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Finite σ models and exact string solutions with Minkowski signature metric

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We consider two-dimensional (2D) σ models with a $(D=2+N)$-dimensional Minkowski signature target space metric having a covariantly constant null Killing vector. These models are UV finite. The $(2+N)$-dimensional target space metric can be explicitly determined for a class of supersymmetric σ models with the N-dimensional "transverse" part of the target space being σ homogeneous Kähler type. The corresponding "transverse" subtheory is an $n=2$ supersymmetric σ model with the exact β function coinciding with its one-loop expression. For example, the finite $D=4$ model has the O(3) supersymmetric σ model as its "transverse" part. Moreover, there exists a nontrivial dilaton field such that the Weyl invariance conditions are also satisfied; i.e., the resulting models correspond to string vacua. Generic solutions are represented in terms of the renormalization group flow in "transverse" theory. We suggest a possible application of the constructed Weyl-invariant σ models to quantization of 2D gravity. They may be interpreted as "effective actions" of the quantum 2D dilaton gravity coupled to a (nonconformal) N-dimensional "matter" theory. The conformal factor of the 2D metric and 2D "dilaton" are identified with the light-cone coordinates of the $(2+N)$-dimensional σ model.

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I. INTRODUCTION

One of the important problems in string theory is to classify possible solutions of the string effective equations, i.e., string vacuum backgrounds which may be represented in terms of Weyl-invariant two-dimensional (2D) σ models (for reviews, see, e.g., Ref. [1]). Since the string equations (or "$\bar{\beta}$ functions") are quite complicated (already at the string tree level), containing all terms in α', the structure of the space of solutions is poorly understood. Among a few classes of solutions which are explicitly known are (1) flat space with linear dilaton [2], (2) group spaces [Wess-Zumino-Witten (WZW) models] [3], (3) "plane-wave" backgrounds [4], (4) backgrounds corresponding to gauged WZW theories [5], and (5) various possible direct products (see, e.g., second paper in Ref. [2]). In contrast with the first three classes of backgrounds (which can be represented in a form essentially independent of α'), the backgrounds of the fourth type are nontrivial functions of α' (see Ref. [6]). There are, of course, many other solutions of the leading-order string equations (see, e.g., Ref. [7]), but their generalizations to all orders in α' (which should exist in perturbation theory) are not explicitly known. One can try to construct new solutions by using various types of duality transformations [8–10]. However, since the exact form of the σ *model* duality transformations is not explicitly known (except in the first two orders in α') [9], all discussed duality rotations of exact string solutions solve string equations only to the leading-order α'. Having found an exact string background, one still confronts an additional problem of identifying a conformal theory which it should correspond to. The solution to this problem is known only in the case of (gauged) WZW theories.

In order to understand better gravitational applications of string theory (e.g., string backgrounds related to cosmology, black hole physics, or possibly to high-energy string scattering [11]), it is important to find new exact solutions which have a physical *Minkowski* signature. A class of such solutions will be described below. In general, the solutions will be nontrivial functions of α'. We shall present a simple algorithm of their construction in terms of the renormalization-group (RG) flow of a nonconformal Euclidean 2D theory. Namely, the following theorem is true [12,13]: Given a nonconformal σ model with an N-dimensional target space with a Euclidean signature metric, there exists a conformal-invariant σ model in $2+n$ dimensions with a Minkowski signature metric. The $(2+N)$-dimensional metric depends on only one of the two extra coordinates (it has a covariantly constant null Killing vector) and is expressed in terms of the "running" coupling of the N-dimensional theory (the "transverse" part of the metric satisfies a first-order renormalization-group-type equation). Thus, starting from an arbitrary N-dimensional Euclidean background, one can construct a $(2+N)$-dimensional string solution with a Minkowski signature.

We shall discuss the 2D supersymmetric generalization of this class of finite σ models and will show that the $(2+N)$-dimensional metric can be explicitly determined in the case when the transverse space is homogeneous Kähler. Then the "transverse" submodel is $n=2$ supersymmetric, and the expression for the exact β function of

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the transverse theory is known (it coincides with the one-loop result) so that the RG equation is easy to integrate.

Conformal-invariant σ models with a null Killing vector are also of interest in connection with the problem of quantizing 2D gravity. If one starts with a 2D model of gravity coupled to a (nonconformal) N-dimensional matter theory, it is expected [14–17] that the couplings of the matter theory should develop a dependence on the conformal factor such that the resulting "quantum action" is represented by an $(N+1)$-dimensional Weyl-invariant σ model. This suggestion suffers from the following difficulty: Since the Weyl-invariance conditions turn out to be *second*-order differential equations in the $(N+1)$th "time" coordinate (conformal factor), there is an ambiguity in choosing a particular solution which satisfies natural initial conditions. This problem is (at least partially) avoided [12,13] if one considers a model of 2D quantum gravity where there is an extra scalar field (2D "dilaton") coupled to 2D curvature (see, e.g., Refs. [18,19]). The central observation is that the corresponding quantum action can be identified with an action of a conformal-invariant $(N+2)$-dimensional σ model with a null Killing vector. The extra scalar field and the conformal factor play the role of the light-cone coordinates v and u. The theory is effectively $N+1$ dimensional since the condition of Killing symmetry implies that couplings are v independent. As a result, the conformal-invariance equations are *first*-order differential equations in u (in fact, the standard RG equations of the "transverse" N-dimensional theory) and their solution satisfying natural initial conditions is unique.

In Sec. II we shall first show that the σ models with a covariantly constant null Killing vector are UV finite in flat 2D space. In contrast with what happens, for example, in WZW models, the divergences will not cancel automatically at each order of perturbation theory, but will be absent on shell (i.e., it will be possible to redefine them away) [12]. The mechanism of finiteness which operates here was already discussed (at the one-loop level) in Ref. [20]. We shall then study the Weyl-invariance conditions [21] on a σ model defined on a curved two-surface and will prove (making use of the general coordinate-invariance identities for the Weyl anomaly coefficients [22]) that there exists a dilaton field such that the σ models with a covariantly constant null Killing vector are Weyl invariant [13]. That means they represent solutions of string effective equations. In contrast with the previously known string solutions with a null Killing vector [4] which have *flat* N-dimensional space, the backgrounds we have found may have an arbitrary transverse space.

