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Exact integrability of strings in D-dimensional de Sitter spacetime

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We show the complete integrability of the string propagation in D-dimensional de Sitter spacetime. We find that the string equations of motion, which correspond to a noncompact $O(D,1)$-symmetric σ model, plus the string constraints, are equivalent to a generalized sinh-Gordon equation. In $D=2$ this is the Liouville equation, in $D=3$ this is the standard sinh-Gordon equation, and in $D=4$ this equation is related to the B_2 Toda model. We show that the presence of instability is a general exact feature of strings in de Sitter space, as a direct consequence of the strong instability of the generalized sinh-Gordon Hamiltonian (which is unbounded from below), irrespective of any approximative scheme. We exhibit Bäcklund transformations for this generalized sinh-Gordon equation, which relate expanding and shrinking string solutions. We find all classical solutions in $D=2$ and physically analyze them. In $D=3$ and $D=4$, we find the asymptotic behaviors of the solutions in the instability regime. The exact solutions exhibit asymptotically all the characteristic features of string instability: namely, the logarithmic dependence of the cosmic time u on the world sheet time τ for $u \to \pm\infty$, the stretching (or the shrinking) of the proper string size, and the proportionality between τ and the conformal time.

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I. INTRODUCTION AND RESULTS

In Ref. [1] a program for studying the classical and quantum string dynamics in curved spacetimes was started.

Since the equations of motion and constraints for strings in curved spacetimes are highly nonlinear (and, in general, nonexactly solvable), we proposed a method (the "strong-field expansion") to study systematically (and approximately) the solutions of the string equations and constraints in the strong-curvature regime, and to find the scattering matrix, mass spectrum, critical dimension, vertex operators, and particle transmutations [1–3]. (For strings propagating on gravitational shock-wave and plane-wave backgrounds, see Refs. [4–7], and for strings propagating on cosmic-string spacetimes, see Ref. [8].)

Strings in D-dimensional de Sitter spacetime were first studied with the method mentioned above [1]. One of the results was that for a large enough Hubble constant, the frequency of the lower string modes, i.e., those with $|n| < \alpha' m H$ (α' being the string tension and m the string mass), becomes imaginary. This was further analyzed [9] as the onset of a physical instability, in which the proper string size starts to grow (precisely like the expansion factor of the de Sitter universe). The perturbative expansion of the solutions around the center of mass of the string is a suitable method for describing the oscillatory or stable behavior of the string. This expansion holds for large world-sheet time τ (asymptotically, for large de Sitter radius, τ being proportional to the cosmic time), but it does not hold for describing the unstable regime, where the proper string amplitudes grow exponentially. In order to describe the highly unstable regime, another approximative scheme, the "small-τ expansion," was then developed, and asymptotic string solutions nonoscillating in time were found [10].

String behavior oscillatory (stable) in time, as well as behavior nonoscillatory (unstable) in time, was found for the large radius of the de Sitter universe. Behavior nonoscillatory in time was also found, in the small-τ expansion, for a small radius R of the Universe [10]. Both large- and small-R behaviors appeared related somehow by the mapping $R \to R^{-1}$.

In order to go further in the understanding of the string dynamics in de Sitter spacetime, we give in this paper a different formulation of this problem and search for its *exact* solvability.

The string equations of motion in curved spacetime are generalized nonlinear σ models. de Sitter spacetime is maximally symmetric; the string equations in D-dimensional de Sitter spacetime correspond to a noncompact $O(D,1)$-symmetric σ model in two dimensions. This model is integrable.

In addition, the two-dimensional (world-sheet) energy-momentum tensor is required to vanish by the constraints. For our purposes here, it is convenient to consider de Sitter spacetime as a D-dimensional hyperboloid embedded in $(D+1)$-dimensional flat Minkowski spacetime of coordinates $(q^0, q^1, \ldots, q^D)$. The complete de Sitter manifold is the hyperboloid

$$-(q^0)^2 + \sum (q^i)^2 = 1 \ .$$

The string system of equations can be then simplified by choosing an appropriate basis for the string coordinates in $(D+1)$-dimensional flat embedding space. The construction of this basis is analogous to the reduction of the O(N)-symmetric nonlinear σ model [12]. We find that the string system of equations is equivalent to a generalized sinh-Gordon equation

$$\partial_\eta \partial_\xi \alpha(\xi,\eta) - e^{\alpha(\xi,\eta)} + e^{-\alpha(\xi,\eta)} \sum_{i=4}^{D+1} u_i v_i = 0 , \qquad (1.1)$$

where

$$\xi = (\sigma+\tau)/2 , \quad \eta = (\sigma-\tau)/2 ,$$

and the function $\alpha(\xi,\eta)$ is defined by the scalar product

$$e^{\alpha(\xi,\eta)} = -\partial_\xi q \cdot \partial_\eta q . \qquad (1.2)$$

The vector fields u_i and v_i take into account embedding dimensions beyond 3 and relate to $\partial_\xi^2 q$ and $\partial_\eta^2 q$, respectively. In $D=2$, we have $u=v=0$, and Eq. (1.1) reduces to the Liouville equation

$$\partial_\eta \partial_\xi \alpha - e^{\alpha(\xi,\eta)} = 0 . \qquad (1.3)$$

In $D=3$, we find $u=u(\xi)$, $v=v(\eta)$, and then, using the reparametrization invariance on the world sheet, we change the variables

$$\frac{d\xi'}{d\xi} = \sqrt{u(\xi)} , \quad \frac{d\eta'}{d\eta} = \sqrt{v(\eta)} ,$$

such that

$$\alpha(\xi,\eta) = \hat{\alpha}(\xi',\eta') + \tfrac{1}{2}\ln[u(\xi)v(\eta)] . \qquad (1.4)$$

Then Eq. (1.1) reduces to the sinh-Gordon equation

$$\partial_{\eta'} \partial_{\xi'} \hat{\alpha} - e^{\hat{\alpha}(\xi',\eta')} + e^{-\hat{\alpha}(\xi',\eta')} = 0 . \qquad (1.5)$$

In $D=4$, we find that the scalar product $u_i \cdot v_i = \cos\beta$ is determined by the equation

$$\partial_{\eta'} \partial_{\xi'} \beta - e^{-\hat{\alpha}} \sin\beta = 0 , \qquad (1.6)$$

and Eq. (1.1) reduces to

$$\partial_{\eta'} \partial_{\xi'} \hat{\alpha} - e^{\hat{\alpha}} + e^{-\hat{\alpha}} \cos\beta = 0 , \qquad (1.7)$$

as was found in Ref. [11].

