Quasiclassical equations of motion for nonlinear Brownian systems

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Following the formalism of Gell-Mann and Hartle, phenomenological equations of motion are derived from the decoherence functional of quantum mechanics, using a path-integral description. This is done explicitly for the case of a system interacting with a "bath" of harmonic oscillators whose individual motions are neglected. The results are compared to the equations derived from the purely classical theory. The case of linear interactions is treated exactly, and nonlinear interactions are also compared, using classical and quantum perturbation theory.

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I. INTRODUCTION

A. Decoherence and the quasiclassical limit

Two of the most puzzling aspects of the quantum theory have, until recently, remained unclear: the proper interpretation of quantum probabilities, and the mechanism by which deterministic classical "laws" can arise from a probabilistic underlying theory. The idea of wave-function collapse, while providing a useful approximate description of most experimental situations, begs the question of why a system which otherwise undergoes purely unitary evolution should suddenly and dramatically be collapsed upon measurement by a scientist. The procedure is highly asymmetric, instantaneous, and irreversible, and moreover requires the existence of a "classical" measuring device outside the system being measured. When one considers a closed system the idea of wave functions collapsing becomes highly ambiguous. There is nothing outside the system to collapse it. The quintessential example of this, of course, is the Universe itself. Clearly, if the fundamental laws of the Universe are quantum mechanical, there can be no separate "classical domain" to explain our observations. Since the classical realm is itself, presumably, merely a limit of the underlying quantum reality, the probabilities must arise directly from the quantum theory itself, without recourse to the *deus ex machina* of the measurement device. And somehow, the various potential futures of the Universe must collapse themselves onto the one possibility which we observe. Quantum cosmology requires a solid formalism for the treatment of closed systems, and work in this field should have that as its goal.

The recent work on the decoherence functional formulation avoids the problems of earlier approaches [1–4]. Physics is described in terms of exhaustive sets of possible histories, coarse grained, with the restriction that these histories must be decoherent. That is, it must be possible to assign probabilities to these histories such that they obey the classical laws of probability with no interference.

Gell-Mann and Hartle have argued that it is possible, in a highly coarse-grained system, to define the classical equation of motion directly from the decoherence functional itself [5]. I will, in this paper, attempt to show that this definition gives the exact classical results, at least for the case of systems interacting with baths of oscillators, and further, that these systems are decoherent in the classical limit. Quantum effects enter as a random, fluctuating force from the effects of neglected degrees of freedom, even in cases where the classical noise would ordinarily be zero. Fluctuations, dissipation, and decoherence turn out to be intimately interlinked.

The linear case has been treated before by a number of people, both classically and quantum mechanically, though not in precisely this same way [6–8]. The correspondence of this quantum system to the classical Langevin equation is thus nothing new. The decoherence of similar systems has also been examined, using a somewhat different definition of decoherence, which for these models generally corresponds to my definition [9]. However, to my knowledge, no one has considered the classical correspondence of these sorts of nonlinear systems, nor the relationship between dissipation, noise, and decoherence in these more general cases. Thus, the results herein are of interest in demonstrating that it is possible to define classical equations of motion directly from the quantum theory in a broad range of systems.

B. Path-integral description of the decoherence functional

We will not, for the purpose of this model, be using the decoherence functional in its most general form. Instead, we will consider only one type of history. Suppose that our system is completely described by a set of generalized coordinates q^β (collectively referred to as q). The most fine-grained possible family of histories would be just the set of all possible paths $q(t)$. We can coarse grain this by dividing the range of the q^β into an exhaustive set of intervals $\Delta^i_{\alpha_i}(t_i)$ at a sequence of times $t_1, t_2, \ldots$, where the α_i are an index labeling the intervals. We can then specify one particular history by which interval was passed through at each time, labeling it by the sequence of indices $\alpha_1, \alpha_2, \ldots$, which I will generally abbreviate as α. Such a history includes all possible paths which pass

through the given set of intervals at the given times.

The decoherence functional is a functional on pairs of histories. The value of this functional on a pair of histories α and α' is given by

$$D(\alpha',\alpha)=\int_{\alpha'}\delta q'\int_{\alpha}\delta q\,\delta(q'_f-q_f)\times\exp(i\{S[q'(t)]-S[q(t)]\}/\hbar)\times\rho(q'_0,q_0)\ . \tag{1.1}$$

Here we are integrating over all paths which pass through the specified sequence of intervals. The functional $S[q(t)]$ is the fundamental action. If $\mathrm{Re}D(\alpha',\alpha)=0$ for $\alpha'\neq\alpha$, then the system is said to be *decoherent,* and obeys classical laws of probability. The diagonal elements $D(\alpha,\alpha)$ are the probabilities of each history α.

The simplest form of this type of history is that where the intervals are completely fine grained in certain variables, and completely coarse grained in others. We divide the coordinates q^β into two groups: x^β (henceforth known as the *system* coordinates), referred to collectively as x, and Q^k (henceforth known as the *reservoir* coordinates), referred to collectively as Q. Our histories will then be complete trajectories $x^\beta(t)$ for the system coordinates, while the reservoir coordinates will be neglected completely.

It is then convenient to break the fundamental action of the system into several parts:

$$S[q(t)]=S_{\mathrm{sys}}[x(t)]+S_{\mathrm{res}}[Q(t)]-\int_{t_0}^{t_f}V(x(t),Q(t))dt\ , \tag{1.2}$$

where $S_{\mathrm{sys}}[x(t)]$ is the action of the system, $S_{\mathrm{res}}[Q(t)]$ is the action of the reservoir, and there is an interaction potential $V(x,Q)$ between them. The decoherence functional is then

$$D[x'(t),x(t)]=\exp(i\{S_{\mathrm{sys}}[x'(t)]-S_{\mathrm{sys}}[x(t)]\}/\hbar)\times\int\delta Q'\int\delta Q\,\delta(Q'(t_f)-Q(t_f))\exp\left\{i\left[S_{\mathrm{res}}[Q'(t)]-S_{\mathrm{res}}[Q(t)]-\int_{t_0}^{t_f}[V(x'(t),Q'(t))-V(x(t),Q(t))]dt\right]/\hbar\right\}\times\rho(x'_0,Q'_0;x_0,Q_0)\ . \tag{1.3}$$

II. LINEAR CASE

The case of a system interacting linearly with a reservoir is a famous one, and has been treated by a number of people, quantum mechanically by Feynman and Vernon and by Caldeira and Leggett, and classically by Zwanzig. For convenience, it is customary to make a number of simplifying assumptions.

