

Summing over inequivalent maps in the string theory interpretation of two-dimensional QCD

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Following some recent work by Gross, we consider the partition function for QCD on a two-dimensional torus and study its stringiness. We present strong evidence that the free energy corresponds to a sum over branched surfaces with small handles mapped into the target space. The sum is modded out by all diffeomorphisms on the world sheet. This leaves a sum over disconnected classes of maps. We prove that the free energy gives a consistent result for all smooth maps of the torus into the torus which cover the target space p times, where p is prime, and conjecture that this is true for all coverings. Each class can also contain integrations over the positions of branch points and small handles which act as "moduli" on the surface. We show that the free energy is consistent for any number of handles and that the first few leading terms are consistent with contributions from maps with branch points.

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I. INTRODUCTION

String theory has progressed tremendously since its beginning in the dual model days of the 1960's. However, the direction of this progression took an unexpected turn about twenty years ago. Instead of string theory being used as a tool to describe strong interactions, it has since served as a means to unify all forces in Nature.

Since the turn away from strong interactions, there have been many technical achievements in string theory. Among the most notable achievements are those of Polyakov [1] and others [2–7], who realized that string amplitudes are given by integrations over all geometries of two-dimensional punctured surfaces. In other words, string theory is basically two-dimensional quantum gravity coupled to matter fields. The path integral is given by the sum over all metrics modded out by diffeomorphisms. If the strings are critical, then the path integral can be modded out by conformal transformations as well. This then leaves an integral over the moduli of the surface, a finite-dimensional space.

But for many reasons, this theory is not QCD. However, QCD stills looks very stringy, at least in the confining phase. Hence a natural question to ask is to what extent do the ideas of Polyakov string theory apply to the strong interactions.

Gross has recently proposed a nice way to start probing this question [8]. His idea is to study matter-free QCD in two dimensions and to determine the stringiness of this particular theory. The great advantage of two dimensions is that the theory is solvable. Hence one can analyze the solutions and decide if they look stringy or not. If this is a string theory then one should be able to interpret the QCD free energy as a sum over maps of two-dimensional surfaces into a two-dimensional target space. Of course two-dimensional QCD is almost a trivial theory and it is not quite clear if everything one learns

from it can be applied to the four-dimensional case. But there might be some general principles that can be extracted from the two-dimensional case that are applicable in four dimensions. In particular, this might lead to a consistent formulation of the measure for the string path integral.

This approach differs from earlier attempts to interpret two-dimensional QCD as a string theory [9–12] in that it leaves out the quark fields. The resulting theory is completely trivial if the Euclidean space is flat and noncompact. However, if the space is compact and topologically nontrivial, then the partition function will have an interesting structure. It is partition functions of this type that Gross proposes to explore.

In Sec. II we review Gross' work on two-dimensional QCD as a string theory. In Sec. III we carry this work out further. We argue that for a toroidal target space, the QCD solutions describe the equivalence classes under diffeomorphisms of smooth maps into the target space. However, unlike Polyakov string theory, there is no integration over world-sheet metrics. We also present evidence that the QCD solutions allow for branched surfaces by showing that this is consistent with lower-order terms in the perturbative expansion. We also argue that the solutions imply the existence of pinched handles and tubes on the surfaces. In Sec. IV we close with a few remarks.

II. GROSS' PICTURE OF TWO-DIMENSIONAL QCD

The particular model that Gross has in mind is a lattice formulation of QCD with a heat kernel action. This model was recently solved by Migdal and Rusakov [13,14] and its partition function for $SU(N)$ is given by

$$Z = \sum_{\text{reps}} (d_r)^{2-2G} \exp(-Ag^2 C_{2R}/N), \quad (2.1)$$

where the sum is over all representations of $SU(N)$, A is the area of the surface, G is the genus of the surface, g/\sqrt{N} is the QCD coupling, d_R is the dimension of the representation and C_{2R} is the value of the quadratic

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Casimir invariant of the representation.