The result that the string propagation in D-dimensional de Sitter spacetime follows a generalized sinh-Gordon equation (in particular, in $D=3$, this is just the standard sinh-Gordon model) shows that the presence of instability is a *general exact* feature of string propagation in de Sitter spacetime, irrespective of any approximation scheme or any particular solution. The strong attractive potential corresponding to the generalized sinh-Gordon equation is unbounded from below, and this indicates that the string time evolution tends to the absolute minima at $\alpha=-\infty$ and $+\infty$. [This unstable behavior is explicitly exhibited by the form of the solutions—see Eqs. (5.13), (5.14), (5.17), and (5.20), for instance.] The function $\exp[\alpha(\sigma,\tau)]$ is a measure of the proper length of the string. The invariant length of the string is given by

$$dS^2 = \frac{1}{2H^2} e^{\alpha(\sigma,\tau)} (d\sigma^2 - d\tau^2) ,$$

and it grows infinitely when $\alpha \to +\infty$. Similarly, when $\alpha \to -\infty$, the string collapses to a point. This infinite stretching and shrinking is a typical feature of string instabilities, here appearing as a direct consequence of the generalized sinh-Gordon equation, without the need of searching for explicit solutions.

The string equations of motion and constraints enjoy an exact symmetry transformation which is defined by a first-order differential equation in (σ,τ) and whose compatibility condition yields the equations of motion. This is a Bäcklund transformation which relates solutions of a expanding string metric of radius R into solutions of the contracting or "dual" string metric of radius $\tilde{R}=R^{-1}$. We relate the solutions α and $\tilde{\alpha}$ of the generalized sinh-Gordon equations through this Bäcklund transformation.

We also analyze solutions of this problem. First, we solve the $D=2$ case. In this case, we find *all* the solutions. The solution corresponding to the center of mass of the string (geodesic motion) appears in a separate sector from the solutions describing the true string properties. In two dimensions there are strings wound around the de Sitter universe and evolving with it. These solutions depend on two arbitrary functions which just reflect the conformal invariance on the world sheet. (There are no further arbitrary functions since there are no transverse degrees of freedom in $D=2$.) Here $\tau \in [0,\pi]$ and a half of the string evolution $(0<\tau<\pi/2)$ corresponds to the expansion time $(0<u<\infty)$ of the de Sitter universe, u being the cosmic time. [Similarly, $(\pi/2<\tau<\pi)$ corresponds to the contraction phase $(-\infty<u<0)$.] (See Fig. 1.) The string can wind n times around de Sitter space (here a circle), n being an integer. In such a case, the string-evolution period is reduced to $\Delta\tau=\pi/2n$ (instead of being $\pi/2$); i.e., in τ, the string expansion (and contraction) is n times faster.

Asymptotically, for $u \to \pm\infty$, the cosmic time u depends logarithmically on τ. In the context of cosmologi-

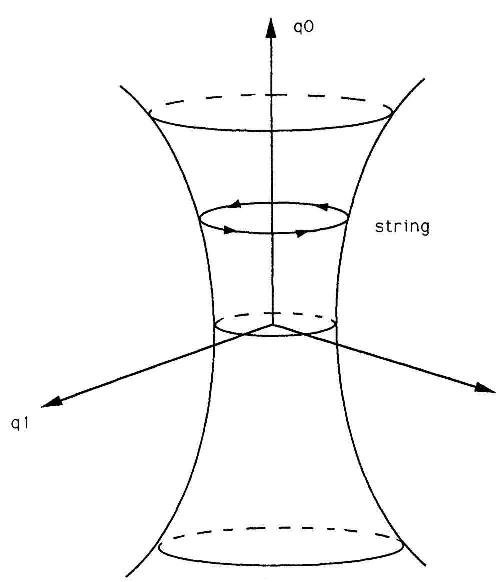

FIG. 1. One-sheet hyperboloid represents the (1+1)-dimensional de Sitter spacetime embedded in a three-dimensional space. The solid circle represents a string solution A [Eq. (3.16)] at a given time.

cal backgrounds, this logarithmic behavior is typical of strings in the de Sitter universe. In these asymptotic regions, the proper length of the string stretches infinitely; i.e., the conformal factor $\exp[\alpha(\sigma,\tau)]$ blows up.

This is the same unstable behavior found in inflationary backgrounds [10], as well as for strings falling into spacetime singularities [7]. In such asymptotic regimes, τ is proportional to the conformal time, another relation typical of strings in the presence of strong gravitational fields or near spacetime singularities: In the asymptotic regimes where the string stretches or shrinks indefinitely (and is thus unstable), the string evolution is governed by the conformal time.

The solution corresponding to the center of mass is the trajectory of a massless particle. Transverse dimensions are absent in $D=2$, and then only massless states appear.

We also analyze solutions for $D=3$ and 4. We find that, for $\tau\to 0$,

$$\alpha(\sigma,\tau)=\pm 2\ln\tau\to\mp\infty\ .$$

This behavior reflects the string instability in de Sitter space, as a consequence of the unboundness of the sinh-Gordon potential.

The study of exact string solutions in a de Sitter universe including higher dimensions will be published elsewhere [13].

This paper is organized as follows. In Sec. II, we formulate the problem and find the reduction of the $O(D,1)$ σ model and constraints to the generalized sinh-Gordon equation in D dimensions. In Sec. III we solve completely the $D=2$ case and analyze the physical meaning of the solutions. In Secs. IV and V we study the $D=3$ and 4 cases, respectively, and find the explicit behavior of the solutions related to the unstable character of the generalized sinh-Gordon Hamiltonian.

II. STRINGS IN *D*-DIMENSIONAL DE SITTER SPACETIME AND A GENERALIZED sinh-GORDON EQUATION

Let us consider D-dimensional de Sitter spacetime with a metric given by

$$dS^2=-dt_0^2+e^{2Ht_0}\sum_{i=1}^{D-1}(dX^i)^2\ . \tag{2.1}$$

Here t_0 is the so-called cosmic time. In terms of the conformal time η,

$$\eta=-\frac{1}{H}e^{-Ht_0}\ ,\quad -\infty<\eta\le 0\ , \tag{2.2}$$

the line element becomes

$$dS^2=\frac{1}{H^2\eta^2}\left[-d\eta^2+\sum_{i=1}^{D-1}(dX^i)^2\right]\ . \tag{2.3}$$

The de Sitter spacetime can be considered as a D-dimensional hyperboloid embedded in a $(D+1)$-dimensional flat Minkowski spacetime with coordinates $(q^0,\ldots,q^D)$:

$$dS^2=\frac{1}{H^2}\left[-(dq^0)^2+\sum_{i=1}^{D}(dq^i)^2\right]\ , \tag{2.4}$$

where

$$\begin{aligned} q^0&=\sinh Ht_0+\frac{H^2}{2}e^{Ht_0}\sum_{i=1}^{D-1}(X^i)^2\ ,\\ q^1&=\cosh Ht_0-\frac{H^2}{2}e^{Ht_0}\sum_{i=1}^{D-1}(X^i)^2\ ,\\ q_{i+1}&=He^{Ht_0}X^i\ ,\quad i=1,\ldots,D-1\ ,\\ &\qquad -\infty<t_0\ ,\quad x^i<+\infty\ . \end{aligned} \tag{2.5}$$

