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Duality in nontrivially compactified heterotic strings

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We study the implications of duality symmetry on the analyticity properties of the partition function as it depends upon the compactification length. In order to obtain nontrivial compactifications, we give a physical prescription to get the Helmholtz free energy for any heterotic string, supersymmetric or not. After proving that the free energy is always invariant under the duality transformation $R \to \alpha'/(2R)$ and getting the zero-temperature theory whose partition function corresponds to the Helmholtz potential, we show that the self-dual point $R_0 = \sqrt{\alpha'/2}$ is a generic singularity like the Hagedorn one. The main difference between these two critical compactification radii is that the term producing the singularity at the self-dual point is finite for any $R \neq R_0$. We see that this behavior at R_0 actually implies a loss of degrees of freedom below that point.

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I. INTRODUCTION

Recently some thinking effort has been devoted to the problem of R duality in string theory (cf., for example, [1, 2]). One of the related topics that seems to have received less attention is that of the implications of duality on the analytic structure of the partition function $Z(R)$ as a function of the compactification radius R. As far as we know this problem has only been fully treated for the noncritical string coupled to conformal matter with $c = 1$ in which $Z(R)$ is $C^\infty(\mathbf{R}^+)$ as a function of R [2, 3]. In the case of critical heterotic strings this problem seems to have been studied only in [4, 5]. In these works a small class of heterotic strings has been treated: trivial toroidal compactifications on $\mathbf{R}^{(\text{critical } d)-1} \times S^1$. On the other hand it is also well known that there exists a completely analogous duality invariance (the so-called β duality in opposition to space-time duality) for the thermal free energies corresponding to the family of supersymmetric heterotic strings. This is the only case in which duality is a mathematically well-defined property of the Helmholtz free energy, although there is always a well-defined duality relationship for the integrand of the free energy represented as an integral over the fundamental region of the modular group, even for the bosonic string. Of course duality is actually an invariance of the spectrum of the theory.

Using the nontrivial relationship between the Helmholtz free energy and the partition function of the same theory on $\mathbf{R}^{(\text{critical } d)-1} \times S^1$ we can get an enormous class of non-supersymmetric strings, which exhibit nontrivial duality as a property of their corresponding partition functions. By nontrivial duality we mean that under the transformation $R \to \text{const}/R$ the solitonic contributions associated with each spin structure interchange between them. In other words, there is a correlation between the solitonic contributions and the spin structures. The study of heterotic strings at finite temperature is then of interest to understand space-time duality as a physical property.

On the other hand, the study of superstrings at finite temperature has interest on its own. A lot of effort has been devoted to this topic (cf., for example, [6–8]), without, in our opinion, getting any conclusive answer for the most important questions: Is there any possibility of a phase transition at the Hagedorn temperature (or before or after)? And if there is, what would the number of physical degrees of freedom be at high temperature? The number of degrees of freedom of the Nambu-Goto string is also of major interest in order to know whether this string has something to do with the features of the deconfinement phase of QCD (cf. [9]).

The existence of β duality for the free energy is a puzzling property of the heterotic string closely related to the latter question. Namely, at the one-loop level β duality on the Helmholtz free energy reads

$$F_{\text{het}}(\beta) = \frac{\pi^2}{\beta^2} F_{\text{het}}\left(\frac{\pi^2}{\beta}\right). \tag{1}$$

The presence of the Hagedorn length β_H together with the β-duality property implies that there exists another critical length $\beta_H^* = \pi^2/\beta_H$ such that although $F_H(\beta)$ diverges for $\beta_H^* < \beta < \beta_H$ it is finite for $\beta < \beta_H^*$. Using $F_{\text{het}}(\beta)$ when $\beta < \beta_H^*$ as the thermodynamical Helmholtz potential some shocking thermodynamical features appear for this would-be high-temperature phase. The most striking one is that, in the limit $\beta \to 0^+$, (1) implies that the free energy behaves as

$$F_{\text{het}}(\beta) \sim \frac{\pi^2}{\beta^2}\Lambda. \tag{2}$$

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Here Λ is the cosmological constant; since $\Lambda = 0$ for a heterotic supersymmetric string (2) implies that no degree of freedom will survive at high temperature. In fact, we will show that (1) and consequently (2) also hold for any heterotic string, even a nonsupersymmetric one. Some authors [10] (cf. also [9]) have pointed out that this could indicate that the heterotic string would be described by a topological theory in that limit. Thermodynamically things appear as though Bose and Fermi statistics were not equivalent at high temperature.

In the present work we will carefully study the behavior of thermal heterotic strings to prove, among other things, that above the Hagedorn length, which is the same for every heterotic string, there is one more critical length at the self-dual point, which is generic too and does not correspond to a divergent term in the free energy.

In Sec. II the prescription given by Atick and Witten in [10] for constructing the free energy of the heterotic string at one loop will be recovered by using what we regard as a much more physical prescription. In fact, by making use of the results of Ref. [1] we will be able to show that every heterotic string enjoys β duality, even in the nonsupersymmetric case. With this new prescription we will explicitly calculate the free energy for the family of nonsupersymmetric heterotic strings in two dimensions, which appear in [11, 12, 5].

Section III will be devoted to the study of the possibility of separately getting the free energy for the bosonic and fermionic sectors of the heterotic string in a manifestly modular-invariant way. This goal is a legitimate one from a purely thermodynamical point of view (e.g., if we are interested in studying a possible Bose condensation). The results so obtained will be used to analyze what would be the structure of the high-temperature phase (or equivalently the number of degrees of freedom at high energy).

In Sec. IV we shall study the behavior of the free energy at some given values of β at which special generic singularities appear [4, 5]. These singularities will be interpreted as coming from contributions of states, which actually behave as ghosts, killing the physical degrees of freedom that are a surplus for duality to hold. Finally, in Sec. V we will summarize the conclusions.

II. GETTING THE FREE ENERGY

In Ref. [10] Atick and Witten gave a prescription for computing the free energy for the heterotic string. In their approach the contribution of an arbitrary set of windings (n, m) is weighted, at one loop level, by a phase given by

$$\begin{aligned} U_1(n,m) &= \frac{1}{2}[-1+(-1)^n+(-1)^m+(-1)^{m+n}], \\ U_2(n,m) &= \frac{1}{2}[1-(-1)^n+(-1)^m+(-1)^{m+n}], \\ U_3(n,m) &= \frac{1}{2}[1+(-1)^n+(-1)^m-(-1)^{m+n}], \\ U_4(n,m) &= \frac{1}{2}[1+(-1)^n-(-1)^m+(-1)^{m+n}], \end{aligned} \tag{3}$$

where $1, 2, 3, 4$ label, respectively, the four spin structures $(+,+), (+,-), (-,-), (-,+)$.