A new class of finite supersymmetric σ models with null Killing vector will be presented in Sec. III. We shall present an explicit expression for the $(2+N)$-dimensional target-space metric (with homogeneous Kähler transverse subspace) which represents an exact solution of superstring theory and consider some of its properties.

A relation to 2D quantum gravity models will be discussed in Sec. IV. In particular, we shall consider a generalization to the case when the σ-model action contains the tachyonic coupling (or a scalar potential).

II. FINITENESS AND WEYL INVARIANCE OF σ MODELS WITH COVARIANTLY CONSTANT NULL KILLING VECTOR

A. Proof of finiteness

The most general $(D=N+2)$-dimensional Minkowski signature metric admitting a covariantly constant null Killing vector can be represented in the form

$$ds^2=\hat{g}_{\mu\nu}dx^\mu dx^\nu=-2dudv+g_{ij}(u,x)dx^i dx^j\ ,$$
$$\mu,\nu=0,1,\ldots,N+1,\quad i,j=1,\ldots,N\ . \tag{1}$$

In fact, starting from the null metric

$$\begin{aligned}ds^2&=\tilde{g}_{\mu\nu}dx^\mu dx^\nu\\ &=-2dudv+g_{ij}(u,x)dx^i dx^j\\ &\quad+2A_i(u,x)dx^i du+K(u,x)du^2\ ,\end{aligned} \tag{2}$$

one can eliminate A_i and K by a change of coordinates which preserves the "null" structure of (2) [23]. Thus the most general null metric is parametrized by the functions $g_{ij}(u,x)$. It is important to keep in mind, however, that (1) considered as a generic form of the metric is written using a specific choice of coordinates v,x^i. For example, if $g_{ij}(u,x)$ is flat as a function of x^i, this does not imply that a generic "null" metric with a flat transverse part is just given by $ds^2=-2dudv+dx^i dx_i$: Transforming the coordinates to make g_{ij} equal to δ_{ij}, we will get back the metric (2) with nonvanishing A_i and K.

To establish the UV finiteness of the corresponding σ model on a flat 2D background, one should check that there exists a vector M_μ such that the β function for the target-space metric $G_{\mu\nu}=\hat{g}_{\mu\nu}$ [Eq. (1)] vanishes up to the M_μ-reparametrization term [24]:

$$\beta^G_{\mu\nu}+2D_{(\mu}M_{\nu)}=0\ . \tag{3}$$

If (3) is satisfied, the divergences can be absorbed into a redefinition of the coordinates x^μ. As we shall see, (3) is indeed satisfied for a particular $g_{ij}(x,u)$ as a function of u. Using the fact that the nonvanishing components of the Christoffel connection and the curvature of $\hat{g}$ are

$$\hat{\Gamma}^i_{jk}=\Gamma^i_{jk}\ ,\quad \hat{\Gamma}^v_{ij}=\tfrac{1}{2}\dot{g}_{ij}\ ,\quad \hat{\Gamma}^i_{ju}=\tfrac{1}{2}g^{ik}\dot{g}_{kj}\ ,$$
$$\dot{g}_{ij}\equiv\frac{\partial g_{ij}}{\partial u}\ , \tag{4}$$

$$\hat{R}_{ijkl}=R_{ijkl}\ ,\quad \hat{R}_{iuju}=T_{ij}\ ,\quad \hat{R}_{uijk}=E_{ijk}\ , \tag{5}$$

$$T_{ij}\equiv-\tfrac{1}{2}(\ddot{g}_{ij}-\tfrac{1}{2}g^{mn}\dot{g}_{im}\dot{g}_{nj})\ ,\quad E_{ijk}=-D_{[j}\dot{g}_{k]i}\ , \tag{6}$$

and that $\beta^G_{iv}=0$, $\beta^G_{uv}=0$ (this follows from the fact that the β^G function is constructed in terms of curvature tensors and covariant derivatives), we can rewrite (3) in the "component" form

$$\beta^g_{ij}+2D_{(i}M_{j)}-2\hat{\Gamma}^v_{ij}M_v=0\ ,$$
$$\bar{\beta}^g_{ij}=\dot{g}_{ij}M_v\ ,\quad \bar{\beta}^g_{ij}\equiv\beta^g_{ij}+2D_{(i}M_{j)}\ , \tag{7}$$

$$\beta^G_{uu}=-2\partial_u M_u\ , \tag{8}$$

$$\beta^G_{iu}=-\partial_i M_u-\partial_u M_i+g^{jk}\dot{g}_{ij}M_k \ , \tag{9}$$

$$\partial_i M_v+\partial_v M_i=0 \ , \quad \partial_u M_v+\partial_v M_u=0 \ . \tag{10}$$

Since all the components of $\beta^G_{\mu\nu}$ do not depend on v, the only v dependence that is possible in M_μ is a linear v term in M_u. Then the general solution of (10) is given by

$$M_v=mu+p \ , \quad M_u=-mv+Q(u,x) \ , \\ M_i=M_i(u,x) \ , \quad p,m=\text{const} \ . \tag{11}$$

For a given $g_{ij}(u,x)$, the components β^G_{uu} and β^G_{iu} are some particular $N+1$ functions of u and x so that one can always satisfy Eqs. (8) and (9) by properly choosing the $N+1$ functions M_u and M_i [once we have solved (8), we can put (9) in the form $\partial_u M_i+h_i^j(u,x)M_j=E_i(u,x)$, which always has a solution].

Having determined M_u and M_i as functionals of g_{ij}, we are left with the final equation (7). It should be interpreted as an equation for $g_{ij}(u,x)$. Using (11) and introducing

$$\tau=m^{-1}\ln(mu+p) \ , \quad m\neq 0 \ , \\ \tau=p^{-1}u \ , \quad m=0 \tag{12}$$

(to get a Weyl-invariant model, one should actually set $m=0$; see below), we can represent (7) in the form

$$\frac{dg_{ij}}{d\tau}=\bar{\beta}^g_{ij} \ . \tag{13}$$

Thus we have proved the following statement: If the metric g_{ij} depends on u in such a way that it satisfies the standard RG equation of the N-dimensional σ model (with some particular reparametrization vectors M_i), then the $(2+N)$-dimensional σ model based on (1) is UV finite to all orders of the loop expansion.