(1) The reservoir variables Q^k are harmonic oscillators, i.e.,

$$S_{\mathrm{res}}[Q(t)]=\sum_k\int_{t_0}^{t_f}\frac{m}{2}(\dot{Q}^{k2}-\omega_k^2Q^{k2})dt\ . \tag{2.1}$$

(2) The initial density-matrix factors

$$\rho(x'_0,Q'_0;x_0,Q_0)=\chi(x'_0,x_0)\phi(Q'_0,Q_0)\ . \tag{2.2}$$

A similar assumption classically is to assume that the initial probability distribution of the reservoir coordinates is independent of the initial state of the system coordinates.

(3) The interaction $V(x,Q)$ is bilinear:

$$V(x,Q)=-\sum_k\gamma_kxQ^k\ . \tag{2.3}$$

I will generally assume that x is a single variable; multivariable systems are a trivial generalization, where the γ_k become matrices.

We will relax these assumptions to a certain degree later on, but for now let us consider this case. The classical case is exactly solvable. In this, the equation of motion for the reservoir variable Q^k is

$$\frac{d^2Q^k}{dt^2}(t)=-\omega_k^2Q^k(t)+(\gamma_k/m)x(t)\ . \tag{2.4}$$

This has a solution

$$Q^k(t)=Q^k(t_0)\cos[\omega_k(t-t_0)]+\frac{\dot{Q}^k(t_0)}{\omega_k}\sin[\omega_k(t-t_0)]+\frac{\gamma_k}{m\omega_k}\int_{t_0}^{t}\sin[\omega_k(t-s)]x(s)ds\ . \tag{2.5}$$

We can then use this in the equation of motion for x:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{x}}\right](t)=\left[\frac{\partial L}{\partial x}\right](t)+\sum_k\gamma_kQ^k(t)=\left[\frac{\partial L}{\partial x}\right](t)+F(t)+\sum_k\frac{\gamma_k^2}{m\omega_k}\int_{t_0}^{t}\sin[\omega_k(t-s)]x(s)ds\ , \tag{2.6}$$

where $F(t)$ is the *Langevin force.* In this case, it is clearly

$$F(t)=\sum_k \gamma_k (Q_k(t_0)\cos[\omega_k(t-t_0)] + \frac{\dot{Q}^k(t_0)}{\omega_k}\sin[\omega_k(t-t_0)]) . \tag{2.7}$$

If we assume that the Q have a thermal probability distribution initially,

$$P(Q)=\prod_k \frac{m\omega_k}{2\pi kT}\exp\left[-\frac{m}{2kT}(\dot{Q}^{k^2}+\omega_k^2 Q^{k^2})\right] , \tag{2.8}$$

which, when averaged over an ensemble, gives

$$\langle Q^k\rangle=0, \quad \langle Q^{k^2}\rangle=\frac{kT}{m\omega_k^2}, \cdots , \tag{2.9a}$$

$$\langle \dot{Q}^k\rangle=0, \quad \langle \dot{Q}^{k^2}\rangle=\frac{kT}{m}, \ldots , \tag{2.9b}$$

then

$$\langle F(t)\rangle=0 , \tag{2.10a}$$

$$\langle F(t)F(s)\rangle=\sum_k \gamma_k^2 \left[\frac{kT}{m\omega_k^2}\right]\cos[\omega_k(t-s)] . \tag{2.10b}$$

Let us compare this to the quantum results. It is interesting to first consider the system in isolation from the reservoir. In this case we would have

$$D[x'(t),x(t)]=\exp(i\{S_{\rm sys}[x'(t)]-S_{\rm sys}[x(t)]\}/\hbar) \times\chi(x_0',x_0) . \tag{2.11}$$

If the action $S_{\rm sys}[x(t)]$ has the usual form

$$S_{\rm sys}[x(t)]=\int_{t_0}^{t_f} L(x(t),\dot{x}(t))dt , \tag{2.12}$$

then we can change variables

$$X(t)=\tfrac{1}{2}[x'(t)+x(t)] , \tag{2.13a}$$

$$\xi(t)=x'(t)-x(t) , \tag{2.13b}$$

and expand the phase in terms of ξ:

$$\begin{aligned} S_{\rm sys}[x'(t)]-S_{\rm sys}[x(t)] &= \int_{t_0}^{t_f}\frac{\partial L}{\partial X}(X(t),\dot{X}(t))\xi(t)+\frac{\partial L}{\partial \dot{X}}(X(t),\dot{X}(t))\dot{\xi}(t)+O(\xi^3)dt \\ &= \int_{t_0}^{t_f}\left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{X}}(X(t),\dot{X}(t))+\frac{\partial L}{\partial X}(X(t),\dot{X}(t))\right]\xi(t)dt-\frac{\partial L}{\partial \dot{X}}(X_0,\dot{X}_0)\xi(t_0)+O(\xi^3) . \end{aligned} \tag{2.14}$$

So the Euler-Lagrange equation of motion appears in the phase of the decoherence function.

One should not put too much weight on this occurrence. This system is not decoherent; substantial interference can still occur between different possible trajectories. There is no particular reason to expect $\xi(t)$ to be small, so it is not correct to neglect higher-order terms. This system, on its own, is still essentially quantum mechanical. It is not even quasiclassical.

