Some computations in background-independent off-shell string theory

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Recently, background-independent open-string field theory has been formally defined in the space of all two-dimensional world-sheet theories. In this paper, to make the construction more concrete, I compute the action for an off-shell tachyon field of a certain simple type. From the computation it emerges that, although the string field action does not coincide with the world-sheet (matter) partition function in general, these functions do coincide on shell. This can be demonstrated in general, as long as matter and ghosts are decoupled.

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I. INTRODUCTION

The beauty of the world-sheet approach to string perturbation theory has led many physicists to hope that "the space of all two-dimensional world-sheet field theories" might be a natural arena for string field theory. This program has been obstructed by (i) the unrenormalizability of the generic world-sheet theory, and other more or less related problems, and (ii) the fact that even given a "space of all two-dimensional theories," one has not known how to formulate a gauge-invariant Lagrangian in that space. In a recent paper [1], the second problem has been resolved in the case of open strings. Dealing with open strings means that we consider two-dimensional Lagrangians of the form

$$L=L_0+L' , \tag{1.1}$$

where L_0 is a bulk action describing a closed-string background and L' is a boundary term describing the coupling to external open strings. L_0 is kept fixed and L' is permitted to vary. For instance, in this paper we will take L_0 to describe the standard closed-string background,

$$L_0=\int_\Sigma d^2\sigma \left[\frac{1}{8\pi}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\eta_{ij}+\frac{1}{2\pi}b^{ij}D_i c_j\right] , \tag{1.2}$$

while L' will be an arbitrary ghost number conserving boundary interaction:

$$L'=\int_{\partial\Sigma}d\theta\, \mathcal{V}(X,b,c) . \tag{1.3}$$

(The notation is standard: Σ is a Riemann surface with metric h, inducing a length element $d\theta$ on the boundary; b and c are the usual antighosts and ghosts; the X^i describe a map to 26-dimensional Minkowski space with Lorentz metric η.) Actually, in the construction in [1] one must introduce a local operator $\mathcal{O}$ of ghost number 1 with

$$\mathcal{V}=b_{-1}\mathcal{O} . \tag{1.4}$$

For the definition of b_{-1} see [1].

The main result of [1] was to describe, given these data, a background-independent, gauge-invariant Lagrangian S on the space of $\mathcal{O}$'s. The construction has many properties that agree with expectations from world-sheet perturbation theory but have previously been hard to understand in the context of a gauge-invariant Lagrangian. For instance, the classical equations of motion derived from S are equivalent to Becchi-Rouet-Stora-Tyutin (BRST) invariance of the world-sheet theory. This is an improved version, which does not assume decoupling of matter and ghosts, of the naive expectation that the equations of motion should assert conformal invariance. (If it is the case that on shell one can always decouple matter and ghosts by a gauge condition on $\mathcal{O}$, then in that gauge the equations of motion derived from S are equivalent to conformal invariance.) Moreover, in expanding around a classical solution, the infinitesimal gauge transformations are generated by the world-sheet BRST operator.

The construction in [1] was, however, purely formal, since it involved correlation functions in the theory (1.1) for arbitrary $\mathcal{V}$, and one certainly must expect ultraviolet divergences. Here is where one faces the fact that we do not know what "the space of two-dimensional field theories" is supposed to be. Nevertheless, the definition of S has the property, noted at the end of [1], that given any concrete family of two-dimensional field theories, the S function can be computed explicitly as a function on the parameter space of that family. The original goal of the present paper was simply to make the ideas of [1] more concrete by computing the S function explicitly for a certain simple family of boundary interactions. This will be accomplished in Sec. II.

From the computation will emerge a simple relation between the S function and the partition function of the matter system. We will explore this relation in Sec. III. The main conclusion is as follows: if matter and ghosts are decoupled, then *on shell,* the S function is equal to the partition function Z of the matter system. This relation has been heuristically expected (and checked in some special cases [2,3]) given the role of world-sheet path integrals in generating effective string interactions.

II. SOME SIMPLE BOUNDARY INTERACTIONS

A. The model

The open-string tachyon corresponds to a boundary interaction of the form

$$\mathcal{V}=T(X)\ . \tag{2.1}$$

For $\mathcal{V}$ of this form, or more generally any $\mathcal{V}$ that depends on X only, not on b and c, it is natural to choose

$$\mathcal{O}=c\mathcal{V}\ , \tag{2.2}$$

with c the component of the ghost field tangent to the boundary. This is the most general situation that we will consider in this paper.