Generalizing these phases to arbitrary genus, one can represent the genus-g contribution to the free energy per unit volume for the heterotic string in a very manageable form, namely [1],

$$F_g(\beta) = \int_{\mathcal{F}_g} d\mu(m) \sum_s \Lambda_s(\tau,\bar{\tau})\theta \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{s}_2 & \mathbf{s}_1 \end{bmatrix} (0|\tilde{\mathbf{\Omega}}_g), \tag{4}$$

where the cosmological constant is given by

$$\Lambda_s = \int_{\mathcal{F}_g} d\mu(m) \sum_s \Lambda_s(\tau,\bar{\tau}) \tag{5}$$

and

$$s \equiv \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix}, \tag{6}$$

$\mathbf{s}_1, \mathbf{s}_2 \in [(\mathbf{Z}/2)/\mathbf{Z}]^g$ being the characteristics defining the 2^{2g} spin structures on the Riemann surface (in fact, only the even-spin structures contribute). The second argument of the Riemann θ function [13] is

$$\begin{aligned} \tilde{\mathbf{\Omega}}_g &= \mathbf{\Omega}_g + \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\ &= \frac{i\beta^2}{2\pi^2}\begin{bmatrix} \tau_1\tau_2^{-1}\tau_1+\tau_2 & -\tau_1\tau_2^{-1} \\ -\tau_2^{-1}\tau_1 & \tau_2^{-1} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

Manipulating (4) one is able to rewrite the genus-g contribution to the free energy as

$$\begin{aligned} F_g(\beta) = \int_{\mathcal{F}_g} d\mu(m) \sum_s \Lambda_s \sum_t & (-1)^{4(\mathbf{s}_2\mathbf{t}_1+\mathbf{s}_1\mathbf{t}_2+\mathbf{t}_1\mathbf{t}_2)} \\ & \times\theta\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \\ 0 & 0 \end{bmatrix}(0|4\mathbf{\Omega}_g). \end{aligned} \tag{8}$$

In [1] it is shown that given this form we have the following duality relation for $F_g(\beta)$:

$$F_g(\beta) = \left(\frac{\pi^2}{\beta^2}\right)^g F_g\left(\frac{\pi^2}{\beta}\right). \tag{9}$$

In particular, for $g = 1$ we get (1).

So every heterotic string such that its free energy is given by the application of the Atick and Witten prescription obeys duality at arbitrary genus (provided only even spin structures contribute to the free energy).

Let us now consider the general form of the one-loop free energy for a heterotic string. By using the four SO(10) conjugacy classes we can write the part of the integrand of the cosmological constant which multiplies Poincaré's invariant measure as (cf., for example, [14])

$$\sum_s \Lambda_s(\tau,\bar\tau) = \tau_2^{-\frac{d-2}{2}}\left[\frac{\bar\theta_2^4}{\bar\eta^{12}}z_s(\tau,\bar\tau)+\frac{\bar\theta_2^4}{\bar\eta^{12}}z_c(\tau,\bar\tau)-\frac{\bar\theta_3^4-\bar\theta_4^4}{\bar\eta^{12}}z_v(\tau,\bar\tau)-\frac{\bar\theta_3^4+\bar\theta_4^4}{\bar\eta^{12}}z_o(\tau,\bar\tau)\right]$$
$$= \tau_2^{-\frac{d-2}{2}}\left[\frac{\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)+\frac{\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)-\frac{\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)\right]. \quad (10)$$

Where z_o, z_v, z_s, z_c are, respectively, the contributions associated with the four conjugacy classes of SO(10) (scalar, vectorial, and two spinorials), and d is the number of noncompact dimensions. For example, for the supersymmetric heterotic string in ten dimensions we have that $z_o = z_c = 0$ and $z_v = z_s = \Theta_{\Gamma_8\oplus\Gamma_8}\eta^{-24}$.

From the modular invariance of (10) we get

$$\frac{\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)+\frac{\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)-\frac{\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)=\frac{\bar\theta_2^4}{\bar\eta^{12}}[T(z_s)+T(z_c)]-\frac{\bar\theta_3^4}{\bar\eta^{12}}[T(z_v)-T(z_o)]+\frac{\bar\theta_4^4}{\bar\eta^{12}}[T(z_v)+T(z_o)] \quad (11)$$

and

$$\frac{\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)+\frac{\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)-\frac{\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)=\frac{\bar\theta_4^4}{\bar\eta^{12}}\bar\tau^{-4}|\tau|^{d-2}[S(z_s)+S(z_c)]$$
$$+\frac{\bar\theta_2^4}{\bar\eta^{12}}\bar\tau^{-4}|\tau|^{d-2}[S(z_v)-S(z_o)]-\frac{\bar\theta_3^4}{\bar\eta^{12}}\bar\tau^{-4}|\tau|^{d-2}[S(z_v)+S(z_o)], \quad (12)$$

where T and S are the two transformations generating the modular group: $T{:}\tau \longrightarrow \tau+1$ and $S{:}\tau \longrightarrow -1/\tau$.