The complete de Sitter manifold is the hyperboloid

$$-(q^0)^2+\sum_{i=1}^{D}(q^i)^2=1\ . \tag{2.6}$$

The coordinates (t_0,x^i) and (η,x^i) cover only half of the de Sitter manifold $q^0+q^1>0$. Further systems of coordinates, which will be used below, are defined by

$$\begin{aligned} q^0&=\sinh u\ ,\quad -\infty\le u\le\infty\ ,\\ q_i&=\hat{q}_i\cosh u\ ,\quad 1\le i\le D\ , \end{aligned} \tag{2.7}$$

where q belongs to the D-dimensional unit sphere

$$\sum_{i=1}^{D}\hat{q}_i^2=1\ .$$

The metric in these coordinates takes the form

$$dS^2=\frac{1}{H^2}[-du^2+(d\hat{q})^2\cosh^2 u]\ . \tag{2.8}$$

We will consider a string propagating in this D-dimensional spacetime. In the conformal gauge, the string action is given by

$$S=\frac{1}{2\pi\alpha'}\int[\partial_\mu q\cdot\partial^\mu q+\lambda(\sigma,\tau)(q\cdot q-1)]d\sigma\,d\tau\ . \tag{2.9}$$

Here we use the scalar product

$$a\cdot b=-a_0b_0+\sum_{i=1}^{D}a_ib_i\ . \tag{2.10}$$

$\lambda(\sigma,\tau)$ is a Lagrange multiplier that enforces the constraint Eq. (2.6), and (σ,τ) parametrizes the string world sheet, as usual. Extremizing the action Eq. (2.9) and eliminating the Lagrange multiplier, we find the equations of motion,

$$q_{\xi\eta}+(q_\xi\cdot q_\eta)q=0\ , \tag{2.11a}$$

with

$$q\cdot q=1\ , \tag{2.11b}$$

where

$$\xi=(\sigma+\tau)/2\ ,\quad \eta=(\sigma-\tau)/2\ ,$$

$$q_\xi=\partial_\xi q\ ,\quad q_\eta=\partial_\eta q\ ,$$

and so on. The string constraints on the world sheet are

$$T_{\xi\xi}=q_\xi^2\simeq 0\ ,$$
$$T_{\eta\eta}=q_\eta^2\simeq 0\ , \quad (2.12)$$
$$T_{\eta\xi}\equiv 0\ .$$

Equations (2.11) describe a noncompact O(D,1) nonlinear σ model in two dimensions. In addition, the (two-dimensional) energy-momentum tensor is required to vanish by the constraints Eqs. (2.12). This system of linear partial differential equations can be simplified by choosing an appropriate basis for the string coordinates in $(D+1)$-dimensional Minkowski spacetime. The construction of this basis is analogous to the reduction of the O(N) nonlinear σ model.

We choose as a basis the vectors

$$e_i=(q,q_\xi,q_\eta,b_4,\ldots,b_{D+1})\ ,\quad 1\le i\le D+1\ , \quad (2.13)$$

where the b_i form an orthonormal set;

$$b_i\cdot b_j=\delta_{ij}$$

and

$$b_i\cdot q=0\ ,$$
$$b_i\cdot q_\eta=0\ , \quad (2.14)$$
$$b_i\cdot q_\xi=0\ .$$

We define

$$e^{\alpha(\xi,\eta)}=-q_\xi\cdot q_\eta\ . \quad (2.15)$$

It is easy to show from Eqs. (2.11b) and (2.12) that

$$q\cdot q_\eta=q\cdot q_\xi=0\ ,$$
$$q_\xi\cdot q_{\xi\xi}=q_\eta\cdot q_{\eta\eta}=0\ . \quad (2.16)$$

In the basis Eq. (2.13), the second derivatives of q are expressed as

$$q_{\xi\eta}=qe^{\alpha(\xi,\eta)}\ , \quad (2.17a)$$

$$q_{\xi\xi}=Bq_\xi+Cq_\eta+\sum_{i=4}^{D+1}u_ib_i\ , \quad (2.17b)$$

$$q_{\eta\eta}=Eq_\xi+Fq_\eta+\sum_{i=4}^{D+1}v\cdot b_i\ . \quad (2.17c)$$

Here

$$u_i=b_i\cdot q_{\xi\xi}\ ,\quad v_i=b_i\cdot q_{\eta\eta}\ . \quad (2.18)$$

The coefficients B, C, E, and F are determined by using Eqs. (2.9), (2.11), (2.16), and (2.17a). We find

$$C=E=0\ ,$$
$$B=\alpha_\xi\ ,\quad F=\alpha_\eta\ . \quad (2.19)$$

Therefore we have

$$q_{\xi\xi}=\alpha_\xi q_\xi+\sum_{i=4}^{D+1}u_ib_i\ ,$$
$$q_{\eta\eta}=\alpha_\eta q_\eta+\sum_{i=4}^{D+1}v_ib_i\ . \quad (2.20)$$

From Eqs. (2.15) and (2.17a), it follows that

$$\alpha_\eta=-e^{-\alpha(\xi,\eta)}q_\xi\cdot q_{\eta\eta}\ . \quad (2.21)$$

Then

$$\alpha_{\eta\xi}=e^{-\alpha}[\alpha_\xi q_\xi\cdot q_{\eta\eta}-q_{\xi\xi}\cdot q_{\eta\eta}-q_\xi\cdot q_{\eta\eta\xi}]\ . \quad (2.22)$$

Inserting Eqs. (2.20) and (2.21) in Eq. (2.22) yields

$$\alpha_{\eta\xi}-e^{\alpha(\xi,\eta)}+e^{-\alpha(\xi,\eta)}\sum_{i=4}^{D+1}u_i\cdot v_i=0\ . \quad (2.23)$$

This is the evolution equation for the function $\alpha(\xi,\eta)$ determining the scalar product $q_\xi\cdot q_\eta$, for all D. This is a generalization of the sinh-Gordon equation. It remains now to find the evolution equations for the fields u_i and v_i. In order to find the equations for u_i and v_i, we express the derivatives of the basis vectors Eq. (2.13) in terms of the basis itself:

$$\frac{\partial e_i}{\partial\xi}=A_{ij}(\xi,\eta)e_j\ ,\quad \frac{\partial e_i}{\partial\eta}=B_{ij}(\xi,\eta)e_j\ . \quad (2.24)$$

For the first three vectors, Eqs. (3.17a), (2.20), and (2.21) yield the coefficients A_{ij} and B_{ij} $(i=1,2,3)$ and $1\le j\le D+1)$. For the remaining vectors, from Eqs. (2.14) and (2.20) we find

$$b_{i\xi}=e^{-\alpha}u_iq_\eta+\sum_{j\ne i}[b_j\cdot(b_i)_\xi]b_j\ ,$$
$$b_{i\eta}=e^{-\alpha}v_iq_\xi+\sum_{j\ne i}[b_j\cdot(b_i)_\eta]b_j\ ,\quad 4\le i,j\le D+1\ , \quad (2.25)$$

where the coefficients $b_j\cdot(b_i)_\xi$ and $b_j\cdot(b_i)_\eta$ depend upon the explicit choice of the vectors $b_4,\ldots,b_{D+1}$.