This still leaves the effects of the reservoir variables and interaction unaccounted for. Let us turn, then, to this portion of the decoherence functional:

$$\begin{aligned} F[x'(t),x(t)] &= \int \delta Q' \int \delta Q\, \delta(Q'(t_f)-Q(t_f))\exp\left\{i\left[S_{\rm res}[Q'(t)]-S_{\rm res}[Q(t)]-\int_{t_0}^{t_f} V(x'(t),Q'(t)) - V(x(t),Q(t))dt\right]/\hbar\right\}\chi(Q_0',Q_0) \\ &= \exp\{iW[x'(t),x(t)]/\hbar\} . \end{aligned} \tag{2.15}$$

$F[x'(t),x(t)]$ is termed the *influence functional* by Feyman and Vernon, and $W[x'(t),x(t)]$ is the *influence phase* [6]. In our simplified model, this is not difficult to evaluate exactly. It is generally assumed that the initial density matrix is in a thermal state. We quote the results of Feynman and Vernon:

$$W[x'(t),x(t)]=\tfrac{1}{2}\int_{t_0}^{t_f} dt \int_{t_0}^{t} ds[x'(t)-x(t)][k(t-s)x'(s)+k(t-s)x(s)] , \tag{2.16}$$

where the real and imaginary parts of $k(t-s)$ are

$$k_R(t-s)=\sum_k \frac{\gamma_k^2}{m\omega_k}\sin[\omega_k(t-s)] , \tag{2.17a}$$

$$k_I(t-s)=\sum_k \frac{\gamma_k^2}{m\omega_k}\coth(\hbar\omega_k/kT)\cos[\omega_k(t-s)] . \tag{2.17b}$$

Changing to our variables X and ξ, we see that

$$W[X(t),\xi(t)]=\sum_k \frac{\gamma_k^2}{2m\omega_k}\int_{t_0}^{t_f}dt\int_{t_0}^{t}ds\left\{2\xi(t)X(s)\sin[\omega_k(t-s)]+i\xi(t)\xi(s)\coth\left[\frac{\hbar\omega_k}{kT}\right]\cos[\omega_k(t-s)]\right\}. \tag{2.18}$$

Thus, we have a real term which is proportional to $\xi(t)$ and an imaginary term which is proportional to $\xi(t)\xi(s)$. The imaginary term is a double integral over a symmetric kernel whose eigenvalues are strictly non-negative; thus, for large ξ the decoherence functional will be diminished by a decaying exponential

$$\exp\left[-\int_{t_0}^{t_f}dt\int_{t_0}^{t}ds\,\xi(t)\xi(s)\cos[\omega_k(t-s)]\right].$$

Since ξ essentially measures how far you are from the diagonal of the decoherence functional, the off-diagonal terms tend to vanish and the system becomes decoherent.

Furthermore, since large ξ is suppressed, it now makes sense to discard terms of $O(\xi^3)$. Thus we can now say

$$\begin{aligned}S_{\rm sys}[x'(t)]-S_{\rm sys}[x(t)]+W[x'(t),x(t)]&\\ =\frac{i}{4}\int_{t_0}^{t_f}dt\int_{t_0}^{t_f}ds\,\xi(t)k_I(t-x)\xi(s)&\\ +\int_{t_0}^{t_f}dt\,\xi(t)e(t)+O(\xi^3),&\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}e(t)=-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot X}(t)\right]+\frac{\partial L}{\partial X}(t)&\\ -\sum_k\frac{\gamma_k^2}{m\omega_k}\int_{t_0}^{t}ds\,X(s)\sin[\omega_k(t-s)].&\end{aligned} \tag{2.20}$$

If we compare this to (2.6), we see that

$$e(t)=0$$

is identical to the ensemble-averaged classical equation of motion. Note that the bath of harmonic oscillators acts as a retarded force on the system. In the limit as we go to a continuum of oscillator frequencies with a high cutoff, this retarded force becomes a dissipative term, i.e., a frictional force. In this limit, Caldeira and Leggett show that for a Debye distribution of oscillator frequencies, the influence phase becomes [8]

$$W[X,\xi]=\int_{t_0}^{t_f}\left[-2\Gamma\dot X\xi(t)+\frac{ikT}{\hbar}\Gamma\xi^2(t)\right]dt, \tag{2.21}$$

where Γ is the usual classical coefficient of friction, defined in terms of γ and the cutoff frequency Ω. See [7,8] for details.

We have seen that the real term of $W[X,\xi]$ corresponds to the classical retarded or (in the limit) dissipative force. The imaginary term also has a classical analog. In the classical case, there is a random stochastic force $F(t)$ given by (2.7), which ensemble averages to zero $\langle F(t)\rangle=0$. As we see in (2.10b), however, the two-time correlation function of this force does not vanish. As $\hbar\to0$, we get $\coth(\hbar\omega/2kT)\to2kT/\hbar\omega$. So the imaginary part of $W[X,\xi]$ has the form

$$\mathrm{Im}W[X(t),\xi(t)]=\int_{t_0}^{t_f}dt\int_{t_0}^{t}ds\,\langle F(t)F(s)\rangle\xi(t)\xi(s). \tag{2.22}$$

Here we observe the subtle linkage between noise, dissipation, and decoherence. In interacting with the many degrees of freedom of the reservoir, the system loses energy. It also is subject to random jostlings from the reservoir oscillators. But one last, purely quantum-mechanical effect is that the state of the system is continually being "measured," and thus the various possible trajectories tend to decohere, at least on a scale large compared to $\hbar$. Later we will see that even in situations where the classical noise vanishes, there is still quantum-mechanical noise. This arises essentially from the zero-point energy of the reservoir oscillators.