These relations determine the transformation properties under both T and S of the combination of z_i's, which appear in (10). However, we should be very careful about the identification of the terms multiplying Jacobi's θ function in (11) and (12). If we consider a linear combination of the form

$$A_1\bar\theta_2^4 + A_2\bar\theta_3^4 + A_3\bar\theta_4^4 = 0, \quad (13)$$

then there exists a trivial solution which correspond to $A_1 = A_2 = A_3 = 0$. But since Jacobi θ functions satisfy the well-known identity

$$\bar\theta_2^4 - \bar\theta_3^4 + \bar\theta_4^4 = 0, \quad (14)$$

we also have a nontrivial solution, namely,

$$A_1 = -A_2 = A_3. \quad (15)$$

From a physical point of view the nontrivial solution would correspond to the fact that when performing a modular transformation we are not recovering the same theory we started with. We need to add to it the spectrum of a supersymmetric heterotic string. Our starting point is that (10) corresponds to the complete theory. Actually if we add a supersymmetric heterotic spectrum we can always write the resulting cosmological constant as in (10). Thus we will only consider the trivial solution of equations (11) and (12). This gives the transformation properties (cf., for example, [15, 14])

$$\begin{aligned} T(z_s)+T(z_c) &= z_s+z_c,\\ T(z_v)+T(z_o) &= z_v-z_o,\\ T(z_v)-T(z_o) &= z_v+z_o,\end{aligned} \quad (16)$$

for the T transformation, and

$$\begin{aligned} S(z_s)+S(z_c) &= \bar\tau^4|\tau|^{2-d}(z_v-z_o),\\ S(z_v)+S(z_o) &= \bar\tau^4|\tau|^{2-d}(z_v+z_o),\\ S(z_v)-S(z_o) &= \bar\tau^4|\tau|^{2-d}(z_s+z_c),\end{aligned} \quad (17)$$

for S. These tranformation properties will be of major interest when dealing with the theory at finite temperature.

To construct the corresponding free energy one can use an analogue model. The free energy of a quantum field with N_B bosonic physical degrees of freedom is given at one loop by [7, 16]

$$F_B(\beta) = -N_B\int_0^\infty ds s^{-1-\frac{d}{2}}\theta_3'\left(0\left|\frac{i\beta^2}{2\pi s}\right.\right)e^{-m^2s/2} =: N_B f_B(\beta), \quad (18)$$

where $f_B(\beta)$ is the free energy per physical degree of freedom and the prime indicates that the zero mode has been suppressed. For a fermionic field of the same mass and N_F physical degrees of freedom we have [17]

$$F_F(\beta) = N_F f_B(\beta) - 2N_F f_B(2\beta). \quad (19)$$

Now it is easy to get the free energy for a field with bosonic and fermionic degrees of freedom:

$$F(\beta) = (N_B - N_F)f_B(\beta) + N_F f_S(\beta), \quad (20)$$

where $f_S(\beta) = f_B(\beta) + f_F(\beta)$. If the cosmological constant vanishes then the free energy can be written

$$F(\beta) = N_F f_S(\beta) = -2N_F\int_0^\infty ds s^{-1-\frac{d}{2}}\theta_2\left(0\left|\frac{2i\beta^2}{\pi s}\right.\right)e^{-m^2s/2}. \quad (21)$$

Weighting the states in the scalar and vectorial conjugacy classes as bosonic quantum fields and both spinorial classes as fermionic ones we obtain, for the integrand of the β-dependent part of the free energy,

$$\chi(\tau,\bar\tau) = \left[\sum_s \Lambda_s(\tau,\bar\tau)\right]\theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right) - \tau_2^{-\frac{d-2}{2}}\frac{2\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)\theta_2\left(0\left|\frac{2i\beta^2}{\pi^2\tau_2}\right.\right). \quad (22)$$

Since $\chi(\tau,\bar\tau)$ is invariant only under the Borel subgroup generated by the transformation T, we have that the free energy is obtained by integrating over the fundamental region of this subgroup, which is the strip $S = \{\tau_1 + i\tau_2 | \tau_2 \geq 0, -1/2 \leq \tau_1 < 1/2\}$. Physically in the analogue model we integrate over the proper time from 0 to $+\infty$, and at the same time we have to impose the left-right level-matching condition as a Kronecker δ that finally becomes an integral over phases. So we have that the β-dependent part of the free energy in the S representation is given by

$$F(\beta) = \int_S \frac{d^2\tau}{\tau_2^2}\left\{ \left[\sum_s \Lambda_s(\tau,\bar\tau)\right] \theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right) - \tau_2^{-\frac{d-2}{2}} \frac{2\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)\theta_2\left(0\left|\frac{2i\beta^2}{\pi^2\tau_2}\right.\right)\right\}. \tag{23}$$

This is in perfect accordance with the field-theoretical result (20).

Our next task will be to go from the S representation for the free energy to the F representation in which it is expressed as an integral over the fundamental region of the modular group of a function, which is invariant under the full modular group (cf., for example, [17]). To do this, we will make use of the coset techniques developed in [18] (see also [11, 19]). Let us separate (23) into two parts:

$$F(\beta) = F_1(\beta) + F_2(\beta) \tag{24}$$

with

$$F_1(\beta) = \int_S \frac{d^2\tau}{\tau_2^2}\left[\sum_s \Lambda(\tau,\bar\tau)\right]\theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right), \tag{25}$$

$$F_2(\beta) = -\int_S \frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}\frac{2\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)\theta_2\left(0\left|\frac{2i\beta^2}{\pi^2\tau_2}\right.\right). \tag{26}$$

Since, as we have imposed above, $\sum_s \Lambda_s$ is a modular invariant quantity, we can obtain the F representation of $F_1(\beta)$ almost inmediately:

$$F_1(\beta) \to \int_{\mathcal F} \frac{d^2\tau}{\tau_2^2}\left[\sum_s \Lambda_s(\tau,\bar\tau)\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|\Omega); \tag{27}$$

here the prime indicates again the absence of the zero mode and $\Omega = \mathbf{\Omega}_g$ with $g = 1$.

$F_2(\beta)$ can be rewritten in a modular invariant way by going from the subgroup of discrete translations B to the congruence subgroup $\Gamma_0(2) \subset \Gamma$ and finally to the full modular group Γ [13, 11, 18]. We get

$$F_2(\beta) \overset{B\to\Gamma_0(2)\to\Gamma}{\longrightarrow} \int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}\left\{-\frac{2\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c)\theta\begin{bmatrix}0\,\frac12\\0\,0\end{bmatrix}(0|4\Omega) - \frac{2\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)\theta\begin{bmatrix}\frac12\,0\\0\,0\end{bmatrix}(0|4\Omega) + \frac{2\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)\theta\begin{bmatrix}\frac12\,\frac12\\0\,0\end{bmatrix}(0|4\Omega)\right\}. \tag{28}$$