Let us now make a number of comments. If g_{ij} corresponds to a finite N-dimensional theory, i.e., $\bar{\beta}^g_{ij}=0$, then one should set $p=0$; i.e., a finite $(2+N)$-dimensional model is found for arbitrary dependence of g_{ij} on u. The above argument for finiteness is simplified in the "one-coupling" case when the transverse metric is proportional to a matrix $\gamma_{ij}(x)$ of a symmetric (constant curvature) space,

$$g_{ij}(u,x)=f(u)\gamma_{ij}(x) \ . \tag{14}$$

The corresponding model is renormalizable for arbitrary $f(u)$. To get more explicit formulas, let us assume that the transverse space is maximally symmetric, i.e.,

$$R_{ijkl}(\gamma)=\frac{R}{N(N-1)}(\gamma_{ik}\gamma_{jl}-\gamma_{il}\gamma_{jk}) \ .$$

Since $\beta^G_{iu}=0$ and the scalar functions (e.g., β^G_{uu}) are x independent, we set $M_i=0$, $\partial_i M_u=0$ and thus solve (7)–(9) by [12]

$$M_v=mu+p \ , \quad M_u=-mv+Q(u) \ , \\ \dot{Q}=-\tfrac{1}{2}\beta^G_{uu}(u) \\ =\tfrac{1}{4}\alpha' N(f^{-1}\ddot{f}-\tfrac{1}{2}f^{-2}\dot{f}^2)+O(\alpha'^3) \ , \tag{15} \\ M_v\dot{f}\gamma_{ij}=\beta(f)\gamma_{ij} \ ,$$

i.e.,

$$\frac{df}{d\tau}=\beta(f) \ , \\ \beta^G_{ij}=\beta^g_{ij}=\beta(f)g_{ij} \ , \quad \beta^g_{ij}=\alpha' R_{ij}+O(\alpha'^2) \ , \tag{16} \\ \beta(f)=a+(N-1)^{-1}a^2f^{-1}+\tfrac{1}{4}(N-1)^{-2}(N+3)a^3f^{-2} \\ +O(a^4f^{-3}) \ , \quad a\equiv\alpha' N^{-1}R \ .$$

Equation (16) has the obvious perturbative solution [we choose the $m=0$ case in (12)]

$$f(u)=bu+(N-1)^{-1}\ln u+O(u^{-1}) \ , \quad b\equiv p^{-1}a \ . \tag{17}$$

The asymptotic freedom corresponds to f (i.e., the inverse coupling of the σ model) growing to infinity at large u. Having found $f(u)$ from (16), one determines Q from (15) and thus solves (7)–(10).

We see that the metric of the transverse space [and thus the full metric (1)] is determined by the β function of the transverse theory. The explicit all-order expressions for the latter are not known in bosonic σ models. On the other hand, there are examples of $n=2$ supersymmetric (n is the number of 2D supersymmetries) σ models with homogeneous symmetric Kähler target spaces for which the exact β function coincides with the one-loop expression [25]. As we shall discuss in Sec. III, the metric of the corresponding *finite* $(2+N)$-dimensional $n=1$ supersymmetric σ models is explicitly given by (14) and the *first* term in (17).

B. Solution of Weyl-invariance conditions

The UV finiteness of a σ model in flat two-space does not in general guarantee that the corresponding model on a curved 2D background is Weyl invariant. The Weyl-invariance conditions for the model,

$$I=\frac{1}{4\pi\alpha'}\int d^2z\sqrt{g}\,[G_{\mu\nu}(x)\partial_a x^\mu\partial^a x^\nu+\alpha' R^{(2)}\phi(x)] \tag{18}$$

(which are equivalent to the string effective equations), have the general structure [21]

$$\bar{\beta}^G_{\mu\nu}=\beta^G_{\mu\nu}+2D_{(\mu}M_{\nu)}=0 \ , \tag{19}$$

$$\bar{\beta}^\phi=\beta^\phi+M^\mu\partial_\mu\phi=0 \ , \\ \beta^\phi=c-\tfrac{1}{2}\alpha' D^2\phi+\tfrac{1}{16}\alpha'^2 R_{\mu\alpha\beta\gamma}R^{\mu\alpha\beta\gamma} \\ +O(\alpha'^3) \ , \quad c=\tfrac{1}{6}(D-26) \ , \tag{20}$$

where M_μ is not arbitrary, but is given by

$$M_\mu=\alpha'\partial_\mu\phi+\tfrac{1}{2}W_\mu \ . \tag{21}$$

Here W_μ is a covariant vector constructed of $G_{\mu\nu}$ only (and determined by the mixing under renormalization of dimension-2 composite operators [21]). To prove that the σ model based on (1) is Weyl invariant, one needs to show that there exists a dilation field ϕ such that M_μ in (3) can be represented in the form (21).

The Weyl anomaly coefficients $\bar{\beta}^G_{\mu\nu}$ and $\bar{\beta}^\phi$ satisfy D differential identities which can be derived from the condition of nonrenormalization of the trace of the energymomentum tensor of the σ model [22]. They can be considered to be a consequence of the target-space reparametrization invariance given that $\bar{\beta}^G_{\mu\nu}$ and $\bar{\beta}^\phi$ are related to a covariant effective action S:

$$\frac{\delta S}{\delta \varphi^A}=k_{AB}\bar{\beta}^B\ , \quad \varphi^A=(G_{\mu\nu}\ ,\phi)\ , \qquad (22)$$

$$2D_\mu\frac{\delta S}{\delta G_{\mu\nu}}-\frac{\delta S}{\delta\phi}D^\nu\phi=0\ . \qquad (23)$$