In the present section, to obtain a family of boundary interactions for which everything can be computed explicitly, we will take $T(X)$ to be a quadratic function of the coordinates

$$T(X)=\frac{a}{2\pi}+\sum_{i=1}^{26}\frac{u_i}{8\pi}X_i{}^2\ , \tag{2.3}$$

with parameters a and u_i. The quadratic nature of the boundary interaction ensures that the world-sheet theory is soluble, so that we will be able to evaluate explicitly all the correlation functions that enter in the definition of the background-independent action S.

Before entering into actual calculations, let us make a few general comments about this family of boundary interactions.

(1) Linear terms in the X_i have been omitted, because as long as the u_i are nonzero, linear terms can be absorbed in a shift in the X_i.

(2) Once the u_i are included, it would be unnatural not to include the constant term $a/2\pi$ in the action. This is because under a change in the normal-ordering prescription used in defining the quantum operator $\sum_{i=1}^{26}u_iX_i{}^2$, this operator will be shifted by a constant. With the tachyon interaction (2.3) is associated a natural 27-parameter family of quantum theories, but the parametrization of the family by 26 u's and one a is not completely natural, having the ambiguity just cited. The background-independent action S is well defined as a function on the 27-dimensional space of quantum theories of this type; identifying it as a function of a and the u's requires a specific normal-ordering recipe.

(3) The world-sheet action is bounded below only if the u_i are positive. We should therefore expect the space-time action S to have singularities for negative u_i.

(4) For $u_i=0$, the theory is invariant under translations of the X_i. The strings propagate in an infinite volume, and the natural object to calculate is the action per unit volume. Taking $u_i>0$ gives a potential energy for the zero mode of the string, so oscillations of the string are limited to a finite volume. If the u_i are positive and small, the action is $\lesssim 1$ for $X_i\lesssim 1/\sqrt{u_i}$. One therefore should expect that

$$S\sim\frac{w}{\prod_i\sqrt{u_i}}\quad\text{for } u_i\to 0\ , \tag{2.4}$$

where w is a constant proportional to the action per unit volume at $u_i=0$.

(5) The theory with $a=u_i=0$ is conformally invariant; this case corresponds to free boundary conditions, $n^\alpha\partial_\alpha X^i=0$ on $\partial\Sigma$ (n^α is the normal vector to the boundary of Σ). There is also a conformally invariant theory at $u_i=\infty$ (with boundary conditions $X=0$ on $\partial\Sigma$). It would be nice to be able to compare the values of the action of two different classical solutions, but the present example is not quite suitable for this, because the theory at $u_i=\infty$ has an action, while the theory at $u_i=0$ has an action per unit volume, as explained in the last paragraph.

In Sec. II B below, we will evaluate basic properties of the model, determining the essential Green's functions and correlation functions and the partition function. Then in Sec. II C, we will compute the action function S for this family of quantum field theories.

B. First properties

Let us focus on a single scalar field X described by the Lagrangian

$$L=\frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X\partial_\beta X+\frac{u}{8\pi}\int_{\partial\Sigma}d\theta\, X^2\ . \tag{2.5}$$

The boundary condition derived by varying this action is

$$n^\alpha\partial_\alpha X+uX=0\quad\text{on }\partial\Sigma\ . \tag{2.6}$$

In keeping with the formulation of [1], we wish to consider this theory on a disc Σ with a rotationally invariant metric. We may as well take this to be the flat metric

$$ds^2=d\sigma_1^2+d\sigma_2^2,\quad \sigma_1^2+\sigma_2^2\le 1\ . \tag{2.7}$$

We also set $z=\sigma_1+i\sigma_2$.

The Green's function of the theory should obey

$$-\frac{1}{2\pi}\partial_z\partial_{\bar z}G(z,w)=\delta^2(z,w)\ , \tag{2.8}$$

along with the boundary condition (2.6). These requirements determine the Green's function to be

$$G(z,w)=-\ln|z-w|^2-\ln|1-z\bar w|^2+\frac{2}{u}-2u\sum_{k=1}^{\infty}\frac{q}{k(k+u)}[(z\bar w)^k+(\bar z w)^k]\ . \tag{2.9}$$

The divergence at $u=0$ reflects the fact that there is a zero mode (the constant mode of X) for $u=0$; the physics behind this was discussed as point (4) at the end of Sec. II A.