The representations can be summarized by Young tableaux. The tableaux are described by m rows, with n_i boxes in row i , which satisfy $n_i \geq n_j$ if $i < j$. The quadratic Casimir invariant for a particular representation is given by

$$C_{2R} = \frac{N}{2} \left[n + \frac{\bar{n}}{N} - \frac{n^2}{N^2} \right], \quad (2.2)$$

where

$$n = \sum_{i=1}^m n_i, \quad \bar{n} = \sum_{i=1}^m n_i(n_i - 2i + 1). \quad (2.3)$$

The partition function only depends on the quantities, N , G and the combination Ag^2 . Naturally, $1/N$ acts as the string coupling and g^2 as the string tension.

The important quantity is the free energy, $-\ln Z$. In Polyakov string theory, this is given as a sum over all connected Riemann surfaces, summing over all moduli of the surface and all matter fields that live on the world sheet. Each term in the sum is weighted by $(g_s)^{2\gamma-2}$, where g_s is the string coupling and γ is the genus of the world sheet. Thus if two-dimensional QCD is to be a string theory, then, at least perturbatively, we should expect the free energy to be comprised of even powers of $g_s = 1/N$. Under this interpretation, the free energy is given by maps of the world sheet of genus γ into the target space of genus G .

Gross has given a beautiful demonstration of why this picture of two-dimensional QCD is correct [8]. Supposing that the map of the world sheet into the target space is continuous, that is, the surface has no tears; then at the very least, the genus of the world sheet γ must be greater than or equal to G . Now consider d_R , which for a representation that has n boxes in the tableaux, behaves as $d_R \sim N^n$ when $N \gg n$. Hence, if $G > 1$ then the partition function is dominated by the representations with a small number of boxes. If we approximate C_{2R} as $Nn/2$, then the free energy can be approximated by

$$F = - \sum_{i=1}^n c_i \left[\frac{1}{N} \right]^{2(G-1)n} \exp(-nAg^2/2), \quad (2.4)$$

where c_i are constants. Gross has interpreted this as follows: each term in the sum represents a map from a world sheet of genus γ which is an n -fold covering of the target space. The string action is basically the Nambu-Goto action, the area swept out by the world sheet multiplied by the string tension, which in this case is nAg^2 . However, there is the caveat that the world sheet is not allowed to fold back on itself, otherwise there would be terms in the sum corresponding to world sheets whose area is not an integer multiple of A . Hence the string action should contain terms that suppress the folds. The first term in the sum has a factor N^{2-2G} , which corresponds to a world sheet with genus $\gamma = G$. Hence, we find that there is no contribution to the world-sheet sum until the genus is large enough so that there can be a smooth map into the target space. Moreover, Gross has pointed out that if the world sheet covers the target space n

times, then the genus of the world sheet must satisfy

$$\gamma - 1 \geq n(G - 1). \quad (2.5)$$

This is clearly satisfied by (2.4).

Gross has also observed that the $1/N$ corrections in C_{2R} lead to terms in the free energy with factors of A/N . He has conjectured that such terms arise from branch points or small handles on the surface. Surfaces with such points will have a larger genus, hence the factors of $1/N$. Integrating over the positions of these points gives the factors of A .

The string theory is described by more than just the action. One also needs to determine the measure. The Nambu-Goto action is invariant under diffeomorphisms, that is, reparametrizations of the world-sheet coordinates; thus one should expect the string functional to be modded out by all diffeomorphisms. This will greatly reduce the integration over maps of the world sheet into the target space. This string theory should not contain integrations over a world-sheet metric either. It does not appear in the Nambu-Goto action, and there is otherwise no reason to introduce it. Hence, we should only consider a fixed world-sheet metric which will be used to define the functional measure. Choosing the world-sheet metric to be the target space metric leads to the area factors in the free energy. In the next section we will further see that this leads to consistent results.