In general, the identity has the structure [22,21]

$$\partial_\mu\bar{\beta}^\phi-\bar{\beta}^G_{\mu\nu}D^\nu\phi-V^{\alpha\beta}_\mu\bar{\beta}^G_{\alpha\beta}=0\ , \qquad (24)$$

where the differential operator $V^{\alpha\beta}_\mu$ depends only on $G_{\mu\nu}$. To the lowest order in α', one finds [26,21]

$$\partial_\mu\bar{\beta}^\phi-\bar{\beta}^G_{\mu\nu}D^\nu\phi+\tfrac{1}{2}D^\nu(\bar{\beta}^G_{\mu\nu}-\tfrac{1}{2}G_{\mu\nu}G^{\lambda\rho}\bar{\beta}^G_{\lambda\rho})+O(\alpha'^2)=0\ . \qquad (25)$$

One of the consequences of (24) is that $\bar{\beta}^\phi=$const once (19) is satisfied. In general, the identity (24) implies that only $\frac{1}{2}D(D+1)+1-D$ of Eqs. (19) and (20) are independent. It may happen, in particular, that if the "transverse" subset of $\frac{1}{2}(D-2)(D-1)$ equations in (19) and the dilaton equation (20) are solved, the remaining D equations (19) are satisfied automatically.

Let us look for solutions of (19) and (20) which have the form [12,13]

$$G_{\mu\nu}=\hat{g}_{\mu\nu}(u,x)\ , \quad \phi=\phi(v,u,x)\ , \quad x^\mu=(v,u,x^i)\ , \qquad (26)$$

where $\hat{g}_{\mu\nu}$ is given by (1). Since $\beta^G_{\mu\nu}$, W_μ, and hence $\beta^G_{\mu\nu}{}'=\beta^G_{\mu\nu}+D_{(\mu}W_{\nu)}$ in (19) are covariant functions of the curvature and its derivatives and since the metric has a Killing vector, it is easy to see that the (μv) component of $\beta^G_{\mu\nu}{}'$ is identically zero. Then (19) gives the following constraint on the dilaton: $\partial_\mu\partial_v\phi=0$, i.e.,

$$\phi=pv+\phi(u,x)\ , \quad p=\text{const}\ . \qquad (27)$$

Here p is an arbitrary integration constant and $\phi(u,x)$ is to be determined. From now on all the functions will depend only on u and x^i. Using (4) and (27), we can represent the nontrivial components of (19) as follows (we shall put $\alpha'=1$):

$$\bar{\beta}^g_{ij}-p\dot{g}_{ij}=0\ , \qquad (28)$$

$$\bar{\beta}^g_{ij}\equiv\beta^G_{ij}+D_{(i}W_{j)}+2D_iD_j\phi\ ,$$

$$\beta^G_{iu}+\tfrac{1}{2}\partial_iW_u+\tfrac{1}{2}\dot{W}_i-\dot{g}_{ij}W^j+2\partial_i\dot{\phi}-\dot{g}_{ij}D^j\phi=0\ , \qquad (29)$$

$$\beta^G_{uu}+\dot{W}_u+2\ddot{\phi}=0\ . \qquad (30)$$

Equation (20) takes the form

$$\begin{aligned}\bar{\beta}^\phi&=c-\gamma\phi+(\partial_\mu\phi)^2+\tfrac{1}{2}W^\mu\partial_\mu\phi+\omega\\&=\tfrac{1}{3}+\bar{\beta}^{\phi\prime}+\tfrac{1}{2}pM^{ij}\dot{g}_{ij}-\tfrac{1}{2}pW_u-2p\dot{\phi}\\&=0\ ,\end{aligned} \qquad (31)$$

$$\bar{\beta}^{\phi\prime}\equiv c'-\gamma'\phi+(\partial_i\phi)^2+\tfrac{1}{2}W^i\partial_i\phi+\omega\ , \quad c'=\tfrac{1}{6}(N-26)\ , \qquad (32)$$

where γ' is the "anomalous-dimension" differential operator, ω is a covariant function of $G_{\mu\nu}$ only, and the M^{ij} term ($M^{ij}=\frac{1}{2}g^{ij}+\cdots$) in (31) originates from the linear in ϕ term

$$-\gamma\phi=-\gamma'\phi-M^{ij}D_iD_j\phi+O(D^3\phi)$$

(see Ref. [13] for details). Being scalar functions of the curvature, γ', ω, and hence $\bar{\beta}^{\phi\prime}$ do not depend on the derivatives of the metric over u. The functions $\bar{\beta}^g_{ij}$ and $\bar{\beta}^{\phi\prime}$ can be interpreted as the Weyl anomaly coefficients of the "transverse" theory defined by $g_{ij}(u,x)$ and $\phi(u,x)$ at fixed u [$\frac{1}{3}$ in (31) corresponds to the central charge contribution of the two light-cone coordinates].

Let us first consider the case of *nonvanishing p*. Then (28) is a first-order differential equation for $g_{ij}(u,x)$ which always has a solution. Eliminating the derivatives of g_{ij} over u from (31) using (28), we find a similar first-order equation for $\phi(u,x)$. Equations (28) and (31) can be interpreted as renormalization-group equations of the "transverse" theory with u playing the role of the RG "time" [12].

Still it is a question of whether or not the solutions of (28) and (31) satisfy also (29) and (30). It is answered positively [13] using the identity (24). Substituting $\bar{\beta}^G_{ij}=0$, $\bar{\beta}^\phi=0$, and the expression (27) for the dilaton into (24), one finds [13]

$$p\bar{\beta}^G_{iu}=0\ , \quad p\bar{\beta}^G_{uu}-\bar{\beta}^G_{iu}D^i\phi-2V^{ju}_u\bar{\beta}^G_{ju}=0\ . \qquad (33)$$

That means that once (28) and (31) are satisfied for nonzero p, the remaining Eqs. (29) and (30) are satisfied as well. The conclusion is that given some initial data $(g_{ij}(x),\phi(x))$ at $u=0$ there exists a u-dependent solution $(g_{ij}(u,x),\phi(u,x))$ of the Weyl-invariance conditions (19)–(21).