We now need to define the quantum operator $X^2(z)$, for $z\in\partial\Sigma$. As $\partial\Sigma$ corresponds to $|z|=1$, we can write $z=e^{i\theta}$, and we write $X^2(\theta)$ for $X^2(z)$. We define

$$X^2(\theta)=\lim_{\epsilon\to 0}[X(\theta)X(\theta+\epsilon)-f(\epsilon)]\ , \tag{2.10}$$

where $f(\epsilon)$ is a function of ϵ, but not u, chosen so that the limit exists. These conditions are obeyed by

$$f(\epsilon)=-2\ln|1-e^{i\epsilon}|^2\ . \tag{2.11}$$

f is uniquely determined up to an additive constant [which is the normal ordering constant mentioned in point (2) at the end of the last subsection] plus terms that vanish for $\epsilon \to 0$. The requirement that f is independent of u ensures that when we compute u derivatives (to evaluate the partition function and the S function) we need not worry about terms coming from the u dependence of the definition of $X^2(\theta)$.

With the above choice of f, the expectation value of $X^2(\theta)$ is

$$\langle X^2(\theta)\rangle = \frac{2}{u} - 4u \sum_{k=1}^{\infty} \frac{1}{k(k+u)} . \tag{2.12}$$

(In the present subsection only, the angular brackets refer to normalized correlation functions.)

Now we can calculate the partition function on the disc of the theory of one scalar field with Lagrangian (2.5). We will call this partition function $Z_1(u)$. From

$$Z_1 = \int DX \exp(-L) , \tag{2.13}$$

and the explicit form of L, we have

$$\begin{aligned} \frac{d}{du} \ln Z_1 &= -\frac{1}{8\pi} \int_0^{2\pi} d\theta \langle X^2(\theta)\rangle \\ &= -\frac{1}{2u} + \sum_{k=1}^{\infty} \frac{u}{k(k+u)} . \end{aligned} \tag{2.14}$$

The Euler Γ function $\Gamma(u)$ obeys ([4], pp. 198–200)

$$\frac{d}{du} \ln \Gamma = -\frac{1}{u} + \sum_{k=1}^{\infty} \frac{u}{k(k+u)} - \gamma , \tag{2.15}$$

with γ being Euler's constant, so

$$\frac{d}{du} \ln Z_1 = \frac{d}{du} \ln \Gamma + \gamma + \frac{1}{2u} . \tag{2.16}$$

Hence (up to an arbitrary multiplicative constant, which can be absorbed by adding a constant to the a parameter in the boundary interaction)

$$Z_1(u) = \sqrt{u} \exp(\gamma u) \Gamma(u) . \tag{2.17}$$

Note the expected small u behavior $Z_1(u) \sim 1/\sqrt{u}$ for $u \to 0$ (where Γ has a simple pole), and the anticipated singularities for negative u (from the other poles of Γ).

For a slightly more general model with several scalar fields X_i and boundary interaction (2.3), the partition function on the disc is therefore

$$Z(u_i;a) = e^{-a} \prod_i Z_1(u_i) . \tag{2.18}$$

The simple a dependence comes from the fact that the a term is just an additive constant in the Lagrangian.

An identity

Let us now pause to evaluate a correlation function that is needed later. First of all, for boundary points $z = e^{i\theta}$, $w = e^{i\theta'}$ the propagator [which we will write as $G(\theta,\theta')$] is

$$\begin{aligned} G(\theta,\theta') = &-2\ln(1-e^{i(\theta-\theta')}) - 2\ln(1-e^{-i(\theta-\theta')}) + \frac{2}{u} \\ &-2u \sum_{k=1}^{\infty} \frac{1}{k(k+u)} [e^{ik(\theta-\theta')} + e^{-ik(\theta-\theta')}] . \end{aligned} \tag{2.19}$$

Expanding $\ln(1-e^{\pm i(\theta-\theta')})$ in a power series and collecting terms, one finds

$$G(\theta,\theta') = 2 \sum_{k\in\mathbb{Z}} \frac{1}{|k|+u} \exp[ik(\theta-\theta')] . \tag{2.20}$$

Now we want to evaluate

$$W = \int_0^{2\pi} \frac{d\theta\, d\theta'}{(2\pi)^2} \cos(\theta-\theta') \langle X^2(\theta) X^2(\theta')\rangle . \tag{2.21}$$