III. DIFFEOMORPHISMS, BRANCHES, HANDLES, AND MODULI

In this section we concentrate on the coefficients that appear in (2.4) and on the higher-order corrections to the quadratic Casimir invariants. We will see that the coefficients count all maps of surfaces that are not connected by diffeomorphisms. We will also see that the \bar{n}/N and the n^2/N^2 terms in C_{2R} can be interpreted as contributions from branched surfaces with handles. For what follows, we will restrict our attention to $G = 1$.

If we continue to approximate C_{2R} as $Nn/2$, then the free energy density is given by

$$\mathcal{F} = - \frac{1}{A} \ln \sum_{\text{reps}} \exp(-Ag^2 n_R / 2). \quad (3.1)$$

Gross has shown this to be equal to

$$\mathcal{F} = - \frac{1}{a} \ln \eta[\exp(-Ag^2/2)], \quad (3.2)$$

where $\eta(q)$ is the Dedekind function,

$$\eta(q) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \quad (3.3)$$

Hence

$$\begin{aligned} \mathcal{F} &= \frac{1}{A} \sum_{n=1}^{\infty} \ln(1 - q^n) \\ &= -g^2 \sum_{n=1}^{\infty} \frac{S_n}{nAg^2} \exp(-Ag^2 n / 2), \end{aligned} \quad (3.4)$$

where S_n is the sum over positive integers that are divisors of n ,

$$S_n = \sum_{q|n} q. \tag{3.5}$$

Hence, for $n = 2$, $S_n = 3$, since 1 and 2 are the divisors of 2.

In the usual Polyakov string theory, one calculates the free energy density by summing over all possible metrics, modded out by all diffeomorphisms. However, the free energy density is basically a zero-point amplitude. Since the torus has a diffeomorphism that is a conformal Killing vector corresponding to constant translations on the surface, then modding out by the diffeomorphisms requires us to divide by the area of the surface $\text{Im}\tau$ [15,16].

Let us assume that for the QCD case, the partition function is just a sum over all smooth maps into the torus modded out by diffeomorphisms. For $G = 1$, it is possible to have an n -fold covering with $\gamma = 1$, so we will assume that every term in (3.4) is from the world sheet with the topology of a torus. We need to decide what measure to use when summing over the maps. This will be especially important when we consider handles and branches. But it is also important for modding out the constant translations. The natural choice is to use the measure that exists on the target space and pull it back to the world sheet. As in Polyakov string theory, the scale is determined by the string tension g^2 , and thus defines the unit of area. Therefore, every contribution to the torus partition should be divided by nAg^2 because of the world-sheet translation invariance. This then accounts for the denominators in (3.4).

It would seem that modding out all maps by diffeomorphisms would leave only one map, since the target space fields have two degrees of freedom, the same number as the space of diffeomorphisms. This is true for $n = 1$, but false for the higher values. It turns out that for these values of n , there are maps that cannot be continuously changed from one to the other, but are not discrete diffeomorphisms of each other either. Let us assume that the target space is parametrized by two vectors (X_1, X_2) , which define the two independent windings on the target space surface. Likewise, let the world sheet be parametrized by two vectors (ω_1, ω_2) , which define the nontrivial windings on its surface. We call this the winding map. Because of reparametrization invariance on the world sheet, the winding vectors can be redefined as

$$(\omega_1 + \omega_2, \omega_2) \text{ or } (\omega_2, -\omega_1). \tag{3.6}$$

The mapping of the world sheet to the target space maps the winding vectors (ω_1, ω_2) to the winding vectors (X_1, X_2) . If the map is an n -fold covering of the surface, then (ω_1, ω_2) will map to multiple numbers of X_1, X_2 or both. For example, a double covered map might be described by $(2X_1, X_2)$. Any map that can be manipulated to this form using the operations in (3.6) is equivalent. It is then easy to see that there are three independent ways to double cover the torus, $(2X_1, X_2)$, $(X_1, 2X_2)$, and $(X_1 + X_2, X_1 - X_2)$. Figure 1 shows these three independent coverings. Checking (3.5), we find that $S_2 = 3$. This