In the particular case when the transverse space is symmetric [i.e., its metric is given by (14)], the symmetry requires that $W_i=0$, $\bar{\beta}^G_{iu}=0$, and that ϕ be x^i independent:

$$\phi=pv+\phi(u)\ .$$

The functions which enter the equations for $f(u)$ and $\phi(u)$ are

$$\beta^G_{ij}=\beta(f)\gamma_{ij}\ , \quad \beta^G_{uu}=\beta^G_{uu}(f)\ , \quad W_u=W_u(f)\ ,$$

$$\bar{\beta}^{\phi\prime}=c'+\omega(f)\ , \quad M^{ij}=\tfrac{1}{2}f^{-1}[1+M(f)]\gamma^{ij}\ .$$

Since Eq. (30) is a consequence of (28) and (31), β^G_{uu} is not independent and we are left with the following two equations for $f(u)$ and $\phi(u)$ [cf. (15) and (16)]:

$$\begin{aligned}&p\dot{f}=\beta(f)\ , \quad p\dot{\phi}=\tfrac{1}{2}c+J(f)\ ,\\&J=\tfrac{1}{2}\omega(f)+\tfrac{1}{8}N[1+M(f)]f^{-1}\beta(f)-\tfrac{1}{4}pW_u\ ,\\&c=\tfrac{1}{6}+c'=\tfrac{1}{6}(N-24)\ .\end{aligned} \qquad (34)$$

As a result, the "scale factor" of the metric $f(u)$ runs according to the standard ("flat space") RG equation, while the dilaton depends on u in such a way as make the total central charge vanish. It is possible to show [12] that if (28) and (30) are satisfied, the central charge of this model

$\bar{\beta}^{\phi}$ is equal to that of the free $(2+N)$-dimensional theory plus the contribution of the linear terms in the dilaton. In fact, since $\bar{\beta}^{\phi}$ is constant on a solution of (28) and (30), it can be computed at any value of u, e.g., $u=\infty$. Given that all higher-loop contributions should vanish in the weak-coupling limit of large u (we are assuming that the transverse σ model is asymptotically free), it is sufficient to compute $\bar{\beta}^{\phi}$ in the leading-order approximation. Representing the dilaton in the form

$$\phi=pv+qu+\Phi(u) , \tag{35}$$

where Φ stands for contributions which are due to σ-model interactions [which depend on the coupling f, i.e., $\Phi(u)=F(f(u))$], we find that the "free theory" and "interaction" contributions cancel separately, giving

$$\bar{\beta}^{\phi}=c-2pq=0 , \quad p\dot{\Phi}=J(f(u)) . \tag{36}$$

Thus one can satisfy the zero total central charge condition for arbitrary N by a proper choice of the constants p and q.

If the "initial" transverse theory is generic, i.e., if $\bar{\beta}^{g}_{ij}$ in (28) is nonvanishing at $u=0$, then the solution exists only for a nonzero p. If, however, the initial theory is Weyl invariant, i.e.,

$$\bar{\beta}^{g}_{ij}(u=0)=0 , \quad \bar{\beta}^{\phi\prime}(u=0)=c''=\text{const} ,$$

there are two possibilities. For $p\neq 0$ the simplest solution of (19) and (20) is the "direct product" one represented by the fixed point of the RG equations (28) and (31), $g_{ij}(u,x)=g_{ij}(x)$,

$$\phi(u,x)=\frac{1}{2p}\left[\frac{1}{3}+c''\right]u+\phi(x) .$$

When the "transverse" theory $(g_{ij}(u,x),\phi(u,x))$ is Weyl invariant at $u=0$ *and* $p=0$, Eqs. (28) and (31) imply that the initial Weyl-invariance conditions (34) are satisfied also for all other values of u. Therefore a solution with (34), (27), and $p=0$ may exist only if the transverse theory is conformal for all u. One can also prove the converse [13]: To get a nontrivial solution with a flat $g_{ij}(u,x)$ (more generally, with a conformal transverse theory), one should set $p=0$. Then [assuming $\phi=\phi(u)$] Eqs. (28) and (31) are satisfied automatically, but since $p=0$, the identities (33) *no longer imply* that (29) and (30) are also satisfied.

Since (28) holds identically, it does not give an equation for $g_{ij}(u,x)$. The same is true for (31): It does not contain terms with u derivatives and being a constant [as a consequence of (19) and (24)] is satisfied for all u if it is satisfied for $u=0$, i.e., if $\frac{1}{3}+c'=0$. Instead of $N+1$ identities in (33) for $p=0$, we are left with just one. As a result, we get N independent equations (29) and (30) [(33) gives a relation between components of (29)] on $\frac{1}{2}N(N+1)+1$ functions $g_{ij}(u,x)$ and $\phi(u,x)$. Their particular solutions in the case when the transverse metric is flat (and correspondence with the "plane-wave" solutions found previously [2]) were studied in detail in Ref. [13]. In that case it is useful to change coordinates, trading the functions $g_{ij}(u,x)$ corresponding to a flat transverse metric for A_i and K in (2), i.e., transforming the metric (1) into the form (2) where $g_{ij}(u,x)$ has its canonical δ_{ij} form.

The above discussion can be generalized to the case of nonvanishing antisymmetric tensor coupling [13]. Namely, there exist similar solutions of the Weyl-invariance conditions with the metric (1), dilaton (27), and the v-independent antisymmetric tensor $\hat{B}_{\mu\nu}$: $\hat{B}_{ij}=B_{ij}(u,x)$, $\hat{B}_{iu}=B_i(u,x)$, $\hat{B}_{\mu\nu}=0$.

III. NEW CLASS OF FINITE SUPERSYMMETRIC σ MODELS WITH MINKOWSKI SIGNATURE TARGET SPACE

In Sec. II we have shown that it is possible to construct conformal-invariant Minkowski signature models in $2+N$ dimensions from nonconformal Euclidean models in N dimensions. Since the metric and dilaton of the $(2+N)$-dimensional theory are essentially the "running" couplings of the transverse theory, their dependence on u is determined by the β functions of the transverse theory. The structure of the β functions is usually simpler in supersymmetric theories, and so it is of interest to generalize the above construction to the supersymmetric case. In particular, we would like to make use of the known fact that there are examples of supersymmetric σ models with homogeneous Kähler Einstein target spaces (i.e., with Ricci tensor proportional to the metric) for which the exact β function coincides with the one-loop expression, i.e., is explicitly calculable [25].