Collecting together (27) and (28) we finally arrive at the modular invariant expression of the Helmholtz free energy for a general heterotic string:

$$F(\beta) = \int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s(\tau,\bar\tau)\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|\Omega) - \int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}\left\{\frac{2\bar\theta_2^4}{\bar\eta^{12}}(z_s+z_c) \times\theta\begin{bmatrix}0\,\frac12\\0\,0\end{bmatrix}(0|4\Omega) + \frac{2\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)\theta\begin{bmatrix}\frac12\,0\\0\,0\end{bmatrix}(0|4\Omega) - \frac{2\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)\theta\begin{bmatrix}\frac12\,\frac12\\0\,0\end{bmatrix}(0|4\Omega)\right\}. \tag{29}$$

After some computations one can show that (29) is the same we would have obtained from the application of the Atick-Witten's prescription as written in (4). Indeed, by using (8) and with the identifications

$$\begin{aligned}
\Lambda_{\frac12,0}(\tau,\bar\tau) &= \tau_2^{-\frac{d-2}{2}}\frac{\bar\theta_2^4}{\bar\eta^{12}}[z_s(\tau,\bar\tau)+z_c(\tau,\bar\tau)],\\
\Lambda_{0,\frac12}(\tau,\bar\tau) &= \tau_2^{-\frac{d-2}{2}}\frac{\bar\theta_4^4}{\bar\eta^{12}}[z_v(\tau,\bar\tau)-z_o(\tau,\bar\tau)],\\
\Lambda_{0,0}(\tau,\bar\tau) &= -\tau_2^{-\frac{d-2}{2}}\frac{\bar\theta_3^4}{\bar\eta^{12}}[z_v(\tau,\bar\tau)+z_o(\tau,\bar\tau)]
\end{aligned} \tag{30}$$

(the contribution from the odd spin structure $\Lambda_{\frac12,\frac12}$ being zero) one gets

$$F_{\rm AW}(\beta) = \int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s\right]\sum_{s_1,s_2}\theta\begin{bmatrix}s_1\,s_2\\0\;\;0\end{bmatrix}(0|4\Omega) - \int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left\{2\Lambda_{0,\frac12}\theta\begin{bmatrix}\frac12\,0\\0\,0\end{bmatrix}(0|4\Omega) + 2\Lambda_{\frac12,0}\theta\begin{bmatrix}0\,\frac12\\0\,0\end{bmatrix}(0|4\Omega) + 2\Lambda_{0,0}\theta\begin{bmatrix}\frac12\,\frac12\\0\,0\end{bmatrix}(0|4\Omega)\right\} \tag{31}$$

that is easily proven to be almost identical to (29). The difference is that in Eq. (29) the zero mode of the Riemann θ function with vanishing characteristics is missing as the result of removing the ultraviolet divergent vacuum energy in the original analogue model. The presence of this zero mode in (31) guarantees that $F_{\rm AW}(\beta)$ obeys (1). In the case of (29) we have that the relation that holds is

$$F(\beta) = \frac{\pi^2}{\beta^2} F\left(\frac{\pi^2}{\beta}\right) - \left(1 - \frac{\pi^2}{\beta^2}\right) \Lambda. \tag{32}$$

However, by simply adding the cosmological constant to (29) we get that $F(\beta)$ obeys (1). This corresponds to dropping the prime in the Riemann θ function with null characteristics. In other words to have duality as a property of the partition function we need to set the result of the decompactification limit to the value of the cosmological constant.

As a particular example, we are now going to apply the prescription to get the free energy for the cases in the family of heterotic strings in two noncompact dimensions, which have been treated in [5] (see also [11] and [12]). The contribution associated with every SO(10) conjugacy class is given by [5]

$$z_o(\tau,\bar\tau) = \frac{1}{2}\bar\theta_2^4(\bar\theta_3^4 + \bar\theta_4^4)[j(\tau) + r_\Gamma(1) - 720], \tag{33}$$

$$z_v(\tau,\bar\tau) = \left[\frac{1}{4}(\bar\theta_2^8 + \bar\theta_3^8 + \bar\theta_4^8) + \frac{1}{2}\bar\theta_3^4\bar\theta_4^4\right][j(\tau) + r_\Gamma(1) - 720], \tag{34}$$

$$z_s(\tau,\bar\tau) = \left[\frac{1}{4}(\bar\theta_2^8 + \bar\theta_3^8 + \bar\theta_4^8) - \frac{1}{2}\bar\theta_3^4\bar\theta_4^4\right][j(\tau) + r_\Gamma(1) - 720], \tag{35}$$

$$z_c(\tau,\bar\tau) = \frac{1}{2}\bar\theta_2^4(\bar\theta_3^4 - \bar\theta_4^4)[j(\tau) + r_\Gamma(1) - 720], \tag{36}$$

where $j(\tau)$ is the modular invariant function and $r_\Gamma(1)$ is the number of lattice vectors with $(\text{length})^2 = 2$, which parametrizes the 24 self-dual Niemeier lattices among which we choose one to compactify the left moving coordinates.

By using (29) we obtain the free energy per unit volume for the family of heterotic strings described above as

$$\begin{aligned} F(\beta) = &-48\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}[j(\tau)+r_\Gamma(1)-720]\theta\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|\Omega) \\ &-\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}[j(\tau)+r_\Gamma(1)-720] \\ &\times\left\{\frac{2\bar\theta_2^{12}}{\bar\eta^{12}}\theta\begin{bmatrix}0\,\frac12\\0\,0\end{bmatrix}(0|4\Omega)+\frac{2\bar\theta_4^{12}}{\bar\eta^{12}}\theta\begin{bmatrix}\frac12\,0\\0\,0\end{bmatrix}(0|4\Omega)-\frac{2\bar\theta_3^{12}}{\bar\eta^{12}}\theta\begin{bmatrix}\frac12\,\frac12\\0\,0\end{bmatrix}(0|4\Omega)\right\}, \end{aligned} \tag{37}$$

where we have set all the constants in front of the integrals to one.