Using the explicit form of the Green's function and doing the angular integrations, this becomes

$$\begin{aligned} W &= 4 \int_0^{2\pi} \frac{d\theta}{2\pi} (e^{i\theta} + e^{-i\theta}) \sum_{k\in\mathbb{Z}} \frac{e^{ik\theta}}{|k|+u} \sum_{k'\in\mathbb{Z}} \frac{e^{ik'\theta}}{|k'|+u} \\ &= 4 \sum_{k\in\mathbb{Z}} \frac{1}{|k|+u} \left[\frac{1}{|k+1|+u} + \frac{1}{|1-k|+u} \right] = 8 \sum_{k\in\mathbb{Z}} \frac{1}{|k|+u} \frac{1}{|k+1|+u} \\ &= 16 \sum_{k\geq 0} \frac{1}{|k|+u} \frac{1}{|k+1|+u} = 16 \sum_{k\geq 0} \left[\frac{1}{k+u} - \frac{1}{k+1+u} \right] = \frac{16}{u} . \end{aligned} \tag{2.22}$$

This is the desired identity.

C. Evaluation of the action

The definition of the background-independent action S in [1] was as follows. Suppose that the boundary interaction is $\mathcal{V} = b_{-1}\mathcal{O}$ where $\mathcal{O}$ has ghost number 1. (S depends on $\mathcal{O}$, not just on $\mathcal{V}$). Let O_i be a basis of ghost number 1 operators, so that $\mathcal{O}$ has an expansion $\mathcal{O} = \sum_i w^i O_i$. Then S is defined[1] in terms of correlation

[1]With a different normalization from that in [1].

functions on the disc by

$$\frac{\partial S}{\partial w^i}=\frac{1}{2}\int_0^{2\pi}d\theta\,d\theta'\,\langle O_i(\theta)\{Q,\mathcal{O}\}(\theta')\rangle\ , \tag{2.23}$$

or equivalently

$$\begin{aligned} dS&=\tfrac{1}{2}\sum_i dw^i\int_0^{2\pi}d\theta\,d\theta'\langle O_i(\theta)\{Q,\mathcal{O}\}(\theta')\rangle \\ &=\tfrac{1}{2}\int_0^{2\pi}d\theta\,d\theta'\langle d\mathcal{O}(\theta)\{Q,\mathcal{O}\}(\theta')\rangle\ . \end{aligned} \tag{2.24}$$

Here and henceforth the angular brackets refer to unnormalized correlation functions (one is not to divide by the partition function).

Obviously, (2.24) determines S up to an additive constant. The fact that S exists is, however, nontrivial. It depends on the fact that the one-form on the right-hand side of (2.24) is closed. In the example we are considering, the proof of this depends on the nontrivial formula (2.22). The general proof depends on Ward identities discussed in [1].

In our case, we have

$$\mathcal{V}=\frac{a}{2\pi}+\frac{1}{8\pi}\sum_i u_iX_i^2\ , \tag{2.25}$$

and $\mathcal{O}=c\mathcal{V}$. In general, for any tachyon field $T(X)$, one has

$$\{Q,cT(X)\}=cc'\left[2\sum_i\frac{\partial^2}{\partial X_i^2}+1\right]T(X) \tag{2.26}$$

(with c' the tangential derivative of c along the boundary), so, in our case,

$$\{Q,\mathcal{O}\}=\frac{1}{8\pi}cc'\left[\sum_i u_i(X_i^2+4)+4a\right]\ . \tag{2.27}$$

The ghost correlation function that we need is therefore $\langle c(\theta)cc'(\theta')\rangle$. Using the fact that the three ghost zero modes on the disc are 1, $e^{i\theta}$, and $e^{-i\theta}$, and normalizing the ghost measure so that $\langle cc'c''(\theta)\rangle=1$, we get

$$\langle c(\theta)cc'(\theta')\rangle=2[\cos(\theta-\theta')-1]\ . \tag{2.28}$$

Consequently, the equation defining S boils down to

$$\begin{aligned} dS=\frac{1}{2}\int_0^{2\pi}&d\theta\,d\theta'\,2[\cos(\theta-\theta')-1] \\ &\times\left\langle\left[\frac{da}{2\pi}+\sum_i\frac{du_iX_i^2(\theta)}{8\pi}\right]\right. \\ &\qquad\times\frac{1}{8\pi}\left.\left[\sum_j u_j[X_j^2(\theta')+4]+4a\right]\right\rangle\ . \end{aligned} \tag{2.29}$$

