltonian is greater than two, then the perturbation theory about the free-field vacuum will be ill defined. One can calculate the anomalous dimension of the interaction Hamiltonian because the coupling constant $g_2(a)$ is not renormalized in the 2D continuum perturbation theory for $g_2^2<4\pi$. Note that there is no plaquette term in the interaction Hamiltonian, since we have studied noncompact QED, which would have a higher scaling dimension than the term we considered.[2]

The 2D sine-Gordon model has the same properties with respect to the coupling constant β: it is perturbatively finite for all β but its free-field perturbation theory is unstable, without additional coupling constant renormalizations, for $\beta^2>8\pi$. The sine-Gordon field theory is equivalent to the grand canonical sum of a Coulomb plasma, and the sine-Gordon phase transition has a nice physical interpretation in terms of the Coulomb gas picture [36]. As β increases, the free ions of the Coulomb gas, represented by vertex operators $\exp(\pm\beta\phi)$ in the sine-Gordon model, collapse to form dipoles and a new gas of interacting dipoles is formed. This can be interpreted in the sine-Gordon model as the appearance of a new dimension-2 renormalizable operator at this fixed point. One can consider the sine-Gordon model for values of $\beta^2>8\pi$ as long as the additional renormalization for the new operator is taken into account [38].

In TLQED, "free ions" are given by fermion charges separated by one link and connected by a flux tube: $\psi^\dagger\exp(gA)\psi$. The "Coulomb gas" in TLQED is then a gas of e^+e^- pairs, where the charges, separated by a single link, interact via Coulomb interactions. The "dipoles" of the strong coupling phase are pairs of e^+e^- flux tubes, with strongly interacting photon fields.

The conjecture is that the nonperturbative $g_2^2(a)=4\pi$

[1]If one formulates QED with one lattice and three continuum dimensions, then the diagrammatic divergences will not be regulated by the lattice, and chiral symmetry breaking will not appear in the strong coupling expansion of the lattice theory. This is shown by choosing a gauge where the lattice gauge field is set to zero.

[2]Plaquette terms are presumably generated perturbatively but are suppressed by powers of the cutoff.

critical point of TLQED, where this phase transition occurs, is the critical point of spontaneous chiral symmetry breaking, where the $\bar{\psi}\psi$ order parameter gets a vacuum expectation value. The bare coupling constant $g_2(a)$ is the "quenched" coupling constant of TLQED, because fermion loop corrections are obviously not included in the bare Hamiltonian. Both the quenched lattice simulations and the quenched planar approximation exhibit chiral symmetry breaking for $\alpha_{\text{bare}} \sim 1$. The phase transition in the quenched planar approximation has been previously compared to the phase transition of the sine-Gordon model by Miransky [6], who interpreted the phase transition of each model as a collapse phenomenon. At the critical point in the quenched planar approximation, the anomalous scaling dimension of the fermion is 1, and the four-fermion term becomes a renormalizable operator [7]. It is tempting to associate the "dipoles" of the strong coupling phase of TLQED with renormalizable four-fermion operator of the quenched planar approximation. We used the strong coupling expansion of TLQED in Sec. VI to calculate explicitly the spontaneous chiral symmetry breaking in the infinite coupling limit.

Recent lattice gauge theory simulations indicate that the UV fixed point of chiral symmetry breaking in the quenched theory may be trivial in the full unquenched theory [12,13].

Some of the results of this paper can immediately be applied to more realistic transverse lattice models. In particular, the construction of staggered fermions and the analysis of chiral symmetry breaking via the strong coupling expansion can be easily generalized to non-Abelian gauge theories.

We now briefly mention two formal areas of the theory that would be interesting to pursue. In Sec. IV, we noted that TLQED can be covariantly (in the 2D sense) bosonized. Bosonization plays a central role in explaining why the Schwinger model is exactly soluble. It would be interesting to understand the continuum limit of this bosonized version of TLQED. It would also be interesting to work in the "opposite" direction—to covariantly bosonize transverse lattice fermions, and then put the two continuous coordinates on a lattice. Naively, this would generate a 4D lattice theory where the fermions are interpreted as bosonic solitons, and the functional integral over fermions is "Gaussian" and easier to simulate.[3] Second, TLQED is an interacting 2D field theory in the form of a combined Schwinger and sine-Gordon model. It might be possible to solve the sine-Gordon "part" by using inverse scattering and/or Bethe ansatz methods. Then the nonintegrable Schwinger terms would have to be treated as perturbations in the space of Bethe ansatz states.

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