With the help of Eqs. (2.18) and (2.20), we get

$$(u_i)_\eta=\sum_{j=4,j\ne i}^{D+1}u_jb_j\cdot(b_i)_\eta\ , \quad (2.26a)$$

$$(v_i)_\xi=\sum_{j=4,j\ne i}^{D+1}v_jb_j\cdot(b_i)_\xi\ , \quad (2.26b)$$

where we have also used Eq. (2.17a). Notice that

$$b_j\cdot(b_i)_\eta=-b_i\cdot(b_j)_\eta$$

since

$$b_i\cdot b_j=\delta_{ij}\ ,\quad 4\le i,j\le D+1\ .$$

Finally, the complete matrices A_{ij} and B_{ij} are given by

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_\xi & 0 & u_4 & \cdots & u_{D+1} \\ e^{\alpha} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{-\alpha}u_4 & 0 & \cdots & b_4\cdot(b_{D+1})_\xi \\ \vdots & \vdots & \vdots & \vdots & b_i\cdot(bj)_\xi & \vdots \\ 0 & 0 & e^{-\alpha}u_{D+1} & b_{D+1}\cdot(b_4)_\xi & \cdots & 0 \end{bmatrix}, \tag{2.27a}$$

$$B=\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ e^{\alpha} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_\eta & v_4 & \cdots & v_{D+1} \\ 0 & v_4 e^{-\alpha} & 0 & 0 & \cdots & b_4\cdot(b_{D+1})_\eta \\ \vdots & \vdots & \vdots & \vdots & b_i\cdot(b_j)_\eta & \vdots \\ 0 & v_{D+1}e^{-\alpha} & 0 & b_{D+1}\cdot(b_4)_\eta & \cdots & 0 \end{bmatrix}. \tag{2.27b}$$

The compatibility condition for Eqs. (2.24) is expressed as

$$\partial_\eta A-\partial_\xi B+[A,B]=0 . \tag{2.28}$$

In the forthcoming sections, we will discuss these equations for $D=2$, 3, and 4.

Let us now investigate the exact symmetries of these equations. For this purpose it is convenient to use the coordinates (X^0,X^i), $i=1,\dots,D-1$, given by Eqs. (2.5). In terms of these coordinates, the string equations of motion and constraints are expressed as

$$\ddot{X}^0-X^{0''}-R\frac{dR}{dX^0}\sum_i[(X'^i)^2-(\dot{X}^i)^2]=0 , \tag{2.29a}$$

$$\ddot{X}^i-X''^i-\frac{2}{R}\frac{dR}{dX^0}(X'^0X'^i-\dot{X}^0\dot{X}^i)=0 , \tag{2.29b}$$

$$(\dot{X}^0)^2+(X'^0)^2-R^2\sum_i[(X'^i)^2+(\dot{X}^i)^2]=0 , \tag{2.30a}$$

$$X'^0\dot{X}^0-R^2\sum_i X'^i\dot{X}^i=0 . \tag{2.30b}$$

Let us now consider the transformation

$$\begin{aligned} X^0&=\tilde{X}^0 ,\\ \dot{X}^i&=\tilde{R}^2\tilde{X}'^i ,\\ X'^i&=\tilde{R}^2\dot{\tilde{X}}^i , \end{aligned} \tag{2.31}$$

where $\tilde{R}\equiv R^{-1}$. That is,

$$X^i_\xi=\tilde{R}^2\tilde{X}^i_\xi ,$$
$$X^i_\eta=-\tilde{R}^2\tilde{X}^i_\eta .$$

Under the transformation (2.31), Eqs. (2.29) and (2.30) remain invariant. This means that if (X^0,X^i) are solutions of Eqs. (2.29) and (2.30) for the metric

$$dS^2=-(dX^0)^2+R^2(dX^i)^2 , \tag{2.32}$$

then $(\tilde{X}^0,\tilde{X}^i)$ are solutions of the same system of equations for the metric

$$\begin{aligned} d\tilde{S}^2&=-(d\tilde{X}^0)^2+\tilde{R}^2(d\tilde{X}^i)^2\\ &=-(dX^0)^2+R^{-2}(d\tilde{X}^i)^2 . \end{aligned} \tag{2.33}$$

Equations (2.31) transform a solution of the nonlinear equations (2.29) and (2.30) into a solution of the same system. This is therefore an *auto-Bäcklund transformation.* [Note that under Eq. (2.31), Eq. (2.29b) for X^i becomes an identity, while the compatibility condition for the transformations (2.31) yields Eq. (2.29b).] In terms of the coordinates (X^0,X^i), the scalar product Eq. (2.15) defining the function $\exp[\alpha,(\sigma,\tau)]$ is expressed as

$$e^{\alpha}=H^2\left[(\dot{X}^0)^2-(X'^0)^2-R^2\sum_{i=2}^{D}[(\dot{X}^i)^2-(X'^i)^2]\right] ,$$

i.e.,

$$e^{\alpha}=H^2\left[X^0_\xi X^0_\eta-R^2\sum_{i=2}^{D}X^i_\xi X^i_\eta\right] . \tag{2.34}$$

Using the transformation (2.31), we have

$$e^{\alpha}=e^{\tilde{\alpha}}-2R^2\sum_{i=2}^{D}X^i_\xi X^i_\eta , \tag{2.35}$$

or

$$e^{\tilde{\alpha}}=e^{\alpha}-2\tilde{R}^2\sum_{i=2}^{D}\tilde{X}^i_\xi\tilde{X}^i_\eta . \tag{2.36}$$

Equation (2.35) [or Eq. (2.36)] connects two solutions of the generalized sinh-Gordon equation (2.23). That is, if $\alpha(\sigma,\tau)$ is a solution of Eq. (2.23) for an expanding metric [Eq. (2.32)], then $\alpha(\sigma,\tau)$ is a solution of the equation

$$\partial_\xi\partial_\eta\tilde{\alpha}-e^{\tilde{\alpha}}-e^{-\tilde{\alpha}}\sum_{i=4}^{D+1}\tilde{u}_i\cdot\tilde{v}_i=0 , \tag{2.37}$$

for a contracting metric (2.37).

This Bäcklund transformation relating α and $\tilde{\alpha}$ is different from the well-known Bäcklund transformation relating two solutions α_0 and $\hat{\alpha}$ of the sinh-Gordon equation [15], namely,

$$\left[\frac{\alpha_0+\hat{\alpha}}{2}\right]_\xi = A\sinh\left[\frac{\alpha_0-\hat{\alpha}}{2}\right],$$

$$\left[\frac{\alpha_0-\hat{\alpha}}{2}\right]_\eta = A^{-1}\sinh\left[\frac{\alpha_0+\hat{\alpha}}{2}\right],$$

where A is an arbitrary parameter.

The transformation (2.31) was suggested by the asymptotic behavior of the string solutions for $R\to\infty$ and $R\to 0$ in the highly unstable regime. (The relation $R\to R^{-1}$ precisely connects these behaviors.) It was called "R duality" [14], since it appears as an extension of the target space duality of string theory in compactified spacetimes with radius R, in which case the string spectra for radii R and $\alpha' h R^{-1}$ are equivalent.

As we will see in Sec. IV, Eq. (2.23) in $D=3$ becomes the sinh-Gordon equation. In this case we have the additional symmetry $\alpha\to-\alpha$.