The correlation functions that arise here can all be evaluated, using

$$\begin{aligned} &\int_0^{2\pi}d\theta\langle X_i^2(\theta)\rangle=-8\pi\frac{\partial Z}{\partial u_i}\ , \\ &\int_0^{2\pi}d\theta\,d\theta'\langle X_i^2(\theta)X_j^2(\theta')\rangle=(8\pi)^2\frac{\partial^2 Z}{\partial u_i\partial u_j}\ , \end{aligned} \tag{2.30}$$

and also (2.22). The result can be written

$$dS=d\left[\sum_i u_iZ-\sum_j u_j\frac{\partial}{\partial u_j}Z+(1+a)Z\right]\ . \tag{2.31}$$

This can be solved by

$$S=\left[\sum_i u_i-\sum_j u_j\frac{\partial}{\partial u_j}+(1+a)\right]Z\ . \tag{2.32}$$

Alternatively, using the fact that the a dependence of Z is an overall factor of e^{-a}, this can be written

$$S=\left[-\sum_j u_j\frac{\partial}{\partial u_j}-\left[a+\sum_j u_j\right]\frac{\partial}{\partial a}+1\right]Z\ . \tag{2.33}$$

This is our final result for the space-time action for this particular family of boundary interactions.

III. RELATION BETWEEN THE ACTION AND THE PARTITION FUNCTION

Because of the way that world-sheet path integrals can be used to compute string interactions, some physicists have suspected (for example, see [2,3]) that the space-time action S in string theory might simply equal the world-sheet partition function on a disc (for open strings) or a sphere (for closed strings). Actually, since ghost zero modes make the usual partition function vanish, the idea is really that, as long as matter and ghosts are decoupled, S would equal the matter partition function, which I will call Z. No one has ever proposed a generalization of this conjecture that makes sense when matter and ghosts are not decoupled.

A little reflection shows that the conjecture is more plausible on shell than off shell. The S function is supposed to be gauge invariant, and on shell Z possesses a well-known gauge invariance: it is invariant under adding to the world-sheet Lagrangian terms of the form $\{Q,\alpha\}$ (where α must obey the severe restriction that $\{Q,\alpha\}$ is independent of the ghosts, as Z is the matter partition function only). There has never been any indication of an off-shell gauge invariance of Z.

If we look back to the final result (2.32) or (2.33) of the last subsection, it is clear that (in this particular approach to background-independent string theory) it is not true in general that $S=Z$. However, these two functions are certainly closely related. In fact, using the formula $Z=\exp(-a)\prod_i Z_1(u_i)$, one finds from (2.32) that

$$\frac{\partial S}{\partial a}=Z-S\ . \tag{3.1}$$

Consequently, on shell or in general as long as the a equation of motion is obeyed, $Z=S$. The purpose of the present section is to show that on shell, $Z=S$ in general, and to get some information about the relation between Z and S off shell.

Let $\mathcal{V}$ be a general boundary interaction constructed from matter fields, and take $\mathcal{O}=c\mathcal{V}$. Let $c^{(n)}=d^nc/d\theta^n$, the nth derivative of c along the boundary. In general, $\{Q,\mathcal{O}\}=\sum_{n=1}^{\infty}cc^{(n)}F_n$, where F_n are some matter operators. (This follows from the fact that $\{Q,c\}=cc^{(1)}$, while $\{Q,\mathcal{V}\}$, being an operator of ghost number 1 without antighosts, can be expanded as a sum of expressions of the type $c^{(n)}F_n$.) Consequently, in evaluating (2.24), the ghost correlation functions that we need are of the form $\langle c(\theta)cc^{(n)}(\theta')\rangle$ for various n. In view of the form of the ghost zero modes on the disc, these correlation functions are all linear combinations of 1, $\cos(\theta-\theta')$, and $\sin(\theta-\theta')$. (2.24) can hence be written

$$dS=\int_0^{2\pi} d\theta\, d\theta' \langle d\mathcal{V}(\theta)[A(\theta')+\cos(\theta-\theta')B(\theta') +\sin(\theta-\theta')C(\theta')]\rangle \ , \tag{3.2}$$

with A, B, and C being suitable linear combinations of the F_n.