III. BOSONIC AND FERMIONIC DEGREES OF FREEDOM IN HETEROTIC STRINGS

In the analogue model the presence of the Hagedorn length is seen as resulting from the exponential growth of the degeneracy of the states in a mass level when the mass goes to infinity. So when this phenomenon happens it does not disappear by increasing the temperature; in fact it worsens. The strange feature implied by duality in the modular invariant expression is that below β_H^* the free energy is finite and then the analogue model interpretation does not hold. One may ask oneself what happens to the density of states we could associate with each of the two statistics. The standard in quantum-field theory is that the free energy diverges as $-\beta^{-d}$ when β goes to zero (see [17] and references therein and also [10]). For a system of bosons and fermions the total free energy behaves in this limit as

$$F_T(\beta) \sim -\left[N_F(1-2^{1-d}) + N_B\right]\beta^{-d}. \tag{38}$$

In particular for a supersymmetric theory,

$$F_T(\beta) \sim -N_T(1-2^{-d})\beta^{-d}, \tag{39}$$

where $N_T = N_F + N_B = 2N_B$. Thus the combined action of bosons and fermions never results in a cancellation between the two types of degrees of freedom.

In this section we will show that it is possible to obtain separated modular invariant expressions for the free energy of each of the the two species of degrees of freedom in the heterotic string: bosonic and fermionic ones. We will see that for each case the corresponding free energy is infinite from β_H to $\beta = 0$. It will be only after combining the two contributions that the strange high-temperature phase appears as though there were tachyons in the fermionic sector having and preserving the fermionic character at high temperature.

To get the modular invariant expression we start using the analogue model as follows from the prescription given in the last section to get

$$\begin{aligned} F_B(\beta) = \int_S \frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}&\left[\frac{\bar\theta_4^4}{\bar\eta^{12}}(z_v - z_o) - \frac{\bar\theta_3^4}{\bar\eta^{12}}(z_v + z_o)\right] \\ &\times\theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right) \end{aligned} \tag{40}$$

for bosons and

$$F_F(\beta) = \int_S \frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}\frac{\bar\theta_2^4}{\bar\eta^{12}}(z_s + z_c)\theta_4'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right) \tag{41}$$

for fermions. Now, the goal is to obtain from (40) and (41) a modular invariant result. It is not straightforward to get manageable expressions. By manageable we mean written in terms of our useful Riemann θ functions. The reason is that the behavior of the Jacobi θ functions under the modular group transformations do not correspond to that of the terms that multiply each of them. To solve this problem we use the easily proven relations

$$\theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right)=\theta_3'\left(0\left|\frac{i(2\beta)^2}{2\pi^2\tau_2}\right.\right)+\theta_2\left(0\left|\frac{i(2\beta)^2}{2\pi^2\tau_2}\right.\right), \tag{42}$$

$$\theta_4'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right)=\theta_3'\left(0\left|\frac{i(2\beta)^2}{2\pi^2\tau_2}\right.\right)-\theta_2\left(0\left|\frac{i(2\beta)^2}{2\pi^2\tau_2}\right.\right) \tag{43}$$

to get the expansions

$$\theta_3'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right)=\theta_3'\left(0\left|\frac{i(2^N\beta)^2}{2\pi^2\tau_2}\right.\right)+\sum_{k=1}^{N}\theta_2\left(0\left|\frac{i(2^k\beta)^2}{2\pi^2\tau_2}\right.\right) \tag{44}$$

and

$$\theta_4'\left(0\left|\frac{i\beta^2}{2\pi^2\tau_2}\right.\right)=\theta_3'\left(0\left|\frac{i(2^N\beta)^2}{2\pi^2\tau_2}\right.\right)+\sum_{k=2}^{N}\theta_2\left(0\left|\frac{i(2^k\beta)^2}{2\pi^2\tau_2}\right.\right)-\theta_2\left(0\left|\frac{i(2\beta)^2}{2\pi^2\tau_2}\right.\right). \tag{45}$$

We are now prepared to consider the case of the bosons; their free energy can be written as

$$F_B(\beta)=F_B(2^N\beta)+\sum_{k=1}^{N}\int_S\frac{d^2\tau}{\tau_2^2}\tau_2^{-\frac{d-2}{2}}\left[\frac{\bar\theta_4^4}{\bar\eta^{12}}(z_v-z_o)-\frac{\bar\theta_3^4}{\bar\eta^{12}}(z_v+z_o)\right]\theta_2\left(0\left|\frac{i(2^k\beta)^2}{2\pi^2\tau_2}\right.\right). \tag{46}$$

We can easily go from the Borel subgroup to $\Gamma_0(2)$ and from this to the full modular group. Following this procedure and taking finally the limit $N\to\infty$ (which is perfectly justified for $\beta\neq 0$), we arrive at

$$F_B(\beta)=\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|\Omega)+\frac{1}{2}\sum_{k=0}^{\infty}\left\{F(2^k\beta)-\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|2^{2k}\Omega)\right\}, \tag{47}$$

where $F(\beta)$ concides with the total free energy given by (29). For the fermionic degrees of freedom we operate along the same lines; its contribution to the free energy can be written as

$$F_F(\beta)=F(\beta)-\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|\Omega)-\frac{1}{2}\sum_{k=0}^{\infty}\left\{F(2^k\beta)-\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\left[\sum_s\Lambda_s\right]\theta'\begin{bmatrix}0\,0\\0\,0\end{bmatrix}(0|2^{2k}\Omega)\right\}, \tag{48}$$

which is indeed equal to $F(\beta)-F_B(\beta)$.

For the case of supersymmetric heterotic strings (47) and (48) drastically simplify. The bosonic contribution to the free energy is simply

$$F_B(\beta)=\frac{1}{2}\sum_{k=0}^{\infty}F(2^k\beta). \tag{49}$$

The singularity structure of (49) is given by an infinite set Σ of values of β labeled by the integer k running in the sum $\Sigma=\{\beta_k\mid\beta_k=2^{-k}\beta_H\}$, where β_H is the Hagedorn length of the heterotic string. β duality for each term of the sum (49) implies a second set of lengths $\Sigma^*=\{\beta_k^*\mid\beta_k^*=\pi^2/\beta_k\}$. Then we find that for each interval (β_k^*,β_k) at least one of the terms of the sum in (49) diverges. It is also easy to show that this family of intervals overlap, so $F_B(\beta)$ diverges for any $\beta<\beta_H$. For the case of fermions the situation is exactly the same. We then see that duality is a property of the total free energy and, at first sight, results from the mutual cancellation between the divergences of the bosonic and fermionic parts so as to leave only an interval of divergence given by (β_H^*,β_H).

However, looking carefully, duality invariance is already sowed in both contributions. Let us take the limit $\beta\to 0^+$ in $F_{B(F)}(\beta)$. By duality symmetry for each term in the sum over k, $F_{B(F)}(\beta)$ will go to zero, and then we have that no bosonic (fermionic) degree of freedom survives [17].