III. STRINGS IN TWO-DIMENSIONAL DE SITTER SPACETIME

Let us now study the case $D=2$. In this case a complete basis is formed by

$$e_i=(q,q_\xi,q_\eta)\ . \tag{3.1}$$

Therefore Eqs. (2.20) and (2.23) become

$$q_{\xi\xi}=\alpha_\xi q_\xi\ ,\quad q_{\eta\eta}=\alpha_\eta q_\eta\ , \tag{3.2}$$

and

$$\alpha_{\eta\xi}-e^{\alpha(\xi,\eta)}=0\ . \tag{3.3}$$

This is the Liouville equation whose general solution is given by

$$\alpha(\xi,\eta)=\ln\left\{\frac{2f'(\xi)g'(\eta)}{[f(\xi)+g(\eta)]^2}\right\}, \tag{3.4}$$

where f and g are arbitrary functions of the indicated variables.

The $D=2$ case can be solved directly from the equations of motion [Eq. (2.11)] and the constraints Eqs. (2.12) in the coordinates (2.7). In this case these coordinates are given by

$$\begin{aligned} &q_0=\sinh u\ ,\quad -\infty\le u\le+\infty\ ,\\ &q_1=\cosh u\,\cos v\ ,\quad 0\le v\le 2\pi\ ,\\ &q_2=\cosh u\,\sin v\ ,\quad 0\le v\le 2\pi\ . \end{aligned} \tag{3.5}$$

The constraints Eqs. (2.12) take the form

$$\begin{aligned} &q_\xi^2=(v_\xi)^2\cosh^2 u-(u_\xi)^2\simeq 0\ ,\\ &q_\eta^2=(v_\eta)^2\cosh^2 u-(u_\eta)^2\simeq 0\ . \end{aligned} \tag{3.6}$$

Therefore we have

$$\begin{aligned} &v_\xi\cosh u=\pm u_\xi\ ,\\ &v_\eta\cosh u=\pm u_\xi\varepsilon\ , \end{aligned} \tag{3.7a}$$

where $\varepsilon^2=1$. In addition,

$$q_\xi\cdot q_\eta=\cosh^2 u v_\xi v_\eta-u_\xi u_\eta=(\varepsilon-1)u_\xi u_\eta\ . \tag{3.7b}$$

The general solution of Eq. (3.7a) is given by

$$\begin{aligned} &v=\pm 2\arctan(e^u)+G(\eta)\ ,\\ &v=\pm 2\epsilon\arctan(e^u)+F(\xi)\ , \end{aligned} \tag{3.8}$$

where F and G are arbitrary functions of the indicated variables. We have here two different cases, depending on whether (A) $\varepsilon=-1$ or (B) $\varepsilon=+1$.

A. Case A: $\varepsilon=-1$

Here we find

$$\begin{aligned} &v=\tfrac{1}{2}[F(\xi)+G(\eta)]\ ,\\ &u=\ln\left[\pm\tan\left[\frac{F-G}{4}\right]\right]. \end{aligned} \tag{3.9}$$

This corresponds to the previous solutions Eq. (3.4). From Eq. (3.7b) we have

$$q_\xi\cdot q_\eta=-e^{-\alpha}=-\frac{1}{2}\frac{F'(\xi)G'(\eta)}{\sin^2[(F-G)/2]}\ . \tag{3.10}$$

That is,

$$\begin{aligned} &f(\xi)=e^{-iF(\xi)}\ ,\\ &g(\eta)=e^{iG(\eta)}\ . \end{aligned} \tag{3.11}$$

B. Case B: $\varepsilon=+1$

In this case

$$\begin{aligned} &q_\xi\cdot q_\eta=0\ ,\\ &\partial_\xi\partial_\eta q=0\ , \end{aligned} \tag{3.12}$$

and, therefore, the parametrization in terms of the field $\alpha(\xi,\eta)$ breaks down. Equations (3.8) yield

$$F=G=\text{const}\equiv C\ .$$

Then

$$u=\ln\left[\pm\tan\left[\frac{v-C}{2}\right]\right]. \tag{3.13}$$

Now we consider the equation of motion [Eq. (3.12)] to find the dependence on ξ and η. From Eqs. (3.5) and (3.13), we find

$$\begin{aligned} &q_0=\mp\cot(v-C)\ ,\\ &q_1=\pm\cot(v-C)\cos C\mp\sin C\ ,\\ &q_2=\pm\cot(v-C)\sin C\pm\cos C\ . \end{aligned} \tag{3.14}$$

Therefore we find

$$v = C - \pi/2 + \arctan[R(\xi) + S(\eta)] \,,$$
$$\cosh u = \{1 + [R(\xi) + S(\eta)]^2\}^{1/2} \,, \tag{3.15}$$

where $R(\xi)$ and $S(\eta)$ are arbitrary functions of the indicated variables.

Let us analyze now the solutions for these two cases. The two arbitrary functions appearing in the solutions for the cases A and B correspond to the conformal invariance of the world sheet. There are no further arbitrary functions since we do not have transverse degrees of freedom in $D=2$.

C. Case A: $\varepsilon = -1$

Using the world-sheet conformal invariance, we can always choose a gauge where

$$v = \sigma \,, \quad 0 < \sigma \le 2\pi \,,$$
$$u = \ln\left[\tan\frac{\tau}{2}\right] \,, \quad 0 < \tau \le \pi \,,$$

i.e.,

$$\sinh u = -\cot\tau \,, \quad -\infty \le u \le +\infty \,. \tag{3.16}$$

This describes a string wound around the de Sitter universe and evolving with it. A half of string evolution $\pi/2 < \tau < \pi$ corresponds to the expansion time $0 < u < \infty$ of the de Sitter universe and similarly for the first half $0 < \tau < \pi/2$, which corresponds to the contraction phase $-\infty < u < \pi$ (see Fig. 1).

Equation (3.16) describes a string wound *once* around de Sitter space (here a circle). More generally, we may have

$$v = n\sigma \,, \quad 0 < \sigma \le 2\pi \,,$$
$$u = \ln\tan\left[\frac{n\tau}{2}\right] \,, \quad 0 < \tau \le \pi/n \,, \tag{3.17}$$

where n is an integer number.

This solution describes a string wound n times around de Sitter space. In this case the string-evolution period is *reduced* to $\Delta\tau = \pi/(2n)$ (instead of being $\pi/2$); that is, in time τ, the string expansion (and contraction) is *n times faster.*

Note that, for $u \to -\infty$,

$$u = \ln\left[\frac{n\tau}{2}\right] \,,$$

i.e.,

$$\tau \to 0^+ \,, \tag{3.18}$$

and that, for $u \to +\infty$,

$$u = -\ln\left[\frac{\pi - n\tau}{2}\right] \,, \tag{3.19}$$

i.e.,

$$\tau \to \pi/n \,.$$

Let us consider the invariant interval

$$dS^2 = \frac{1}{H^2}[-du^2 + \cosh^2 u \, dv^2] \tag{3.20}$$

between two points on the string. For the solution Eq. (3.17), we have

$$dS^2 = \frac{n^2}{H^2 \sin^2 n\tau}(d\sigma^2 - d\tau^2) \,. \tag{3.21}$$

In the asymptotic regions $\tau \to 0^+$ and $\tau \to \pi^-/n$, the conformal factor blows up. The proper length of the string stretches infinitely as

$$\Delta X = \begin{cases} \dfrac{1}{H\tau}\Delta\sigma & \text{for } \tau \to 0^+ \,, \\[2mm] \dfrac{1}{H(\pi/n - \tau)}\Delta\sigma & \text{for } \tau \to \pi^-/n \,. \end{cases} \tag{3.22}$$

This is analogous to the unstable behavior found in D-dimensional inflationary backgrounds [9,10], as well as for strings falling into spacetime singularities [7].