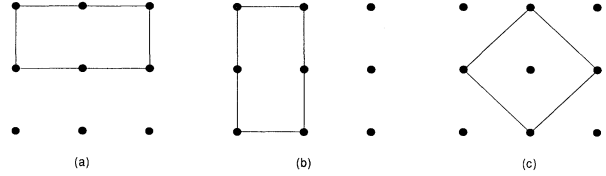


FIG. 1. Three distinct toroidal world sheets that double cover the target space. The dots represent the lattice of the toroidal target space.

suggests that S_n counts the number of independent maps of the torus into the torus.

We now give a simple proof that S_p does count the maps for p prime. Given that the area of the target space is A , then the area of the world sheet whose winding map is given by $(aX_1 + bX_2, cX_1 + dX_2)$ is $(ad - bc)A$. $ad - bc$ is invariant under the transformations in (3.6). Suppose that $ad - bc = p$, where p is prime. Let us further suppose that none of the integers a, b, c , or d are zero. Then there are two possible scenarios. Either none of these integers are divisible by p , or two of them are.

Let us consider the first case. Suppose that $d > b > 0$, which one can always impose using the operations in (3.6). Since p is prime, d and b must be relatively prime. (That is, the only common factor is 1.) Next perform the diffeomorphism $(\omega_1, \omega_2) \rightarrow (\omega_1, \omega_2 - \omega_1)$ n times, such that $d - nb$ is as small as possible but greater than 0. The new winding map is

$$(aX_1 + bX_2, (c - na)X_1 + (d - nb)X_2), \tag{3.7}$$

where $d - nb \equiv d' \leq b$, with an equality only if $b = 1$. Since b and d are relatively prime, then, by construction, b and d' are also relatively prime. If $d' = 1$, then do the operation

$$(\omega_1, \omega_2) \rightarrow (\omega_1 - \omega_2, \omega_2) \tag{3.8}$$

b times, leaving the winding map $(pX_1, (c - na)X_1 + X_2)$. If $d' \neq 1$ then carry out the diffeomorphism in (3.8) m times such that $b' = b - md'$ is as small as possible but greater than zero. This gives the new map

$$(a'X_1 + b'X_2, c'X_1 + d'X_2), \tag{3.9}$$

where $0 < d' < d$, $0 < b' < b$, and d' is relatively prime with b' . We then repeat the process until we are left with the map

$$(pX_1, qX_1 + X_2). \tag{3.10}$$

q cannot be a multiple of p , since this would mean that a and c were multiples of p . Of course, q can be adjusted such that $0 < q < p$, by acting with the diffeomorphism $\omega_2 \rightarrow \omega_2 + \omega_1$ enough times. But clearly two maps in the form (3.10) cannot be connected by a diffeomorphism if $q_1 \neq q_2 \pmod{p}$. Furthermore, the original map could have been transformed into the map $(X_1 + q'X_2, pX_2)$. Hence, every map of this form is equivalent to one in the form (3.10). Thus we find that there are $p - 1$ distinct maps

that can be found from the first class of maps.

Turning to the second case, if a and b are the two integers that are divisible by p , the diffeomorphism $\omega_1 \rightarrow \omega_1 + \omega_2$ will lead to a map where none of the integers are divisible by p . A similar argument holds for c and d divisible by p . Hence this subclass will not lead to new maps. However, if a and c are divisible by p , then they will remain that way under any diffeomorphism. Furthermore, b and d must be relatively prime. Hence, using the previous argument we can find a diffeomorphism that sets c to zero, giving the map (pX_1, X_2) . Likewise, if b and d are divisible by p , then they stay that way under diffeomorphisms and the map is equivalent to (X_1, pX_2) . Therefore, combining all possible scenarios we find a total of $p + 1$ distinct maps. Examining (3.5), we see that this is precisely S_p if p is prime.