For the general case, computing the $\beta\to 0^+$ limit of $F_B(\beta)$ and $F_F(\beta)$, we get the paradoxical results

$$F_B(\beta)\sim\frac{\pi^2}{\beta^2}\left(\frac{4}{3}\Lambda\right), \tag{50}$$

$$F_F(\beta)\sim-\frac{\pi^2}{\beta^2}\left(\frac{1}{3}\Lambda\right). \tag{51}$$

The solution of this paradox is that the equivalence between the corresponding analogue models and the modular invariant expressions is broken. A related question is that of the value of β (i.e., the energy scale) at which the intruder fermionic (bosonic) degrees of freedom corrupt the bosonic (fermionic) free energy. We postpone this problem until the next section.

IV. THE SINGULARITIES OF THE FREE ENERGY OF A HETEROTIC STRING

In quantum-field theory the equivalence between the free energy of a quantum field and the contribution to the vacuum energy, the cosmological constant, of the same field with Euclidean time is a very well known fact (cf., e.g., [17] and references therein). In the case of strings [7, 18, 10], this relationship is not direct. Actually, it depends on whether the string can be described by its field content. Until now the answer has not been too precise; in general, the fact that any string model has a Hagedorn temperature has been used to break any relationship at higher temperatures. From the work done in [4, 5] it has been concluded that in some compactifications this equivalence is broken by a different kind of singularities of the partition function.

The strategy is to find the relationship between the modular-invariant free energy gotten by using the coset technique and the cosmological constant of a zero-temperature heterotic theory. Then we will study the analytic behavior of the free energy through the mass formulas of the corresponding zero-temperature theory. After that we will see that, at the Hagedorn temperature, the analogue model presents the same divergent behavior as the modular invariant extension. The breakdown point will be the self-dual radius below which we claim that the equivalence between the modular invariant result and the analogue model does not hold any more.

In what follows we will treat this problem for the general free energy of a heterotic string. To prove that the manifestly modular-invariant free energy corresponds in every case to the cosmological constant of a heterotic string with Euclidean time it is useful to realize that, after some Riemann θ-function gymnastics, the integrand of (31) can be written (we now drop the multiplicative factor $\tau_2^{-(d-2)/2}$)

$$\begin{aligned}\chi(\tau,\bar\tau) = &-\frac{\bar\theta_3^4-\bar\theta_4^4}{\bar\eta^{12}}\left\{z_o\left(\theta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}-\theta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}\right)+z_v\left(\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\theta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}\right)\right\}\\ &-\frac{\bar\theta_3^4+\bar\theta_4^4}{\bar\eta^{12}}\left\{z_o\left(\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\theta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}\right)+z_v\left(\theta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}-\theta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}\right)\right\}\\ &+\frac{\bar\theta_2^4}{\bar\eta^{12}}\left\{z_s\left(\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}-\theta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}\right)+z_c\left(\theta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}+\theta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}\right)\right\}\\ &+\frac{\bar\theta_2^4}{\bar\eta^{12}}\left\{z_s\left(\theta\begin{bmatrix}\frac12 & 0\\ 0 & 0\end{bmatrix}+\theta\begin{bmatrix}\frac12 & \frac12\\ 0 & 0\end{bmatrix}\right)+z_c\left(\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}-\theta\begin{bmatrix}0 & \frac12\\ 0 & 0\end{bmatrix}\right)\right\},\end{aligned} \tag{52}$$

where all the Riemann θ functions are evaluated at $(0|4\Omega)$. With this form of the free energy it is also easy to read back the mass formulas and constraints for each sector in the corresponding theory at zero temperature. The general structure of Eq. (52) can be sketched as $(v,\Gamma_v)+(o,\Gamma_o)+(s,\Gamma_s)+(c,\Gamma_c)$ where v,o,s,c are the conjugacy classes of SO(10) and Γ_i with $i=v,o,s,c$ are sets of vectors (which, in general, do not close under addition) defined in the following way: Corresponding to each of the two terms multiplying the contribution of the conjugacy classes of SO(10) we define two sets $\Gamma_{i,1}$ and $\Gamma_{i,2}$ such that $\Gamma_i=\Gamma_{i,1}\bigcup\Gamma_{i,2}$. Now, for each class, we can characterize these sets as

$$\begin{aligned}\Gamma_{i,1}&=\{(\mathbf{p}_L,\mathbf{p}_R)|\mathbf{p}_{L,R}=[\mathbf{v}_{L,R},p_{L,R}(\beta)]\}\,,\\ \Gamma_{i,2}&=\{(\mathbf{p}_L,\mathbf{p}_R)|\mathbf{p}_{L,R}=[\mathbf{w}_{L,R},p_{L,R}(\beta)+\delta_{L,R}(\beta)]\}\,.\end{aligned} \tag{53}$$

Here the first entry corresponds to the vectors giving the generalized θ functions contained in z_i, which may eventually depend on a radius of compactification, although there are always components that correspond to internal compact dimensions at fixed radii, at least 16 dimensions in the left-moving sector. The second one is a function of β corresponding to the momentum in Euclidean time and $\delta_{L,R}(\beta)$ is a shift.

The mass formulas and constraints for the four bosonic sectors in (52) are

$$\frac14 m_{i,1}^2=N_R^i+N_L+\frac12 p_{R,i+1}^2+\frac12 p_{L,i+1}^2+\frac{\beta^2}{4\pi^2}(2m+1)^2+\frac{\pi^2}{4\beta^2}(2n+1)^2-\frac32, \tag{54}$$

$$N_L-N_R^i+\frac{2m+1}{2}(2n+1)+\frac12 p_{L,i+1}^2-\frac12 p_{R,i+1}^2-\frac12=0, \tag{55}$$

$$\frac14 m_{i,2}^2=N_R^i+N_L+\frac12 p_{R,i+1}^2+\frac12 p_{L,i+1}^2+\frac{\beta^2}{4\pi^2}(2m)^2+\frac{\pi^2}{4\beta^2}(2n)^2-\frac32, \tag{56}$$