It is interesting to relate the string solution to the cosmic and conformal times t_0 and η respectively.

From Eqs. (2.7) and (3.16), we find

$$q^0 = -\cot\tau \,, \quad q^1 = \frac{\cos\sigma}{\sin\tau} \,, \quad q^2 = \frac{\sin\sigma}{\sin\tau} \,, \tag{3.23}$$

and then, using Eqs. (2.2) and (3.23), we have

$$-e^{-Ht_0} = H\eta = \frac{\sin\tau}{\cos\tau - \cos\sigma} \,. \tag{3.24}$$

The coordinates (t_0, X_1) or (η, X_1) only cover the sector $\sigma \le \tau$ of the string world sheet. (For simplicity, we have taken here $n=1$.)

In the asymptotic region $\tau \to \pi^-$, the conformal time η behaves like

$$\eta_{\tau \to \pi^-} = \frac{\tau - \pi}{2H \sin\sigma/2} \to 0^- \,, \quad 0 < \sigma < \pi \,. \tag{3.25}$$

In this asymptotic regime, the string time τ is proportional to the conformal time η. This relation is typical of strings in strong gravitational fields such as inflationary backgrounds or a string falling into spacetime singularities: In the asymptotic region where the string stretches indefinitely (and is thus unstable), string evolution is governed by the conformal time.

Note that, alternatively to Eq. (3.16), one could take the choice

$$v = \tau \,,$$
$$u = \ln(\tan\sigma/2) \,. \tag{3.26}$$

However, the physical meaning of this choice is not clear to us.

D. Case B: $\varepsilon = +1$

Let us consider now case B [Eq. (3.15)]. Using the world-sheet conformal invariance, we can always choose

$$v = C - \pi/2 \pm \arctan\tau \,,$$
$$\cosh u = (1 + \tau^2)^{1/2} \,. \tag{3.27}$$

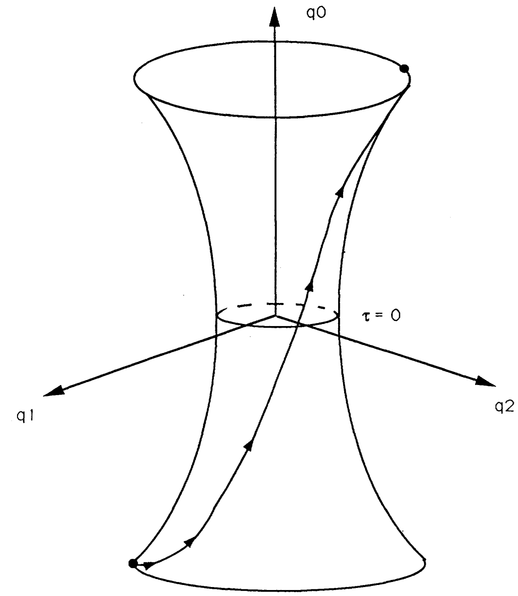

FIG. 2. Same as in Fig. 1, but now the string solution B [Eq. (3.28)] is drawn. This is in fact a geodesic.

Therefore the string motion Eq. (3.14) yields

$$q^0=\tau \; , \quad q^1=-\tau\cos C-\sin C \; , \quad q^2=-\tau\sin C+\cos C \; . \tag{3.28}$$

This solution actually describes a particle trajectory since it was possible to gauge away the parameter σ. Equation (3.28) describes a geodesic in two-dimensional de Sitter spacetime, that is, the trajectory of a massless particle. Since transverse dimensions are absent, only massless states appear in this two-dimensional case. The solution Eq. (3.27) or (3.28) is a particular case of the center-of-mass solution described in Ref. [1] when $D=2$ and $m=0$. When τ goes from $-\infty$ to $+\infty$, the light rays go from $q^0=-\infty$ to $q^0=+\infty$. At the same time, the angle v varies through an interval of π: $v(-\infty)=v(+\infty)\pm\pi$. In Eq. (3.27) the signs $\pm$ correspond to a motion in the positive or negative direction of the de Sitter spatial circle (see Fig. 2). It should be noted that traveling from $q^0=\tau=-\infty$ to $+\infty$, the particle goes over half of the de Sitter circle.

The solutions described in cases A and B contain *all the string solutions* in two-dimensional de Sitter spacetime.

IV. STRINGS IN THREE-DIMENSIONAL DE SITTER SPACETIME

In $D=3$ we have a four-dimensional embedding Minkowski spacetime where the antisymmetric Levi-Civita tensor allows us to construct a vector $b\equiv b_4$ orthogonal to the vectors q, q_ξ and q_η, namely,

$$b_a\equiv e^{-\alpha}\epsilon_{abcd}q_b(q_\xi)_c(q_\eta)d \; . \tag{4.1}$$

The vectors (q,q_ξ,q_η,b) form a basis. In addition,

$$b_a b^a=1 \; . \tag{4.2}$$

In this case Eqs. (2.25) take the form

$$b_\eta=ve^{-\alpha}q_\xi \; , \quad b_\xi=ue^{-\alpha}q_\eta \; , \tag{4.3}$$

where $u\equiv u_4$ and $v\equiv v_4$. Here the compatibility condition Eq. (2.28) yields

$$0=\partial_\eta A-\partial_\xi B+[A,B]$$
$$=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha_{\xi\eta}+uve^{-\alpha}-e^{\alpha} & 0 & \partial_\eta u \\ 0 & 0 & -\alpha_{\xi\eta}+e^{\alpha}-uve^{-\alpha} & -\partial_\xi v \\ 0 & e^{-\alpha}\partial_\xi v & e^{-\alpha}\partial_\eta u & 0 \end{bmatrix} . \tag{4.4}$$

This implies

$$u=u(\xi) \; , \quad v=v(\eta) \; ,$$

and

$$\alpha_{\xi\eta}-e^{\alpha}+uve^{-\alpha}=0 \; . \tag{4.5}$$

In particular, in the $D=2$ case, $b=0$, and then, from Eq. (4.3), it follows $u=v=0$; that is, in $D=2$, we have

$$\alpha_{\xi\eta}-e^{\alpha}=0 \; ,$$

which is the Liouville equation, as we have found in Sec. II.