As for nonprime n , we have checked the first few values of S_n and have found that they agree with the number of distinct n -fold maps. Hence, we conjecture that this is true for all n .

Let us now examine the higher-order corrections to C_{2R} and consider their full implications. We can get a hint to what they might mean by realizing that Gross' condition (2.5) is actually a consequence of the Riemann-Hurwitz relation [17]

$$\gamma - 1 = n(G - 1) + B/2, \tag{3.11}$$

where B is the branching number. The branching number of a map is the sum of the branching numbers for each point. At each point p , one can find a local coordinate z which is mapped onto another local coordinate w on the other Riemann surface. Under the map, w is given by $w = z^n$. The branching number for this point is $b(p) = n - 1$. A point with nonzero $b(p)$ is called a ramification point and its image on the target space is called a branch point.

From (3.11) we immediately see that the branching number is even. Therefore, if we only consider maps with ramification points whose branching number is one, then the number of ramification points is even. We propose that the \bar{n}/N terms in the quadratic Casimir invariants are somehow related to these points. At a ramification point of the world sheet the map will locally double cover the target space. Hence if there are any such points at all, the entire target space must be at least double covered. Since the single covered target space does not have these points, we should expect $\bar{n} = 0$ for $n = 1$, which is in fact the case.

The first nonzero values for \bar{n} occur at $n = 2$. In this case, $\bar{n} = 2, -2$ for the two representations, and therefore, the contribution to the partition function is

$$\exp \left[-\frac{Ag^2}{2}(2 - 4/N^2) \right] [\exp(-Ag^2/N) + \exp(Ag^2/N)]. \tag{3.12}$$

Notice that (3.12) is an even function of $1/N$, hence the contribution to the perturbative string expansion has only even powers of the string coupling. In fact, the same is true for all $n \leq N$, since for every Young tableaux

with $n_1 \leq N$, there exists a transposed tableaux whose value of \bar{n} has the opposite sign. This is illustrated in Fig. 2, which shows a tableaux and its transpose. \bar{n} is twice the sum of the numbers that appear in the boxes. Expanding the term inside the square brackets in (3.12), we find

$$2 + \frac{1}{N^2}(Ag^2)^2 + \frac{2}{N^4 4!}(Ag^2)^4 + \dots$$

and therefore the free energy density has the contribution

$$\frac{1}{A} \left[\frac{3}{2} + \frac{1}{N^2}(Ag^2)^2 + \frac{2}{N^4 4!}(Ag^2)^4 + \dots \right] \exp(-Ag^2) \tag{3.13}$$

from terms that double cover the target space. The Riemann-Hurwitz relation (3.11), suggests that the $1/N^2$ term can be attributed to the contribution of a surface with two ramification points. The points connect two otherwise disconnected world sheets. Each additional ramification point leads to another factor of $1/N$. Each point also has a factor of Ag^2 associated with it because a surface with the points at new positions corresponds to a new world sheet which is not connected to the old one by a diffeomorphism. Hence, it is necessary to integrate over all positions of the ramification points. These positions are basically the "moduli" of the surface. Pulling back the metric of the target space, we find that every ramification point leads to a factor of Ag^2 , the area of the target space multiplied by the string tension, up to symmetry factors.

The symmetry factors are as follows. There is a factor of $\frac{1}{2}$ for the two world sheets and a factor of $1/n!$ for a surface with n ramification points, since they are indistinguishable. There is also a factor of 4 which comes from the cuts that connect the branch points on the target space. A cut joining the two points could wind either way around the cycles of the torus. One cut cannot be deformed into the other, hence we find a factor of 2 for each cycle, or a factor of 4 altogether. If there are more than two branch points, there is only an overall factor of 4, and not 4 for each pair of points, because all of these possible cuts can be continuously deformed into one of four types. Putting these factors together, the total symmetry factor for n points is $2/n!$, which agrees with (3.13).