$$N_L-N_R^i+2mn+\frac12 p_{L,i+1}^2-\frac12 p_{R,i+1}^2-\frac12=0, \tag{57}$$

where $i=0,1$ [$v\equiv 0(\mathrm{mod}\ 2)$, $o\equiv 1(\mathrm{mod}\ 2)$]; N_L is a positive integer and $N_R^o\in\mathbf{Z}^+$, $N_R^v\in\mathbf{Z}^++1/2$; again $p_{R(L),i}$ denotes the vectors associated with each z_i. For the four fermionic sectors we have

$$\frac14 m_{j,1}^2=N_R^j+N_L+\frac12 p_{R,j}^2+\frac12 p_{L,j}^2+\frac{\beta^2}{4\pi^2}(2m)^2+\frac{\pi^2}{4\beta^2}(2n+1)^2-1, \tag{58}$$

$$N_L-N_R^j+m(2n+1)+\frac12 p_{L,j}^2-\frac12 p_{R,j}^2-1=0, \tag{59}$$

$$\frac{1}{4}m_{j,2}^2 = N_R^j + N_L + \frac{1}{2}p_{R,i}^2 + \frac{1}{2}p_{L,j}^2 + \frac{\beta^2}{4\pi^2}(2m+1)^2 + \frac{\pi^2}{4\beta^2}(2n)^2 - 1, \tag{60}$$

$$N_L - N_R^j + (2m+1)n + \frac{1}{2}p_{L,j}^2 - \frac{1}{2}p_{R,j}^2 - 1 = 0, \tag{61}$$

where now $j = 0, 1$ [$s \equiv 0 (\text{mod } 2)$, $c \equiv 1 (\text{mod } 2)$]; and N_R^j, N_L^j are positive integers. From these formulas one can show that the Hagedorn length is the same for every heterotic string (cf. [21]); it comes from the sector associated with the scalar conjugacy class and z_v with the following set of quantum numbers: $N_L = N_R = 0$, $m = n = 0$ or $m = n = -1$, and $p_{L,v}^2 = p_{R,v}^2 = 0$.

The existence of a vanishing value of the momentum in the compact dimensions associated with the vectorial conjugacy class in the theory at zero-temperature results from the induced U(1) Kaluza-Klein bosons. That these bosons must appear glued to this class is the result of modular invariance because $T(z_v) = z_v$ and then if $(p_{L,v}; p_{R,v})$ is a vector we must have $p_{L,v}^2 - p_{R,v}^2 \equiv 0 (\text{mod } 2)$. Since z_v, in general, receives contributions from vectors $(p_{L,v}; p_{R,v}) = (\tilde{p}_{L,v}, p_{L,v}(R); \hat{p}_{R,v}, p_{R,v}(R))$, where $\tilde{p}$ and $\hat{p}$ are momenta corresponding to fixed-size dimensions (at least 16 dimensions in the left sector), and $p_{L,v}(R)$, $p_{R,v}(R)$ are left and right momenta depending on the value of the radii; it would appear that there is a way of getting $T(z_v) = z_v$ by having $p_{L,v}^2 - p_{R,v}^2 \equiv 1 (\text{mod } 2)$ and $\hat{p}^2 - \tilde{p}^2 \equiv 1 (\text{mod } 2)$. This is impossible because this would imply that in the limit in which the radii of compactification go to infinity z_v goes to zero and then the corresponding theory in the decompactification limit could not be modular invariant because of the absence of the contribution of the vectorial conjugacy class. On the contrary, $T(z_o) = -z_o$ implies that $p_{L,o}^2 - p_{R,o}^2 \equiv 1 (\text{mod } 2)$.

The Hagedorn temperature, which corresponds to the length of the Euclidean time $\beta_H = \pi(\sqrt{2} + 1)$, and its dual $[1/T_H^* = \beta_H^* = \pi(\sqrt{2} - 1)]$ are singularities at which the free energy diverges, respectively, to the left and to the right in a plot $F(\beta)$ vs β.

In the same sector we can also find that for $\{N_L = 1,\ N_R = 0,\ m = 0, n = -1$ or $m = -1, n = 0$ and $p_{L,v}^2 = p_{R,v}^2 = 0\}$ or $\{N_L = N_R = 0,\ m = 0, n = -1$ or $m = -1, n = 0$ and $p_{L,v}^2 = 2,\ p_{R,v}^2 = 0\}$, we have another critical length at $\beta = \pi$, the self-dual point of the free energy. This value of β gives $m^2 = 0$ but such that $m^2 > 0$ for every $\beta \neq \pi$.

Both lengths are generic for every heterotic string, supersymmetric or not, as long as we have Kaluza-Klein U(1) bosons. Of course there may be more critical lengths as the Hagedorn one and their duals that depend on R [8] and consequently are not generic. Furthermore, there is an intermediate case. When there is a vector of $(\text{length})^2 = 0$ in one of the spinorial representations[1] we have a class of heterotic string theories for which two more critical lengths, $\beta = \pi\sqrt{2}$ and its dual $\beta^* = \pi/\sqrt{2}$, can be found independently of the radii and such that $m^2 > 0$ for every $\beta \neq \pi\sqrt{2}, \pi/\sqrt{2}$.

For example, in the first case, in which only Hagedorn and the self-dual point are generic temperatures, we find the nonsupersymmetric $O(16)\times O(16)$ [22, 14] and, in general, any heterotic theory for which z_s and z_c do not receive a contribution of vectors with $(\text{length})^2 = 0$. In particular, the model presented in Sec. II whose free energy is given by (37) does not present these singularities, since the corresponding generalized θ functions associated with z_s and z_c start with powers of q higher than zero [see (35) and (36)]. On the other hand, for the old heterotic string [23] we have the opposite situation because the $E_8\times E_8$ lattice, which is common to all conjugacy classes, do have a vector with $p_L^2 = 0$.