We can straightforwardly generalize to the case where the potential is nonlinear in x, but still linear in Q. Suppose that

$$V(x,Q)=-\sum_k a_k(x)Q^k. \tag{2.23}$$

Here the influence phase is

$$\begin{aligned}W[x'(t),x(t)]&\\ =\sum_k\int_{t_0}^{t_f}dt\int_{t_0}^{t}ds\{[a_k(x'(t))-a_k(x(t))]&\\ \times[a_k(x'(s))k_k(t-s)&\\ -a_k(x(s))k_k^*(t-s)]\},&\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}k_k(t-s)=\frac{1}{2m\omega_k}\{\sin[\omega_k(t-s)]&\\ +i\coth(\hbar\omega_k/2kT)\cos[\omega_k(t-s)]\}.&\end{aligned} \tag{2.25}$$

We can again separate the real and imaginary parts, and change to variables X and ξ. We then get, to $O(\xi^3)$,

$$W[X(t),\xi(t)]=-\frac{1}{2m}\sum_k\int_{t_0}^{t_f}dt\int_{t_0}^{t}ds\{a_k'(X(t))a_k(X(s))\xi(t)\sin[\omega_k(t-s)] -i\coth\left[\frac{\hbar\omega_k}{2kT}\right]a_k'(X(t))a_k'(X(s))\xi(t)\xi(s)\cos[\omega_k(t-s)]\}\ . \tag{2.26}$$

Again, we see that the real term has the same form as the classical retarded force, which becomes dissipative in the limit of continuous frequencies and high cutoff. The imaginary term again corresponds to a double integral over the two-time correlation function of the classical stochastic force. It is strictly non-negative, and exponentially damps the decoherence functional for large ξ.

III. NONLINEAR EXAMPLES

The problem with potentials nonlinear in Q is that the path integral is no longer solvable in closed form. Thus, it is difficult to be certain that this correspondence with the classical equation of motion which holds in the linear case is truly universal. We can, however, consider weak couplings, and solve for the equation of motion using perturbation theory. We can then compare the classical perturbative equation to that derived from the influence functional.

A. Classical and quantum perturbation theory

Let us consider a system coupled to a bath of harmonic oscillators with a potential of the form

$$V(x,Q)=-\varepsilon\sum_k V_k(x,Q^k)\ , \tag{3.1}$$

where $V_k(x,Q^k)$ can be nonlinear in x and Q^k. In general, such a problem cannot be solved exactly. However, if the coupling is weak ($\varepsilon\ll 1$), then we can make a perturbation expansion, at least for reasonably well-behaved potentials.

The total Lagrangian is

$$L_{\text{total}}(x,\dot{x},Q,\dot{Q})=L(x,\dot{x})+\frac{m}{2}\sum_k[(\dot{Q}^k)^2-\omega_k^2(Q^k)^2] -\varepsilon V_k(x,Q^k)\ . \tag{3.2}$$

Let us suppose that the trajectory $x(t)$ is known. Then the equation of motion for the kth harmonic oscillator is

$$\frac{d^2Q^k}{dt^2}=-\omega_k^2Q^k+\frac{\varepsilon}{m}\frac{\partial V_k}{\partial Q}(x(t),Q^k)\ . \tag{3.3}$$

If we then write Q^k as an expansion

$$Q^k(t)=Q_0^k(t)+\varepsilon Q_1^k(t)+\varepsilon^2Q_2^k(t)+\cdots \tag{3.4}$$

and equate equal powers of ε, we get a series of equations

$$\frac{dQ_0^k}{dt}=-\omega_k^2Q_0^k\ , \tag{3.5a}$$

$$\frac{dQ_1^k}{dt}=-\omega_k^2Q_1^k+\frac{1}{m}\frac{\partial V_k}{\partial Q}(x(t),Q_0^k(t))\ , \tag{3.5b}$$

$$\frac{dQ_2^k}{dt}=-\omega_k^2Q_2^k+\frac{1}{m}Q_1^k(t)\frac{\partial^2V_k}{\partial Q^2}(x(t),Q_0^k(t))\ , \tag{3.5c}$$

etc., where we have Taylor expanded $V_k(x,Q_0^k+\varepsilon Q_1^k+\cdots)$ in powers of ε.

Now we have equations for each $Q_i^k(t)$ in terms of the lower-order functions. Notably, the lowest-order equation is now a simple harmonic oscillator, and we can solve for it easily in terms of the initial conditions

$$Q_0^k(t)=A_k\cos(\omega_kt)+B_k\sin(\omega_kt)\ , \tag{3.6}$$

where $A_k=Q^k|_{t_0}$ and $B_k=(1/\omega_k)(dQ^k/dt)|_{t_0}$.

The higher-order equations are driven oscillators. We can solve for them exactly, matching initial conditions:

$$Q_1^k(t)=\frac{1}{m\omega_k}\int_{t_0}^{t}\sin[\omega_k(t-s)]\frac{\partial V_k}{\partial Q}(x(s),Q_0^k(s))ds\ , \tag{3.7a}$$

$$Q_2^k(t)=\frac{1}{m\omega_k}\int_{t_0}^{t}\sin[\omega_k(t-s)]\times\frac{\partial^2V_k}{\partial Q^2}(x(s),Q_0^k(s))Q_1^k(s)ds\ , \tag{3.7b}$$

and so forth.

Having found the motion of the harmonic oscillators in terms of $x(t)$, we can now turn around and find the equation of motion for x. This is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}(t)=\frac{\partial L}{\partial x}(t)+\varepsilon\sum_k\frac{\partial V_k}{\partial x}(x(t),Q^k(t))\ . \tag{3.8}$$

$Q^k(t)$ is the expansion that we solved for above, and it will depend on the earlier behavior of x, in general. Note that causality is strictly obeyed. This classical causality follows as a result of more fundamental quantum causality, as discussed by Gell-Mann and Hartle [5].