To proceed further, we will use the following trick [which will enable us to avoid to have to explicitly find the generalization of the key formula (2.22)]. Suppose that the matter system consists of two decoupled subsystems. Let Z_1 and Z_2 be the partition functions of the two subsystems, so the combined matter partition function is $Z=Z_1Z_2$. Let O_i be a basis of local operators for the first system, and let $\tilde{O}_j$ be an analogous basis for the second system. The boundary interaction is then $\mathcal{V}=\sum_i x^i O_i+\sum_j y^j O_j$, with x^i and y^j being coupling constants of the first and second systems, respectively. So

$$d\mathcal{V}=\sum_i dx^i O_i+\sum_j dy^j \tilde{O}_j \ . \tag{3.3}$$

We can expand A in terms of the O's and $\tilde{O}$'s with some unknown coefficients:

$$A=-\sum_i V^i_{(1)}(x)O_i-\sum_j V^j_{(2)}(y)\tilde{O}_j \ . \tag{3.4}$$

B and C can be similarly expanded. (3.2) can therefore be written out in terms of two-point functions of O's and $\tilde{O}$'s. The terms in (3.2) involving $\langle O_i\tilde{O}_j\rangle$ can be very simply evaluated to give

$$dZ_1\, V^k_{(2)}\frac{\partial}{\partial y^k}Z_2+V^i_{(1)}\frac{\partial}{\partial x^i}Z_1\, dZ_2 \ . \tag{3.5}$$

The $\langle O_iO_j\rangle$ terms contribute a one-form of the general form

$$\sum_i dx^i a_i(x)\, Z_2 \ , \tag{3.6}$$

and the $\langle \tilde{O}_i\tilde{O}_j\rangle$ terms contribute a one-form that can be written

$$Z_1\sum_j dy^j \tilde{a}_j(y) \ , \tag{3.7}$$

with unknown functions a_i and $\tilde{a}_j$. So

$$dS=dZ_1\, V^k_{(2)}\frac{\partial Z_2}{\partial y^k}+V^i_{(1)}\frac{\partial Z_1}{\partial x^i}dZ_2 +\sum_i dx^i a_i\, Z_2+Z_1\sum_j dy^j\tilde{a}_j \ . \tag{3.8}$$

These functions can be related using the fact that $d^2S=0$. Calculating the exterior derivative of the one-form on the right-hand side of (3.8), and setting to zero the coefficient of $dx^i dy^j$ gives

$$0=dZ_1\left[dy^i\tilde{a}_i-d\left[V^k\frac{\partial Z_2}{\partial y^k}\right]\right] +\left[d\left[V^i_{(1)}\frac{\partial Z_1}{\partial x^i}\right]-dx^i a_i\right]dZ_2 \ . \tag{3.9}$$

Using the fact that some of the quantities entering here depend only on the x's and some depend only on the y's, this implies that

$$\begin{aligned} d\left[V^i_{(1)}\frac{\partial Z_1}{\partial x^i}\right]-a_i\,dx^i&=-g\,dZ_1 \ ,\\ dy^i\tilde{a}_i-d\left[V^j_{(2)}\frac{\partial Z_2}{\partial y^j}\right]&=g\,dZ_2 \end{aligned} \tag{3.10}$$

for some constant g. Using these formulas to express the a_i and $\tilde{a}_j$ in terms of the Z's and V's, (3.8) can be rewritten

$$dS=d\left[\left[V^i_{(1)}\frac{\partial}{\partial x^i}+V^j_{(2)}\frac{\partial}{\partial y^j}+g\right]Z_1Z_2\right] \ . \tag{3.11}$$

And so

$$S=\left[V^i_{(1)}\frac{\partial}{\partial x^i}+V^j_{(2)}\frac{\partial}{\partial y^j}+g\right]Z_1Z_2 \ . \tag{3.12}$$

So far, we have considered "matter" to consist of two decoupled systems, but this restriction is unnecessary. To any matter system of interest one can always add an auxiliary decoupled system and carry out the above analysis. Then the auxiliary system can be suppressed by setting its couplings to a fixed value. The conclusion is that in general there is a vector field V on the space of world-sheet theories, and a constant g, such that

$$S=\left[V^k\frac{\partial}{\partial x^k}+g\right]Z \ . \tag{3.13}$$

Moreover, V vanishes for classical solutions, since it was constructed from A and so ultimately from $\{Q,\mathcal{O}\}$. Therefore, on shell S is a constant multiple of Z, as promised. Our earlier formula (2.33) serves to illustrate (3.13) for some particular boundary interactions.

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