It is convenient to transform the variables as

$$\alpha(\xi,\eta)=\hat{\alpha}(\xi',\eta')+\ln[f(\eta)g(\xi)] \; , \tag{4.6}$$

where

$$\frac{d\xi'}{d\xi}=g(\xi) \; , \quad \frac{d\eta'}{d\eta}=f(\eta) \; ,$$

and

$$f^2(\eta)=v(\eta) \; , \quad g^2(\xi)=u(\xi) \; . \tag{4.7}$$

Then Eq. (4.5) takes the sinh-Gordon form

$$\hat{\alpha}_{\xi'\eta'}-e^{\hat{\alpha}}+e^{-\hat{\alpha}}=0 \; . \tag{4.8}$$

In this way the string equations and constraints in three-dimensional de Sitter spacetime reduce to the sinh-Gordon equation.

The string coordinates q and its derivatives q_ξ and q_η are related to $\alpha(\xi,\eta)$ through

$$e^{\hat{\alpha}}=-q_\xi\cdot q_\eta \; , \quad b\cdot q_{\xi\xi}=1 \; , \quad b\cdot q_{\eta\eta}=1 \; , \tag{4.9}$$

where

$$b_a=e^{-\hat{\alpha}}\epsilon_{abcd}q_b(q_\xi)_c(q_\eta)d \; ,$$

and we dropped the primes in ξ' and η'. In order to make these equations explicit, it is convenient to perform a stereographic projection of the three-dimensional unit sphere $\hat{q}$ [see, for example, Eq. (2.7)]:

$$W=\frac{\hat{q}_1+i\hat{q}_2}{1+\hat{q}_3}\ ,\qquad \overline{W}=\frac{\hat{q}_1-i\hat{q}_2}{1+\hat{q}_3}\ . \tag{4.10}$$

Using (u,v,w) as coordinates of de Sitter spacetime, Eqs. (2.12) and (4.9) are expressed as

$$\left[\frac{u_\xi}{\cosh u}\right]^2=4\frac{W_\xi \overline{W}_\xi}{(1+W\overline{W})^2}\ ,\qquad \left[\frac{u_\eta}{\cosh u}\right]^2=4\frac{W_\eta \overline{W}_\eta}{(1+W\overline{W})^2}\ , \tag{4.11}$$

$$e^{\alpha}=u_\xi u_\eta-2\frac{(W_\xi\overline{W}_\eta+W_\eta\overline{W}_\xi)}{(1+W\overline{W})^2}\cosh^2 u\ , \tag{4.12}$$

$$\begin{aligned}e^{\alpha}=&\frac{2i}{1+W\overline{W}}(W_\xi\overline{W}_\eta-\overline{W}_\xi W_\eta)\cosh^2 u\\ &\times\{u_{\xi\xi}[\cosh^2 u-2u_\xi u_\eta/(e^{\alpha}-2u_\xi u_\eta)]\\ &\quad+(u_\xi)^2(\tanh u)[\sinh^2 u-2-e^{\alpha}/(e^{\alpha}-2u_\xi u_\eta)]\\ &\quad+e^{\alpha}u_\xi\alpha_\xi/(e^{\alpha}-2u_\xi u_\eta)\}\ ,\end{aligned} \tag{4.13a}$$

and

$$\begin{aligned}e^{\alpha}=&\frac{2i}{1+W\overline{W}}(W_\xi\overline{W}_\eta-\overline{W}_\xi W_\eta)\cosh^2 u\\ &\times\left\{u_{\eta\eta}\left[\cosh^2 u\frac{-2u_\xi u_\eta}{e^{\alpha}-2u_\xi u_\eta}\right]\right.\\ &\quad+(u_\eta)^2(\tanh u)\left[\sinh^2 u-2-\frac{e^{\alpha}}{e^{\alpha}-2u_\xi u_\eta}\right]\\ &\quad\left.+e^{\alpha}u_\eta\alpha_\eta/(e^{\alpha}-2u_\xi u_\eta)\right\}\ ,\end{aligned} \tag{4.13b}$$

where $\alpha(\xi,\eta)$ obeys the sinh-Gordon equation

$$\alpha_{\xi\eta}-e^{\alpha}+e^{-\alpha}=0\ , \tag{4.14}$$

where we dropped the notation $\hat{\Lambda}$ in α.

V. STRINGS IN FOUR-DIMENSIONAL DE SITTER SPACETIME

Here we have a five-dimensional embedding Minkowski spacetime, where one has to choose two orthogonal vectors b_4 and b_5 forming a basis together with the vectors q,q_ξ,q_η. In $D=4$, Eqs. (2.26a) take the form

$$(u_4)_\eta=u_5 b_5\cdot(b_4)_\eta\ ,\qquad (u_5)_\eta=u_4 b_4\cdot(b_5)_\eta=-b_5\cdot(b_4)_\eta u_4\ . \tag{5.1}$$

A consequence of these equations is

$$\partial_\eta[(u_4)^2+(u_5)^2]=0\ . \tag{5.2}$$

Then we write

$$u_4=U(\xi)\cos\overline{\beta}(\eta,\xi)\ ,\qquad u_5=U(\xi)\sin\overline{\beta}(\eta,\xi)\ , \tag{5.3}$$

and, from Eqs. (5.1) and (5.3), we have

$$b_5\cdot(b_4)_\eta=\frac{(u_4)_\eta}{u_5}=-\overline{\beta}_\eta\ . \tag{5.4}$$

Similarly, by using Eq. (2.26b), we find

$$v_4=V(\eta)\cos\gamma(\xi,\eta)\ ,\qquad v_5=V(\eta)\sin\gamma(\xi,\eta)\ , \tag{5.5}$$

and then we have

$$b_5\cdot(b_4)_\xi=-\gamma_\xi\ . \tag{5.6}$$

We are now in a position to write the complete matrices A and B [Eqs. (2.27)] of the linear system Eq. (2.24):

$$A=\begin{bmatrix}0&1&0&0&0\\0&\alpha_\xi&0&u_4&u_5\\e^{\alpha}&0&0&0&0\\0&0&u_4e^{-\alpha}&0&-\gamma_\xi\\0&0&u_5e^{-\alpha}&\gamma_\xi&0\end{bmatrix}\ , \tag{5.7a}$$

$$B=\begin{bmatrix}0&0&1&0&0\\e^{\alpha}&0&0&0&0\\0&0&\alpha_\eta&v_4&v_5\\0&v_4e^{-\alpha}&0&0&-\overline{\beta}_\eta\\0&v_5e^{-\alpha}&0&\overline{\beta}_\eta&0\end{bmatrix}\ . \tag{5.7b}$$

Here the compatibility condition Eq. (2.28) yields the nontrivial equations

$$\alpha_{\xi\eta}-e^{\alpha}+(u_4v_4+u_5v_5)e^{-\alpha}=0\ , \tag{5.8a}$$

$$(\gamma-\overline{\beta})_{\xi\eta}+(u_5v_4-u_4v_5)e^{-\alpha}=0\ . \tag{5.8b}$$

Equation (5.8a) is just Eq. (2.23) for $D=4$.