We treat this same problem quantum mechanically by trying to find the influence functional $F[x'(t),x(t)]$ as a perturbation expansion. Assume that the reservoir starts in a definite initial state $|a\rangle$, with wave function $\phi_a(Q)$. Then

$$F_a[x',x]=\int\delta(Q'(t_f)-Q(t_f))\exp\left\{\frac{i}{\hbar}\{S_{\rm res}[Q'(t)]-S_{\rm res}[Q(t)]\}\right\}$$
$$\times\left[1+\frac{i\varepsilon}{\hbar}\int_{t_0}^{t_f}[V(x'(t),Q'(t))-V(x(t),Q(t))]dt\right.$$
$$+\left(\frac{i\varepsilon}{\hbar}\right)^2\int_{t_0}^{t_f}\int_{t_0}^{t}[V(x'(t),Q'(t))-V(x(t),Q(t))]$$
$$\left.\times[V(x'(s),Q'(s))-V(x(s),Q(s))]ds\,dt+\cdots\right]\phi_a[Q'(t_0)]\phi_a^*[Q(t_0)]\delta Q'\delta Q$$
$$=1+\frac{i\varepsilon}{\hbar}\int_{t_0}^{t_f}\{V_{aa}[x'(t)]-V_{aa}[x(t)]\}dt$$
$$-\left(\frac{\varepsilon^2}{\hbar^2}\right)\sum_b\int_{t_0}^{t_f}\int_{t_0}^{t}\{V_{ab}[x'(t)]V_{ba}[x'(s)]e^{-i\omega_{ba}(t-s)}-V_{ab}[x(t)]V_{ba}[x'(s)]e^{-i\omega_{ba}(t-s)}$$
$$-V_{ba}[x'(t)]V_{ab}[x(s)]e^{i\omega_{ba}(t-s)}+V_{ba}[x(t)]V_{ab}[x(s)]e^{i\omega_{ba}(t-s)}\}ds\,dt+\cdots$$
$$=1+\frac{i\varepsilon}{\hbar}\int_{t_0}^{t_f}V'_{aa}[X(t)]\xi(t)dt-\left(\frac{\varepsilon^2}{\hbar^2}\right)\sum_b\int_{t_0}^{t_f}\int_{t_0}^{t}\{V'_{ba}[X(t)]V'_{ab}[X(s)]\xi(t)\xi(s)\cos[\omega_{ba}(t-s)]$$
$$-2iV'_{ba}[X(t)]V_{ab}[X(s)]\xi(t)\sin[\omega_{ba}(t-s)]\}ds\,dt$$
$$+O(\varepsilon^3)+O(\xi^3)\,. \tag{3.9}$$

Here we have defined the functions

$$V_{aa}(x)=\langle a|V(x,\hat{Q})|a\rangle=\int\phi_a(Q)\phi_a^*(Q)V(x,Q)dr\,, \tag{3.10a}$$

$$V_{ba}(x)=\langle b|V(x,\hat{Q})|a\rangle=\int\phi_a(Q)\phi_b^*(Q)V(x,Q)dr\,. \tag{3.10b}$$

In our case, we assume that the reservoir is a collection of harmonic oscillators initially in a thermal state. In this case, the states $|a\rangle$ become the ordinary Fock states $|n\rangle$ and the influence functional is

$$F[x'(t),x(t)]=\sum_n\rho_{nn}F_n[x'(t),x(t)]\,, \tag{3.11}$$

where

$$\rho_{nn}=\prod_k[1-\exp(-\hbar\omega_k/kT)]\exp(-n_k\hbar\omega_k/kT)\,. \tag{3.12}$$

B. Polynomial potentials

We will specifically consider a potential of the form (3.1) where the individual potentials are polynomials in Q^k. We will see that it is convenient to separate the even and odd terms:

$$V_k(x,Q^k)=\sum_{l=0}^{N}a_{kl}(x)(Q^k)^{2l+1}+\sum_{l=1}^{N}b_{kl}(x)(Q^k)^{2l}\,. \tag{3.13}$$

The $a_{kl}(x)$ and $b_{kl}(x)$ are arbitrary functions of x, only assuming that the potential as a whole remains relatively well behaved, integrable, etc. For convenience, I will drop the index k for the rest of this derivation. It should be understood that the final result is to be summed over all the oscillators:

$$W[x'(t),x(t)]=\sum_k W_k[x'(t),x(t)]\,. \tag{3.14}$$

From the Eq. (3.5a), we can write down the equations of motion for a classical oscillator $Q(t)=Q_0(t)+\varepsilon Q_1(t)+\cdots$. We then plug in the solutions (3.6) and (3.7a) to get

$$Q_0(t)=A\cos(\omega t)+B\sin(\omega t)=\beta e^{i\omega t}+\beta^* e^{-i\omega t}\,, \tag{3.15a}$$

$$Q_1(t)=\frac{1}{m\omega}\int_{t_0}^{t}\sin[\omega(t-s)]$$
$$\times\left[\sum_{j=0}^{N}(2j+1)a_j[x(s)]Q_0^{2j}(s)\right.$$
$$\left.+\sum_{j=1}^{N}2jb_j[x(s)]Q_0^{2j-1}(s)\right]ds\,, \tag{3.15b}$$

etc., where $\beta=(A-iB)/2$. The equation of motion for x is then

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right]=\left[\frac{\partial L}{\partial x}\right]+\varepsilon\left[\sum_{j=0}^{N}a_j'[x(t)]Q_0^{2j+1}(t)+\sum_{j=1}^{N}b_j'[x(t)]Q_0^{2j}(t)\right]$$
$$+\varepsilon^2\left[\sum_{j=0}^{N}(2j+1)a_j'[x(t)]Q_0^{2j}(t)+\sum_{j=1}^{N}2jb_j'[x(t)]Q_0^{2j-1}(t)\right]Q_1(t)+O(\varepsilon^3)$$
$$=\left[\frac{\partial L}{\partial x}\right]+\varepsilon\tau_1(t)+\varepsilon^2\tau_2(t)+O(\varepsilon^3)\ . \tag{3.16}$$

We are interested in the ensemble-averaged equation. We can make use of the fact that

$$\langle\beta^m\beta^{*n}\rangle=\delta_{mn}n!\left[\frac{kT}{2m\omega^2}\right]^n\ . \tag{3.17}$$

So only the even terms contribute to the first-order component of the Eq. (3.16). $Q_0^{2j}(t)$ is readily found then with a binomial expansion of

$$(\beta e^{i\omega t}+\beta^* e^{-i\omega t})^{2j}=\sum_{i=0}^{2j}\begin{bmatrix}2j\\ i\end{bmatrix}(\beta e^{i\omega t})^i(\beta^* e^{-i\omega t})^{2j-i}\ , \tag{3.18}$$

$$\begin{bmatrix}n\\ i\end{bmatrix}=\frac{n!}{i!(n-i)!}\ ,$$

yielding

$$\langle\tau_1(t)\rangle=\sum_{j=1}^{N}\frac{2j!}{j!}b_j'(x(t))\left[\frac{kT}{2m\omega^2}\right]^j\ . \tag{3.19}$$