If the world sheet covers the target space three or more

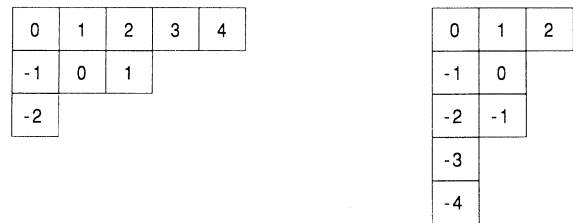


FIG. 2. Young tableaux for a representation and its transpose. \bar{n} is given by twice the sum of numbers in the boxes.

times then the counting becomes quite complicated. This is because many of the possible world sheets will be equivalent to each other but determining the equivalence is rather arduous. We have managed to work out the factors for a triple covering of the target space with two ramification points and have found agreement with the result from the free energy. In this case, the ramification points connect a surface that covers the target space once with a surface that covers the target space twice. There are three representations that have three boxes in the tableaux, and their respective values of \bar{n} are 6,0,-6. Plugging these values into the free energy and expanding in $1/N$, we find that the contribution to the free energy density for a surface that triple covers the target space with two ramification points is

$$\frac{1}{A} \frac{1}{N^2} 8(Ag^2)^2 \exp(-3Ag^2/2). \tag{3.14}$$

The possible maps are shown in Fig. 3. We have used the translational invariance on the world sheet to fix one ramification point to the origin. The first six figures lead to a factor of $6Ag^2$, coming from the integration of the second ramification point over the surface. (The integration in these figures is over one cover of the target space.) There are two maps for each surface shown in Fig. 1, coming from the two possible ways to draw the branch cuts. (There are two and not four since we have distinguished the points by fixing one to the origin.) The last two surfaces in Fig. 3 contain branch cuts that wrap more than once around the cycles of the smaller sheet. For these surfaces it is only necessary to consider the double covered surface in Fig. 1(a). It turns out that maps of this type that use the other two surfaces in Fig. 1 are equivalent to the first. To see this, one can break up the world sheet into separate regions and show that the different regions are connected to each other in the same way for either of the mappings. This is illustrated in Fig. 4, where we compare maps containing the double covered

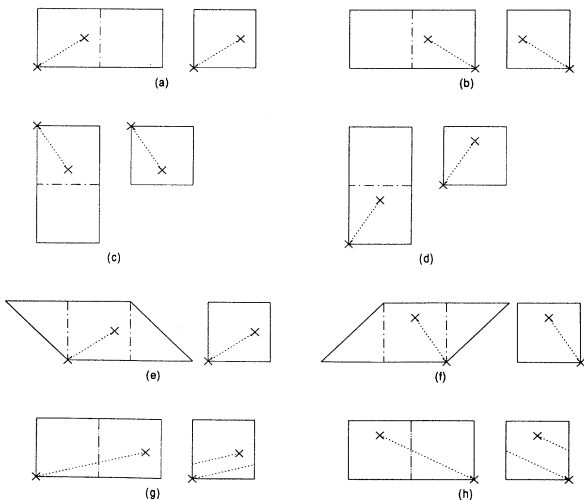


FIG. 3. Triple covered surfaces with two ramification points. The short dashed lines are the cuts connecting the two surfaces. The parallelograms have periodic boundary conditions.