The two-dimensional ($n=1$) supersymmetric σ model can be constructed for an arbitrary metric $G_{\mu\nu}$ of a D-dimensional target space. Its superfield action is given by [27]

$$I=\frac{1}{4\pi\alpha'}\int d^2z\, d^2\theta\, G_{\mu\nu}(X)\mathcal{D}X^{\mu}\bar{\mathcal{D}}X^{\nu} , \tag{37}$$

where

$$X^{\mu}=x^{\mu}+\bar{\theta}\psi^{\mu}+\tfrac{1}{2}\bar{\theta}\theta F^{\mu} , \quad \mathcal{D}=\frac{\partial}{\partial\bar{\theta}}+\bar{\theta}\gamma^{a}\partial_a .$$

The component form of the action is

$$I=\frac{1}{4\pi\alpha'}\int d^2z[G_{\mu\nu}(x)\partial_a x^{\mu}\partial^a x^{\nu}+G_{\mu\nu}(x)\bar{\psi}^{\mu}\gamma^{a}D_a\psi^{\nu} +\tfrac{1}{6}R_{\mu\nu\lambda\rho}\bar{\psi}^{\mu}\psi^{\lambda}\bar{\psi}^{\nu}\psi^{\rho}] . \tag{38}$$

For the metric with the null Killing vector (1), we can represent (37) in terms of the real superfields U, V, and X^i:

$$I=\frac{1}{4\pi\alpha'}\int d^2z\, d^2\theta[-2\mathcal{D}U\bar{\mathcal{D}}V+g_{ij}(U,X)\mathcal{D}X^i\bar{\mathcal{D}}X^j] . \tag{39}$$

The component form of (39) can be found either directly from (39) or by substituting the expressions (1) and (4)–(6) into (38).

Equations (3)–(17) have a straightforward generalization to the supersymmetric case. In particular, the solution $g_{ij}(u,x)$ of the condition of finiteness [Eq. (13)] is determined by the β function of the "transverse" part of

(39), i.e., of the supersymmetric model with the metric $g_{ij}(u,x)$ for constant u. As is well known [28], if the transverse space is Kähler, the N-dimensional model is $n=2$ supersymmetric. If it is also a compact homogeneous Einstein space [e.g., $S^2=\mathrm{SO}(3)/\mathrm{SO}(2)$ or CP^m], then it is very plausible that its β function is exactly calculable and is given by the one-loop expression [25] (it is easy to check that the four-loop correction to the β function [32] vanishes for such spaces; see, e.g., [36]). This was actually proved in Ref. [25] for the following classes of Kähler manifolds: symmetric spaces,

$$M_1=\mathrm{SO}(m+2)/\mathrm{SO}(m)\times\mathrm{SO}(2)\ ,\quad N=2m\ ,$$

$$M_2=\mathrm{SU}(m+k)/\mathrm{SU}(m)\times\mathrm{SU}(k)\times\mathrm{U}(1)\ ,\quad N=2mk\ ,$$

$$M_3=\mathrm{Sp}(m)/\mathrm{SU}(m)\times\mathrm{U}(1)\ ,\quad N=m^2+m\ ,$$

$$M_4=\mathrm{SO}(2m)/\mathrm{SU}(m)\times\mathrm{SO}(2)\ ,\quad N=m^2-m\ ,$$

and nonsymmetric spaces,

$$M_5=\mathrm{SU}(m+1)/[\mathrm{U}(1)]^m\ .$$

In that case the transverse part of the metric (14), the β function (16), and the solution of (13) are given simply by

$$g_{ij}(u,x)=f(u)\gamma_{ij}(x)\ ,\quad \beta(f)=a\ ,$$
$$f(u)=bu\ ,\quad b=p^{-1}a\ . \tag{40}$$

The constant $a>0$ is determined by the geometry of the transverse space [25] [$a_1(m=1)=2$, $a_1(m\geq 2)=m$, $a_2=m+k$, $a_3=m+1$, $a_4=m-1$]. The constant b is arbitrary and can be absorbed into redefinition of the coordinates u and v. Then the final expression for the Minkowski signature metric of the finite $(2+N)$-dimensional supersymmetric σ model is

$$ds^2=-2dudv+u\gamma_{ij}(x)dx^idx^j \tag{41}$$

(we have assumed $u>0$). Note that while the transverse model (with fixed constant u) is $n=2$ supersymmetric, the full $(2+N)$-dimensional model apparently has only $n=1$ supersymmetry. The nonzero components of the curvature of the metric (41) can be found from (5) and (6):

$$\hat{R}^{i}_{jkl}=R^{i}_{jkl}(\gamma)\ ,\quad \hat{R}_{iuju}=\frac{b}{4u}\gamma_{ij}\ . \tag{42}$$

All curvature invariants are singular at $u=0$. It is still possible that this singularity is harmless in string theory (cf. Ref. [29]).

The simplest nontrivial example of the finite models we have constructed corresponds to the case when the transverse theory is represented by the O(3) supersymmetric σ model [30]. The resulting metric (41) is that of *four*-$(2+N=4)$ dimensional space with the transverse part being proportional to the metric on S^2:

$$ds^2=-2dudv+u(d\theta^2+\sin^2\theta d\varphi^2)\ . \tag{43}$$

This metric is conformal to the standard metric on the product of the two-dimensional Minkowski space and two-sphere. The corresponding geodesic equations can be easily integrated with the conclusion that the part of space with $u>0$ is not geodesically complete (replacing the factor u by the modulus $|u|$ apparently introduces additional singularities at $u=0$).

To find out whether the constructed finite supersymmetric models can be identified with the exact solutions of the superstring effective equations, we need to check that these σ models correspond to Weyl-invariant theories on a curved 2D background. It is straightforward to add to (37) the dilaton-coupling term $\int d^2z d^2\theta E^{-1}R^{(2)}\phi(X)$ (E^{-1} is the determinant of the $n=1$ supervielbein) and to generalize the expressions for the Weyl-invariance conditions (19)–(21) and the identity (24) to the case of $n=1$ supersymmetric σ models [31]. Then the argument in Sec. II B can be repeated to prove that for an arbitrary "initial" ($u=0$) transverse Euclidean $n=1$ supersymmetric model, there exist such a metric $g_{ij}(u,x)$ and dilaton $\phi(u,x)$ that the corresponding $n=1$ supersymmetric model with metric (1) is Weyl invariant, i.e., represents a string vacuum.