The second-order component is more complicated. Plugging expression (3.15b) for $Q_1(t)$ into (3.16), doing a binomial expansion for the powers of $Q_0(t)$ and $Q_0(s)$, pairing $e^{mi\omega t}$ and $e^{-mi\omega t}$ terms, and ensemble averaging gives us

$$\langle\tau_2(t)\rangle=\frac{1}{m\omega}\int_{t_0}^{t}\left\{\sum_{k=0}^{N}\sum_{i,j=k}^{N}\sin[(2k+1)\omega(t-s)]C_{ijk}(t,s)\right.$$
$$-\sum_{k=1}^{N}\sum_{i,j=k}^{N}\sin[(2k-1)\omega(t-s)]C_{ijk}(t,s)+\sum_{k=1}^{N}\sum_{i,j=k}^{N}\sin[2k\omega(t-s)]D_{ijk}(t,s)$$
$$\left.-\sum_{k=1}^{N}\sum_{i,j=k}^{N}\sin[(2k-2)\omega(t-s)]D_{ijk}(t,s)\right\}ds\ , \tag{3.20}$$

where

$$C_{ijk}(t,s)=a_i'[x(t)]a_j[x(s)](i+j)!(2i+1)(2j+1)\begin{bmatrix}2i\\ i-k\end{bmatrix}\begin{bmatrix}2j\\ j-k\end{bmatrix}\left[\frac{kT}{2m\omega^2}\right]^{i+j}\ , \tag{3.21a}$$

$$D_{ijk}(t,s)=b_i'[x(t)]b_j[x(s)](i+j-1)!4ij\begin{bmatrix}2i-1\\ i-k\end{bmatrix}\begin{bmatrix}2j-1\\ j-k\end{bmatrix}\left[\frac{kT}{2m\omega^2}\right]^{i+j-1}\ . \tag{3.21b}$$

We can collect together and combine those terms with the same sine factor to get

$$\langle\tau_2(t)\rangle=\frac{1}{m\omega}\int_{t_0}^{t}\left\{\sum_{k=0}^{N}\sum_{i,j=k}^{N}\sin[(2k+1)\omega(t-s)]E_{ijk}(t,s)+\sum_{k=0}^{N}\sum_{i,j=k}^{N}\sin[2k\omega(t-s)]F_{ijk}(t,s)\right\}ds\ , \tag{3.22}$$

where

$$E_{ijk}(t,s)=a_i'[x(t)]a_j[x(s)](i+j)!\left[\frac{kT}{2m\omega^2}\right]^{i+j}(2k+1)(i+j+1)\begin{bmatrix}2i+1\\ i-k\end{bmatrix}\begin{bmatrix}2j+1\\ j-k\end{bmatrix}\ , \tag{3.23a}$$

$$F_{ijk}(t,s)=b_i'[x(t)]b_j[x(s)](i+j-1)!\left[\frac{kT}{2m\omega^2}\right]^{i+j-1}2k(i+j)\begin{bmatrix}2i\\ i-k\end{bmatrix}\begin{bmatrix}2j\\ j-k\end{bmatrix}\ . \tag{3.23b}$$

We are also interested in the correlation function $\langle F(t)F(s)\rangle$, where $F(t)$ is the force due to the interaction with the reservoir. To second order this is

$$\langle F(t)F(s)\rangle=\varepsilon^2\left\{\sum_{k=0}^{N}\sum_{i,j=k}^{N}2\cos[(2k+1)\omega(t-s)]G_{ijk}(t,s)+\sum_{k=1}^{N}\sum_{i,j=k}^{N}2\cos[2k\omega(t-s)]H_{ijk}(t,s)\right.$$
$$\left.+\sum_{i,j=1}^{N}H_{ij0}(t,s)\right\}+O(\varepsilon^3)\ , \tag{3.24}$$

$$G_{ijk}(t,s)=a_i'[x(t)]a_j'[x(s)](i+j+1)!\left[\frac{kT}{2m\omega^2}\right]^{i+j+1}\begin{bmatrix}2i+1\\ i-k\end{bmatrix}\begin{bmatrix}2j+1\\ j-k\end{bmatrix}\ , \tag{3.25a}$$

$$H_{ijk}(t,s)=b_i'[x(t)]b_j'[x(s)](i+j)!\left[\frac{kT}{2m\omega^2}\right]^{i+j}\begin{bmatrix}2i\\ i-k\end{bmatrix}\begin{bmatrix}2j\\ j-k\end{bmatrix}\ . \tag{3.25b}$$

We can subtract off-the-average values to get

$$\langle F(t),F(s)\rangle=\langle F(t)F(s)\rangle-\langle F(t)\rangle\langle F(s)\rangle\ , \tag{3.26}$$

where $\langle F(t)\rangle$ is the first-order ensemble-averaged force from (3.19).