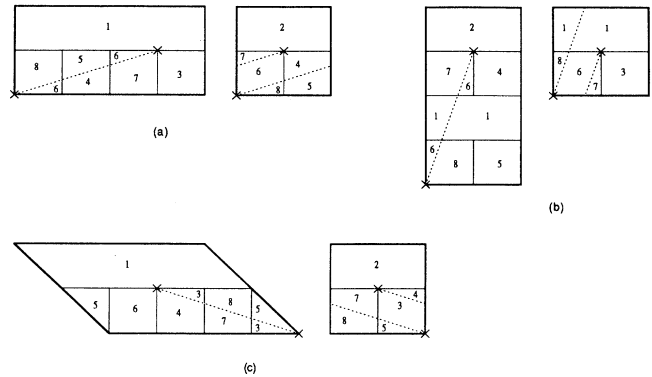


FIG. 4. Identical triple covered surfaces with two ramification points.

surfaces pictured in Fig. 1. By examining the eight regions in Fig. 4(a) with those in Figs. 4(b) and 4(c), one finds that the three surfaces are identical. (The reader is encouraged to verify this by tracing closed loops around the surfaces.) Therefore, there is a factor of $2Ag^2$ from the last two maps in Fig. 3, and hence the total sum of factors agrees with (3.14).

Finally, consider the last term in C_{2R} , $-N(n^2/N^2)/2$. This term appears to be associated with small handles on the surface and with pinched tubes connecting different parts of the world sheet. It is convenient to rewrite $n^2/2N^2$ as

$$\frac{n}{2N^2} + \frac{n(n-1)}{2N^2}. \tag{3.15}$$

It is then consistent to say that the first term in (3.15) is related to small handles and the second term is related to pinched tubes. For every small handle on the surface, the genus is increased by one, thus one expects every handle to come with a factor of $1/N^2$. Furthermore, the position of the handle needs to be integrated over, since each position corresponds to a different surface. Therefore, each handle has a factor of nAg^2 , the area of the world sheet using the target space metric. If the handles are infinitesimally small, then their positions are the *only* moduli. If the handle has finite length, it would have two points associated with it, corresponding to the points where the handle is attached to the surface. In this case, one would integrate over both points, but with a factor of $\frac{1}{2}$, since the ends of the handle are indistinguishable. By shrinking the length of the handle, the two points coalesce into one, but the factor of $\frac{1}{2}$ remains. Finally, interchanging the positions of two handles gives back the same world sheet. Hence, for n_h handles there is a symmetry factor of $1/n_h!$. Therefore, if these small handles exist, then every term in the free energy that comes from a surface that covers the target space n times, should be multiplied by $\exp(Ag^2 n/2N^2)$. This is precisely the contribution from the first term in (3.15).

If one allows infinitesimally small handles, then con-

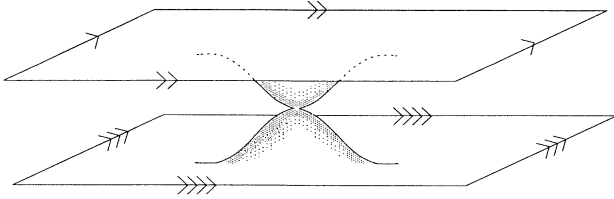


FIG. 5. Two tori joined by a pinched off tube. The arrows indicate which edges are identified on the world sheet. Note that the surface folds back on itself at the pinch.

sistency requires that there be infinitesimally small tubes connecting different points of the world sheet. These points on the world sheet must map to the same point on the target space. Each tube should come with a factor of $1/N^2$, since the genus will be increased by one. Therefore, if we consider a not necessarily connected surface with area nA , and then put in a tube, this will lead to a factor of $Ag^2n(n-1)/2N^2$, where Ag^2 is from the integration over the target space, and $n(n-1)/2$ comes from choosing two of the n sections of the world sheet. (By section, we mean a part of the surface that covers the target space exactly once.) Again, interchanging two tubes leaves the surface invariant. Therefore, if these pinched tubes exist, then surfaces of area nA also come with a factor of $\exp[Ag^2n(n-1)/2N^2]$ in the partition function. This agrees with the second part of (3.15).

Figure 5 shows such a tube connecting two parts of a world sheet. By examining the figure, one notes that actually the surface has folded back onto itself. But the fold takes place at a single point, not along an extensive line. Hence, the constraints on the world sheet should read that finite length folds are suppressed.