When the term producing the Hagedorn temperature is substracted from the free energy, the remaining would-be renormalized free energy presents a singularity at the self-dual point. The term contributing to the free energy associated with this singularity is

$$\begin{aligned} I(\beta) &= -2\frac{\pi\sqrt{2}}{\beta}\int_1^\infty d\tau_2 \tau_2^{-\frac{d+1}{2}} \exp\left[-\frac{\pi}{2}\tau_2\left(\frac{\beta^2-\beta_0^2}{\beta\beta_0}\right)^2\right] \\ &= \frac{2\pi\sqrt{2}}{\beta}\left[\pi\left(\frac{\beta^2-\beta_0^2}{\beta\beta_0}\right)^2\right]^{\frac{d-1}{2}} \\ &\quad \times \Gamma\left[\frac{1-d}{2}, \frac{\pi}{2}\left(\frac{\beta^2-\beta_0^2}{\beta\beta_0}\right)^2\right], \end{aligned} \tag{62}$$

where $\beta_0 = \pi$. Whenever $d \geq 2$ and even, $I(\beta)$ is finite for any value of β, in particular, at the corresponding singular point, it has a finite jump in the $d-1$ derivative. For d odd it suffers from a logarithmic singularity in the $d-1$ derivative at β_0. When $d = 0, 1$ the contribution of $I(\beta)$ to the free energy is divergent at $\beta = \beta_0$ but finite at any other point. It is worth noticing that $\beta_0^{-1}I(\beta)$ is always invariant under the exchange $\beta \leftrightarrow \beta_0$. This implies that near β_0, $\beta_0^{-1}I(\beta)$ is a function of $|\beta - \beta_0|$.

When the singularities at $\pi\sqrt{2}$ and $\pi/\sqrt{2}$ appear the associated terms contribute to the free energy as in (62) with the opposite sign and a factor 2 multiplying the argument of the exponential function in the integrand. Finally, we should note that all these terms must be multiplied by integral degeneracy factors. For example for the ten-dimensional supersymmetric heterotic string we would have a factor given by $24 + 2 \times 240 = 504$ multiplying $I(\beta)$ with $\beta_0 = \pi$.

[1]This suffices since duality symmetry interchanges both spinorial conjugacy classes and then both chiralities must have a null vector in each of the associated sets of vectors.

V. CONCLUSIONS

Let us go back in time to Ref. [5] where the dependence of the partition function on the compactification scale was studied for a particular class of two-dimensional heterotic strings. There we showed that no critical radius as the Hagedorn one appears. There is, of course, a singularity, but of a quite different kind. Furthermore, it is easy to show that the origin of this soft singularity is the unstability of the vacuum represented as

$$\begin{aligned}-RM(R) &= -R\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}J(\tau)\\ &\qquad\times\sum_{m,n\in\mathbf{Z}}{}' \exp\left(-\frac{2\pi R^2}{\tau_2}|m\tau+n|^2\right)\\ &= -4\pi R\left(1-\frac{1}{2R^2}\right)+4\pi R\left|1-\frac{1}{2R^2}\right| . \end{aligned} \tag{63}$$

Here $J(\tau)=j(\tau)-744$. Below the Planck length scale this way of representing nothingness breaks down. A nonvanishing contribution appears, which corresponds to a massless field in two dimensions with positive Helmholtz free energy. In fact $-48[RM(R)+\frac{4\pi}{R}]$ equals (choosing adequate units to eliminate π's and other constant factors) the partition function in $\mathbf{R}\times S^1$ of the nonsupersymmetric heterotic string described in [24, 11], which has Atkin-Lehner symmetry [11]. The term $-RM(R)$ acts effectively as though it represented the contribution of a ghost field for the net number of bosonic degrees of freedom of the theory in $\mathbf{R}^2$ so as to destroy them. This phenomenon is generic for this kind of trivial compactifications in which the solitonic contribution is a common factor for all the conjugacy classes; the larger d is, the softer the singularity. One would like to know whether this phenomenon at the self-dual radius appears for other kind of compactifications, and if so whether it is related to "heteroticity," i.e., the property of being a hybrid of a bosonic and a fermionic string.

What we have shown in the present work is that by substracting from the free energy a term $G(\beta)$, which gives the infinite background for $\beta_H^* < \beta < \beta_H$ associated with the Hagedorn singularity (whose contribution is negligible when $\beta < \beta_H^*$) we can see a completely analogous phenomenon for a large family of heterotic strings gotten by compactifying nontrivially one of the originally uncompact dimensions. This nontrivial compactification has been dictated by the process of getting the Helmholtz free energy for any heterotic string, supersymmetric or not. Looking at (62) when $d=2$ we see that at the self-dual point (in this case, with $\beta=2\pi R$, $\sqrt{\alpha'/2}$) there is a finite jump in the first derivative with a sign that is only consistent with a loss of degrees of freedom. In particular for the supersymmetric heterotic string the self-dual point corresponds to a cusp pointing to the minus infinity direction in the graph of $F(\beta)-G(\beta)$ vs β. When $d>2$ this cusp softens to give a cup. In the mass formulas the presence of this critical compactification length depends upon the existence of the U(1) Kaluza-Klein bosons associated with the Cartan subalgebra of the gauge group.

Regarding the problem from a thermodynamical point of view we have shown that the theory presented in [5] whose free energy is given by (37) is an example of a theory with a thermal free energy, which is dual and at the same time is a monotonically increasing function of β in the would-be high-temperature phase. The complementary situation can be exemplified by the O(16)×O(16) theory whose cosmological constant is positive [22].

Another related point is that of the associated density of states as a function of the compactification scale. The main issue is to try to get the density of states by inverse Laplace transforming the partition function as a function of R. To do that we need to know the analytic continuation of $Z(R)$ to the complex R plane, but how can we perform the analytic continuation of a real function involving an absolute value? For example, in the case of the Atkin-Lehner symmetric theory, if we try to naively analytically continue $-RM(R)$ to the R complex plane by directly substituting into the solitonic sum the real R variable by a complex R, we would have obtained a complex function having, in addition to two singular points at $\pm\sqrt{\alpha'}$, a dense countable set of singularities located over the imaginary axis (in fact all of the same type). In the same way, if we substitute R by iR into the last part of (63) we get that on the imaginary axis there are only three singular points located at 0 and $\pm i\sqrt{\alpha'}$. The lesson to be learned is that, in general, analytically continuing each term in the series in the integrand is not equivalent to the continuation of the integrated result. May be we can relax this requirement and demand only some kind of procedure to perform the inverse Laplace transform without really writing down the analytical continuation.

Finally it would also be interesting to know how string-field theory [25] might deal with these singularities when one looks at the dependence of the action on the parameters of the background target, in particular, regarding the structure of the possible unitary transformation relating by duality different backgrounds of the same class. Of course, a prerequisite would be having a heterotic superstring field theory.

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