We can compare this result to that obtained from our quantum-mechanical procedure. Suppose that the reservoir begins in a definite state $|n\rangle$. Then the influence functional is given by (3.9),

$$F_n[X(t),\xi(t)]=1+\varepsilon\alpha_{n1}[X(t),\xi(t)]+\varepsilon^2\alpha_{n2}[X(t),\xi(t)]+\ldots\ , \tag{3.27}$$

and in the thermal case by (3.11),

$$F[X(t),\xi(t)]=1+\varepsilon\alpha_1[X(t),\xi(t)]+\varepsilon^2\alpha_2[X(t),\xi(t)]+\cdots=\sum_n\rho_{nn}F_n[X(t),\xi(t)]\ , \tag{3.28}$$

where

$$\alpha_i[X(t),\xi(t)]=\sum_n\rho_{nn}\alpha_{ni}[X(t),\xi(t)]\ . \tag{3.29}$$

The influence phase is then

$$W[X(t),\xi(t)]=-i\hbar\ln F[X(t),\xi(t)]=-i\hbar\varepsilon\alpha_1[X(t),\xi(t)]-i\hbar\varepsilon^2\{\alpha_2[X(t),\xi(t)]-\tfrac{1}{2}\alpha_1^2[X(t),\xi(t)]\}+\cdots\ . \tag{3.30}$$

From (3.9), then, we see that we must find an expression for $\langle m|r^l|n\rangle$. This will, in general, be a polynomial in n, for certain values of m, and zero for the rest. In comparing to the classical result, we need keep only the highest power of n, since the lower powers will be higher order in $\hbar\omega/kT$ as we let $\hbar\to0$. This will be

$$\langle m|r^l|n\rangle=\begin{bmatrix}l\\ k\end{bmatrix}\left[\frac{\hbar}{2m\omega}\right]^{l/2}n^{l/2}+\ldots\ ,\quad m=n+l-2k,2k\le l\ , \tag{3.31}$$

$$\langle m|r^l|n\rangle=\begin{bmatrix}l\\ k\end{bmatrix}\left[\frac{\hbar}{2m\omega}\right]^{l/2}m^{l/2}+\ldots,\quad m=n-l+2k,2k\le l\ , \tag{3.32}$$

and zero otherwise.

We can then use the fact that, as $\hbar\to0$,

$$\sum_n\rho_{nn}n^l\approx l!\left[\frac{kT}{\hbar\omega}\right]^l\ . \tag{3.33}$$

Thus from Eq. (3.9) we get

$$\alpha_1[X(t),\xi(t)]=\sum_n\rho_{nn}\frac{i}{\hbar}\int_{t_0}^{t_f}\sum_j b_j'[X(t)]\langle n|r^{2n}|n\rangle\xi(t)dt=\frac{i}{\hbar}\int_{t_0}^{t_f}\sum_j b_j'[X(t)]\xi(t)\frac{2j!}{j!}\left[\frac{kT}{2m\omega^2}\right]^j dt\ , \tag{3.34}$$

which agrees exactly with the first-order term in the classical equation of motion (3.19).

Similarly, we can calculate the second-order term to get

$$\alpha_2[X(t),\xi(t)]=-\frac{1}{\hbar^2}\left\{\sum_{k=0}^{N}\sum_{i,j=k}^{N}\int_{t_0}^{t_f}\int_{t_0}^{t}2\cos[(2k+1)\omega(t-s)]G_{ijk}(t,s)\xi(t)\xi(s)ds\,dt\right.$$
$$+\sum_{k=1}^{N}\sum_{i,j=k}^{N}\int_{t_0}^{t_f}\int_{t_0}^{t}2\cos[2k\omega(t-s)]H_{ijk}(t,s)\xi(t)\xi(s)ds\,dt$$
$$+i\sum_{k=1}^{N}\sum_{i,j=k}^{N}\int_{t_0}^{t_f}\int_{t_0}^{t}\sin[(2k+1)\omega(t-s)]E_{ijk}(t,s)\xi(t)ds\,dt$$
$$+i\sum_{k=1}^{N}\sum_{i,j=k}^{N}\int_{t_0}^{t_f}\int_{t_0}^{t}\sin[2k\omega(t-s)]F_{ijk}(t,s)\xi(t)ds\,dt$$
$$\left.+\sum_{i,j=1}^{N}\int_{t_0}^{t_f}\int_{t_0}^{t}H_{ij0}(t,s)\xi(t)\xi(s)ds\,dt\right\}. \tag{3.35}$$

Here we have used the same definitions of E_{ijk}, etc., where the classical system variable x has become the quantum variable X.

We can clearly see from this the exact correspondence with the classical equation of motion, at least to second order in ε. The real part of $W[X(t),\xi(t)]$ is just an integral of the classical retarded force, just as in the linear case, and the imaginary part consists of a double integral

$$\int_{t_0}^{t_f}\int_{t_0}^{t}[\langle F(t)F(s)\rangle-\langle F(t)\rangle\langle F(s)\rangle]\xi(t)\xi(s)ds\,dt\ ; \tag{3.36}$$

note that the $-\langle F(t)\rangle\langle F(s)\rangle$ comes from subtracting $\alpha_1^2/2$ from the second-order term. Again, we note the non-negativity of this imaginary part; the presence of noise both makes the behavior unpredictable and causes different trajectories to decohere. So we see that in perturbation theory, the nonlinear problem has exactly the same classical correspondence as the linear problem.

IV. MORE GENERAL CASES

Though the above discussion is fairly general, it leaves unexamined the far broader range of possible strong, nonlinear interactions, as well as the possibilities of nonoscillator reservoirs. This is, of course, a product of computational convenience, as it is very difficult to get analytical answers in other cases. Are there any arguments that can be made for more general systems?

In any case where the action can be decomposed,

$$S[x(t),Q(t)]=S_{\rm sys}[x(t)]+S_{\rm res}[Q(t)]+S_{\rm int}[x(t),Q(t)]\ , \tag{4.1}$$

it is possible formally to write the decoherence functional in the form

$$D[x(t),x'(t)]=\exp\left[\frac{i}{\hbar}\{S_{\rm sys}[x(t)]-S_{\rm sys}[x'(t)]+W[x(t),x'(t)]\}\right]. \tag{4.2}$$

If we restrict ourselves, for the moment, to systems in a factorizable pure state,

$$\rho(x',Q';x,Q)=\Psi^*(x')\Psi(x)\Phi^*(Q')\Phi(Q)\ , \tag{4.3}$$

then the influence phase is defined simply by (2.15):

$$\exp\{iW[x'(t),x(t)]/\hbar\}=\int\delta Q'\int\delta Q\,\delta[Q'(t_f)-Q(t_f)]$$
$$\times\exp\frac{i}{\hbar}\{S_{\rm res}[Q'(t)]-S_{\rm res}[Q(t)]+S_{\rm int}[x'(t),Q'(t)]-S_{\rm int}[x(t),Q(t)]\}$$
$$\times\Phi^*(Q_0')\Phi(Q_0)\ . \tag{4.4}$$