We close this section by noting that, for the gauge group $U(N)$, the quadratic Casimir invariant is given by

$$C_{2R} = \frac{N}{2} \left[n + \frac{\bar{n}}{N} \right]. \quad (3.16)$$

Hence the string theory corresponding to this version of QCD would have ramification points but not small handles.

IV. DISCUSSION

To summarize, we have given a string interpretation for all terms found in the perturbative expansion of the QCD partition function given in (2.1). The free energy is given by a sum over maps from the world sheet modded out by diffeomorphisms. Unlike the Polyakov string, there is no integration over world-sheet metrics. The free energy contains a sum over branched surfaces with small handles, which are inequivalent under diffeomorphisms. We have verified the consistency of this interpretation for the lower-order terms in the expansion.

We close with a few remarks. We first note that there is actually a natural way to suppress the folds. This is accomplished by introducing an extrinsic curvature term in the action. For a two-dimensional surface mapped into another two-dimensional surface, this is given by

$$\int d^2\xi \sqrt{G} G^{ab} \partial_a n \partial_b n. \quad (4.1)$$

G_{ab} is the induced metric and n is basically the normal “vector” to the surface, which in this case is $n = \pm 1$. At a fold, the derivative normal to the fold on the surface is a δ function. Hence the integral in (4.1) is $L\delta(0)$, where L is the length of all folds on the surface. Hence finite-length folds will be suppressed by this term. This suggests that an analogous term might appear in the four-dimensional case as well, although, in this case, the bending of the surface will lead to finite results.

Our second remark concerns nonperturbative effects. The partition function in (2.1) is an even function of $1/N$ only up to terms with N boxes in the tableaux. The fact that the entire sum will not be an even function is then a nonperturbative result. This then complies with Shenker’s observation that nonperturbative string effects are on the order of $\exp(-1/g_s)$ and not $\exp(-1/g_s^2)$. If the latter case were true then the complete partition function would be an even function of $1/N$. Another way to understand what determines the order of the nonperturbative effects is to realize that there are corrections to the perturbative sum when the number of coverings of the target space is N or greater. This is because tableaux have been summed over that do not correspond to physical representations of $SU(N)$. Hence, the free energy will have correction terms of the form $\exp(-Ag^2N/2)$ multiplied by moduli factors.

TABLE I. Multiplicative factors from integrations over positions of the ramification points. $q = Ag^2$.

Covers	2 Ram. pts.	4 Ram. pts.	6 Ram. pts.
1	0	0	0
2	$2q^2$	$(1/12)q^4$	$(1/360)q^6$
3	$8q^2$	$(20/3)q^4$	$(91/45)q^6$
4	$30q^2$	$102q^4$	$(383/3)q^6$
5	$80q^2$	$(2288/3)q^4$	$(24140/9)q^6$
6	$180q^2$	$3773q^4$	$(180331/6)q^6$
7	$336q^2$	$14232q^4$	$(3349714/15)q^6$
8	$620q^2$	$(133616/3)q^4$	$(11174816/9)q^6$
9	$960q^2$	$119904q^4$	$5558312q^6$
10	$1590q^2$	$(584517/2)q^4$	$(252779965/12)q^6$

Finally, while it is difficult to verify that the free energy gives a single counting of branched maps into the torus, it is not too hard to actually calculate the terms in the free energy. Hence one can turn this around and simply postulate what the distinct maps are by reading off the terms in the free energy. Using the symbolic manipulator program MAPLE, we were able to calculate such terms for surfaces that cover the target space up to 10 times and with as many as six ramification points. These results are shown in Table I. This is then another instance where quantum field theory can be used to explore questions in geometry.

Note added. After this paper was completed, we learned that Gross and Taylor were able to prove that the free energy counts the number of independent maps (without branch points or small handles) for any number of coverings. They also extended this to target spaces with genus $G > 1$ [18].

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