Let us now specialize to the case when the transverse metric is symmetric Kähler. Then we can apply the discussion of the symmetric transverse-space case in Sec. II A. We conclude that the Weyl-invariance conditions are again given by Eqs. (34) and (36). The equation on f is the same RG equation as in the finiteness condition, and so its solution is represented by (40) and (41). Since the transverse model is $n=2$ supersymmetric, we can make use of the result [31] that the dilaton coupling is not renormalized in the $n=2$ supersymmetric case (in the minimal-subtraction scheme). That means that M and ω in J in (34) should vanish. As a result, the dilaton ϕ is given by [cf. (35) and (36)]

$$\phi(v,u)=pv+qu+\Phi(u)\ ,$$
$$\bar{\beta}^{\phi}=c-2pq=0\ , \tag{44}$$
$$c=\tfrac{1}{4}(N-8)\ ,$$

$$\dot{\Phi}=I(f(u))\ ,$$
$$I=p^{-1}J=\frac{N}{8p}f^{-1}\beta(f)-\frac{1}{4}W_u=\frac{N}{8u}-\frac{1}{4}W_u\ , \tag{45}$$
$$f=bu\ ,$$

where we have used that in superstring theory $c=\frac{1}{4}(D-10)$. Note that differentiating the equation for Φ in (45) and comparing with (30) gives

$$W_u=-\frac{N}{2p}\frac{d}{du}(f^{-1}\beta)-2\beta^G_{uu}$$
$$=\tfrac{1}{2}Nu^{-2}-2[\tfrac{1}{4}Nu^{-2}+O(\alpha'^3u^{-3})]$$
$$=O(\alpha'^3u^{-3}) \tag{46}$$

[the one-loop term in β^G_{uu}, i.e., $\alpha' R_{uu}$, is given by (42); see also (15)]. Higher-loop corrections to W_u and to β^G_{uu} are thus directly related. It is easy to see that there is no two-loop term in β^G_{uu} in the case of symmetric transverse space; in the bosonic case, both β^G_{uu} and W_u are nonvanishing in the tree-loop approximation [12].

It is possible that W_u is actually vanishing in the present model. Though the results of Ref. [32] (a comparison of the perturbative expression for the β^G function with superstring effective equations) imply that W_μ contains a nonzero four-loop term in a general $n=1$ supersymmetric model, W_μ does vanish in $n=2$ supersymmetric models [30]. If $W_u=0$, then the exact expression for the dilaton is [see (44) and (45)]

$$\phi(v,u)=\phi_0+pv+qu+\tfrac{1}{8}N\ln u \ . \tag{47}$$

The resulting backgrounds (41), (44), or (47) thus represent exact solutions of superstring effective equations with a nontrivial dilaton. Note that the string coupling e^{ϕ} is

$$e^{\phi}=Au^{N/8}e^{qu+pv}\ , \quad A=e^{\phi_0}\ . \tag{48}$$

It goes to zero in the strong-coupling region $u\rightarrow 0$ of the transverse σ model, i.e., is *small* near the singularity $u=0$. If $N<8$, the constant q in (44) and (47) is negative (we are assuming $u>0$, $v>0$, $p>0$) so that the string coupling is also vanishing in the small coupling region $u\rightarrow\infty$. In the case of the critical dimension $D=10$ (or $N=8$), q must vanish. Then the string coupling is inversely proportional to the σ-model coupling f^{-1}:

$$e^{\phi}=A'fe^{pv}\ .$$

IV. APPLICATION TO 2D QUANTUM GRAVITY

As is well known, the classical gravitational action in $d=2$ is trivial before one accounts for the (nonlocal) quantum anomaly term. Introducing an extra scalar field ("2D dilaton") coupled to the scalar curvature, one obtains a nontrivial theory (though still with no propagating degrees of freedom). This theory seems simpler and better defined as a starting point for a (perturbative) quantization. By redefining the fields, one can represent the general action in the form [18,19]

$$S=-\tfrac{1}{2}\int d^2x\sqrt{g}\,[\partial^\mu\varphi\partial_\mu\varphi+q\varphi R+V(\varphi)]\ . \tag{49}$$

For example, the metric-dilaton action, which generates the σ-model Weyl anomaly coefficients in the case of $D=2$ target space and which has a classical "black hole" solution [33]

$$S=-\tfrac{1}{2}\int d^2x\sqrt{g}\,e^{-2\phi}[R+4(\partial\phi)^2+c]\ , \tag{50}$$

can be represented as (49) with $V=c\exp(\varphi/q)$. By a further redefinition, it can be put into the form

$$S=-\tfrac{1}{2}\int d^2x\sqrt{\hat{g}}\,(\hat{R}v+c)\ ,$$
$$\hat{g}_{\mu\nu}=vg_{\mu\nu}\ , \quad v=e^{-2\phi}\ . \tag{51}$$

Let us now switch to "world sheet" notation and consider the metric-scalar $(\hat{\gamma},v)$ gravitational theory coupled to some extra N "matter" scalar fields which is described by the σ model

$$I_0=\frac{1}{4\pi}\int d^2z\sqrt{\hat{\gamma}}[pv\hat{R}^{(2)}+g_{ij}(x)\partial_a x^i\partial^a x^j+T(x)]\ . \tag{52}$$

In the conformal gauge

$$\hat{\gamma}_{ab}=e^{-2u/p}\gamma_{ab}\ ,$$

(52) takes the form

$$I_0=\frac{1}{4\pi}\int d^2z\sqrt{\gamma}[-2\partial_a v\partial^a u+g_{ij}(x)\partial_a x^i\partial^a x^j+pvR^{(2)}+T(x)e^{-2u/p}]\ . \tag{53}$$