From Eqs. (5.3) and (5.5), we find

$$\alpha_{\xi\eta}-e^{\alpha}+U(\xi)V(\eta)e^{-\alpha}\cos\beta=0\ , \tag{5.9a}$$

$$\beta_{\xi\eta}-U(\xi)V(\eta)e^{-\alpha}\sin\beta=0\ , \tag{5.9b}$$

where $\beta(\eta,\xi)=\overline{\beta}-\gamma$.

As for the $D=3$ case, it is convenient to perform the transformation defined by Eqs. (4.6) and (4.7), which yields

$$\hat{\alpha}_{\xi'\eta'}-e^{\hat{\alpha}}+e^{-\hat{\alpha}}\cos\beta=0\ , \tag{5.10a}$$

$$\beta_{\xi'\eta'}-e^{-\hat{\alpha}}\sin\beta=0\ . \tag{5.10b}$$

It must be noted that when $\beta=0$, Eqs. (5.9) [or, equivalently, Eq. (5.10)] give Eq. (4.5) [or its equivalent Eq. (4.8)] of the $D=3$ case.

Equation (5.10) can be derived from the (reduced) Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\alpha)^2+\tfrac{1}{2}(\partial_\mu\beta)^2-V(\alpha,\beta)\ ,\quad V(\alpha,\beta)=-e^{\alpha}-e^{-\alpha}\cos\beta\ . \tag{5.11}$$

In Ref. [11] this Lagrangian was derived from the string equations in de Sitter spacetime. This potential can be related to the B_2 Toda field theory upon changing $\beta\rightarrow i\beta$.

The effective models derived in 2+1 and 3+1 dimensions [Eqs. (4.14) and (5.10)] have, as potentials,

$$U_{D=2+1}(\alpha)=-2\cosh\alpha\ ,\quad U_{D=3+1}(\alpha)=-e^{\alpha}-e^{-\alpha}\cos\beta\ , \tag{5.12}$$

respectively. They are unbounded from below, and they indicate that the string time evolution should tend to the absolute minima at $\alpha=+\infty$ and $-\infty$ (with $|\beta|<\pi/2$ in the 3+1 case). It is then interesting to study the string behavior near such strongly attractive points $\alpha=\pm\infty$.

Let us start by the (2+1)-dimensional case (i.e., the standard sinh-Gordon equation). By choosing the time origin such that $\alpha=+\infty$ at $\tau=0$, we find, from Eq. (4.14),

$$e^{\alpha(\sigma,\tau)}_{\tau\rightarrow 0}=\frac{2}{\tau^2}+B(\sigma)+\frac{\tau^2}{10}[3B^2(\sigma)+B''(\sigma)-1]+O(\tau^4)\ , \tag{5.13}$$

where $B(\sigma)$ is an arbitrary periodic function of σ.

Now in order to find $q(\sigma,\tau)$ near $\tau=0$, we must satisfy Eqs. (2.11), (2.12), and (2.15). A consistent solution near $\tau=0$ has the form

$$q(\sigma,\tau)=\frac{V_0(\sigma)}{\tau}+V_1(\sigma)+\tau V_2(\sigma)+\tau^2V_3(\sigma)+\tau^3V_4(\sigma)+O(\tau^4)\ , \tag{5.14}$$

where $V_1(\sigma)=0$, $V_0(\sigma)$ is a null vector with V_0' having unit norm:

$$V_0\cdot V_0=0\ ,\quad V_0'\cdot V_0'=1\ ,\quad V_0''\cdot V_0''=\tfrac{3}{2}B(\sigma)\ . \tag{5.15}$$

The vector $V_2(\sigma)$ is given by

$$V_2(\sigma)=-\tfrac{1}{2}V_0''(\sigma)-\frac{B(\sigma)}{2}V_0(\sigma)\ . \tag{5.16}$$

It follows from Eqs. (5.15) and (5.16) that

$$2V_2\cdot V_0=1\ ,\quad V_0'\cdot V_2=V_0\cdot V_2'=0\ ,\quad V_2\cdot V_2=-B(\sigma)/8\ .$$

The vector V_3 must obey

$$V_0\cdot V_3=V_0\cdot V_3'=0\ .$$

Finally, we have, for $V_4(\sigma)$,

$$V_4(\sigma)=\frac{1}{4}\left[V_2''+BV_2+\frac{3B(\sigma)^2+B''(\sigma)-1}{10}V_0\right]\ .$$

Let us consider now the string behavior near the strongly attractive points $\alpha=+\infty$ in the (3+1)-dimensional case. The absolute minimum at the point $\alpha=-\infty$, $\beta=0$ represents the limit of a collapsing string configuration, whereas the absolute minimum at the line $\alpha=+\infty$, $|\beta|<\pi/2$ describes an expanding string. The asymptotic behavior of the solution near the shrinking point $\alpha\rightarrow-\infty$, $\beta\rightarrow 0$ is given by

$$\begin{aligned}\alpha_{\tau\rightarrow 0}&=-\ln\left[\frac{2}{\tau^2}\right]-\tau[C_1\cos(\sqrt{7}\ln\tau)+C_2\sin(\sqrt{7}\ln\tau)+C_3]+O(\tau^2)\ ,\\ \beta_{\tau\rightarrow 0}&=\sqrt{\tau}\left[A\cos\left[\frac{\sqrt{7}}{2}\ln\tau\right]+B\sin\left[\frac{\sqrt{7}}{2}\ln\tau\right]\right]\ ,\end{aligned} \tag{5.17}$$

where the coefficients C_1, C_2, and C_3 in $\alpha(\sigma,\tau)$ are related to the A and B coefficients as

$$\begin{aligned}C_1&=\frac{1}{176}[2\sqrt{7}AB+9(A^2-B^2)]\ ,\\ C_2&=\frac{1}{176}[18AB-\sqrt{7}(A^2-B^2)]\ ,\\ C_3&=-\frac{1}{4}(A^2+B^2)\ .\end{aligned} \tag{5.18}$$

For $\alpha\rightarrow+\infty$ (stretching of the string), β decouples from α; the asymptotic behavior is given by

$$\alpha=-\ln\left[\frac{\tau^2}{2}\right]+O(\tau^2)\ ,\quad \beta=a+b\tau-\frac{\sin a}{24}\tau^4\ . \tag{5.19}$$

In order to find $q(\sigma,\tau)$, we must satisfy Eqs. (2.11), (2.12), and (2.15). We find

$$q(\sigma,\tau)=V_0(\sigma)\left[\frac{1}{\tau}-\left[\frac{A^2(\sigma)-B^2(\sigma)}{88}\right]C(\tau)-\frac{A(\sigma)B(\sigma)}{44}S(\tau)-\left[\frac{A^2(\sigma)+B^2(\sigma)}{4}\right]\right]+O(\tau,\tau C^2,\tau S^2)\ , \tag{5.20}$$

where

$$C(\tau)=\cos(\sqrt{7}\ln\tau)\ ,\quad S(\tau)=\sin(\sqrt{7}\ln\tau)\ ,$$

and $A(\sigma)$ and $B(\sigma)$ are the coefficients of the $\tau\rightarrow 0$ behavior of β. For $A=0$ and $B=0$ (i.e., $\beta=0$), we recover the (2+1)-dimensional case.

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