This model is renormalizable on a flat background with g_{ij} "running" with a cutoff. Once all the fields are quantized, one may expect that the "effective action" will be represented by a general σ model in $2+N$ dimensions $x^\mu=(u,v,x^i)$. The model should be Weyl invariant with respect to the background metric γ_{ab} since the 2D metric itself is an integration variable [14–16]. We are implicitly assuming that the theory can be regularized in a way covariant with respect to the original metric $\hat{\gamma}$ so that all the elements of the theory (the action, the measure, and the regularization) depend only on the full $\hat{\gamma}$ (and that we are in the phase where the 2D metric has a zero expectation value). To determine the "effective action," we need to find a solution of the Weyl-invariance conditions for the metric, dilaton, and tachyon couplings of the $(2+N)$-dimensional theory such that at the classical limit they reduce to the couplings in (53). It seems natural to impose an additional assumption that the dependence of the couplings on v in the "effective action" should remain as simple as in (53); i.e., the target-space metric and the tachyon should be v independent (the metric will have a Killing vector), while the dilaton will be at most linear in v. It is precisely such solutions of the metric and dilaton Weyl-invariance conditions (19) and (20) that we have studied in Sec. II (let us first ignore the tachyon-coupling term). We have found that the action

$$I=\frac{1}{4\pi}\int d^2z\sqrt{\gamma}\{-2\partial_a v\partial^a u+g_{ij}(u,x)\partial_a x^i\partial^a x^j+[pv+\phi(u,x)]R^{(2)}\} \tag{54}$$

defines a Weyl-invariant quantum theory if the metric g_{ij} and dilaton ϕ depend on u according to the first-order RG equations (28) and (31). The result that g_{ij} starts running with u according to the RG equation $\dot{g}_{ij}\sim R_{ij}+\cdots$ is very natural given that $u(z)$ is proportional to the conformal factor of the 2D metric (which should be coupled to a covariant cutoff). At the same time, one would also expect to find the conformal anomaly term $\sim K(u,x)(\partial u)^2$, but it is missing in (54). Note, however, that such term can be generated by a redefinition of the field v. As discussed in Ref. [13], there is, in fact, an *equivalent* solution of the conformal-invariance conditions (19)–(21) with $\phi(u,x)=0$, but with the metric (1) containing the additional term $K(u,x)du^2$ [cf. (2)]. The difference between the theory (52) and the standard 2D gravity coupled to a σ model [where both the anomaly term $K(u,x)\partial_a u\partial^a u$ and $\phi(u,x)R^{(2)}$ should appear in the quantum action [16]] is due to the presence of the extra scalar field v.

Let us now study the solutions of the Weyl-invariance condition for the tachyon coupling [34,21] [cf. (19)–(21)]:

$$\bar{\beta}^T = -\gamma T + (\alpha' \partial^\mu \phi + \tfrac{1}{2} W^\mu)\partial_\mu T - 2T + b(T)$$
$$= -\tfrac{1}{2}\alpha' D^2 T + \alpha' \partial^\mu \phi \partial_\mu T - 2T + O(\alpha'^3) + b(T)$$
$$= 0 \; . \quad (55)$$

γ is the same differential operator which appeared in (31). $b(T)$ represents "nonperturbative" corrections which are of higher order in T. If there were no v coordinate so that the metric of the $(1+N)$-dimensional space and the dilaton were given by $ds^2 = K du^2 + ds_N^2$ and $\phi = Ku + \cdots$, then (55) would reduce to a second-order equation in u [17], $-\frac{1}{2}K^{-1}\ddot{T} + \dot{T} + \cdots = 0$, which would reproduce the standard RG equation only in the "semiclassical" limit of a large anomaly coefficient K. On the other hand, if the metric $G_{\mu\nu}$ is given by (1) and the dilaton is linear in v [Eq. (27)], then for a v-independent tachyon $T = T(u,x)$, Eq. (55) takes a form similar to (28) and (31); i.e., it becomes a *first-order* RG-type equation (cf. Ref. [17]):

$$p\dot{T} = \bar{\beta}^{T'} \; . \quad (56)$$

$\bar{\beta}^{T'}$ (containing only derivatives over x^i) denotes the Weyl anomaly coefficient of the "transverse" theory with the coupling $T(u,x)$ and $u =$const playing the role of the RG "time." The simplest example of a solution of (55) and (56) is found if $T = T(u)$. Let us first ignore the "nonperturbative" term $b(T)$. Then [cf. (53)]

$$p\dot{T} = -2T \; , \quad T = T_0 e^{-2u/p} \; . \quad (57)$$

Equivalent solutions in the context of the 2D gravity model were discussed in Ref. [19]. Now it is possible show that T in (57) solves the full Eq. (56) (with all higher-order terms included), i.e., that there are no nonperturbative divergences in the model

$$I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} [-2\partial_a v \partial^a u + p v R^{(2)} + T(u)] \; . \quad (58)$$

In fact, v plays the role of a Lagrange multiplier which makes u effectively nonpropagating so that there are no quantum corrections in the theory (see also Ref. [35]). Then the condition of conformal invariance is equivalent to the classical conformal-invariance relation (57). To reconcile this conclusion with the expected presence of $O(T^2)$ and $O(\partial T \partial T)$ terms in $\bar{\beta}^T$, $\bar{\beta}^\phi$, and $\bar{\beta}^G$, one is to note that a derivation of such terms [or a proof of correspondence with $O(T^3)$ terms in the effective action] presumes an analytic continuation in momenta and is not, strictly speaking, valid in the case when T depends just on one variable (the question of nonperturbative terms in the β functions should be addressed separately for each 2D theory corresponding to a particular scalar potential T; see Ref. [34]).

In conclusion, we have suggested a connection between the conformal-invariant $(2+N)$-dimensional σ models and the 2D scalar quantum gravity coupled to nonconformal "transverse" N-dimensional σ models. The conformal factor of the 2D metric is identified not with time, but with the light-cone coordinate u; this makes the corresponding Weyl-invariance conditions first order in u. Given that the target-space metric corresponding to 2D gravity plus scalar matter models has a natural Minkowski signature [18], it seems important to try to clarify further the connection between the "Minkowski" conformal theories and 2D quantum gravity.

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