By bringing the integral over the final condition Q_f,Q_f' to the front, we can rewrite this as a product of two path integrals:

$$\exp\{iW[x'(t),x(t)]/\hbar\} = \int\int dQ_f dQ'_f \delta(Q'(t_f)-Q(t_f)) \times \left[\int \delta Q' \exp\left[\frac{i}{\hbar}\{S_{\rm res}[Q'(t)]+S_{\rm int}[x'(t),Q'(t)]\}\Phi^*(Q'_0\right]\right] \times \left[\int \delta Q \exp\left[\frac{i}{\hbar}\{-S_{\rm res}[Q(t)]-S_{\rm int}[x(t),Q(t)]\}\Phi(Q_0)\right]\right]$$
$$= \int\int dQ_f dQ'_f \delta(Q'(t_f)-Q(t_f))\Phi^*_{x'(t)}(Q'_f)\Phi_{x(t)}(Q_f)$$
$$= \int \Phi^*_{x'(t)}(Q_f)\Phi_{x(t)}(Q_f)dQ_f = \langle \Phi_{x'(t)}|\Phi_{x(t)}\rangle \ , \tag{4.5}$$

where $|\Phi_{x(t)}\rangle$ and $|\Phi_{x'(t)}\rangle$ are the states that $|\Phi\rangle$ will evolve into under the influence of the interaction, given the trajectories $x(t)$ and $x'(t)$, respectively.

Clearly, $\langle \Phi_{x'(t)}|\Phi_{x(t)}\rangle \leq 1$, which implies equally clearly that $\mathrm{Im}W[x'(t),x(t)]\geq 0$. So the non-negativity that we saw in cases I and II above is generally true. This is also clearly the case for mixed states, since we can represent any mixed state as

$$\rho(x',Q';x,Q)=\sum_i p_i \Psi^*_i(x')\Psi_i(x)\Phi^*_i(Q')\Phi_i(Q) \ , \tag{4.6}$$

where

$$\sum_i p_i = 1, \quad p_i \geq 0 \ ; \tag{4.7}$$

so if the $F_i[x'(t),x(t)]<1$, then clearly

$$\exp\{iW[x'(t),x(t)]/\hbar\}=\sum_i p_i F_i[x'(t),x(t)]\leq 1 \ , \tag{4.8}$$

and $\mathrm{Im}W[x'(t),x(t)]\geq 0$ still holds. Also, $\mathrm{Im}W[x'(t),x(t)]=0$ for $x'(t)=x(t)$. Thus, without assuming *anything* about the interaction *or* the reservoir, we see that there will be a maximum at $\xi(t)=0$, and that the off-diagonal $\xi(t)\neq 0$ terms will tend to be suppressed. This is not surprising, as one expects almost any sort of interaction with neglected degrees of freedom to result in the loss of phase coherence. However, it does show how these highly simplified models might actually demonstrate behavior important to the rise of classical physics from quantum mechanics in physical systems.

For example, in considering quantum gravity, decoherence might arise from neglected gravitational degrees of freedom. The usual semiclassical treatment of quantum gravity, which omits the "back action" of mass energy on the curvature of space-time, cannot exhibit this effect. The weakness of the gravitational interaction would in general make it less important in causing decoherence than stronger forces, such as electromagnetism; but it might well become important in quantum cosmology.

There are, of course, still questions. All that has been demonstrated is the non-negativity of $\mathrm{Im}W[x'(t),x(t)]$. Can there not be zeros for some choice of $\xi(t)\neq 0$? And *how strongly,* in general, are the off-diagonal terms suppressed?

There can certainly be zeros for nonzero $\xi(t)$ in some cases. Indeed, if we consider the form of $\mathrm{Im}W[x'(t),x(t)]$ for the linear case

$$\mathrm{Im}W[x'(t),x(t)]\sim \int_{t_0}^{t_f}\int_{t_0}^{t_f}\xi(t)\xi(s)\cos[\omega(t-s)]ds\,dt \tag{4.9}$$

(for a one-oscillator "reservoir" of frequency ω), there are an infinite number of choices of $\xi(t)$ which make this zero. Thus, one cannot call this system truly decoherent. However, as the number of oscillator frequencies is increased, the number of possible choices of $\xi(t)$ is further and further restricted, so that as the reservoir becomes infinite only $\xi(t)=0$ remains. One would expect similar behavior in the more general case. While it is certainly possible to construct cases where $\mathrm{Im}W[x'(t),x(t)]$ has many zeros even for a very large reservoir, in practice one expects $\mathrm{Im}W[x'(t),x(t)]>0$ for $x(t)\neq x'(t)$, as the degrees of freedom of the reservoir are increased.

Similarly, the strength with which off-diagonal terms will be suppressed depends on the details of the system. However, one would expect that $|\Phi_{x(t)}\rangle$ and $|\Phi_{x'(t)}\rangle$ differ more in the case of strong interactions than small, and hence that $\langle \Phi_{x'(t)}|\Phi_{x(t)}\rangle$ would be more strongly suppressed in general.

V. CONCLUSIONS

It is clear that it is possible to define a "classical" equation of motion directly from the underlying quantum theory, and that, at least in many cases, this corresponds closely to the equation obtained from the classical theory. While correspondences of this sort have often been demonstrated in the past, never before has there been a rigorous, *a priori* technique for deriving them.

Using the formalism of Gell-Mann and Hartle, we can now see classical physics as, very simply, a limit of the underlying quantum theory; and we can systematically determine, at least in principle, the deviations from strict classical equations due to quantum effects. Using the decoherence functional as a criterion for determining if an effect is experimentally observable, we can once and for all avoid the problem of collapsing the wave function; there is no longer any need for an independent "classical

realm" of measurement.

Note added. After the completion of this research, I learned that Bei Lok Hu, Juan Pablo Paz, and Yuhong Zhang had studied a very similar class of nonlinear Brownian systems more or less simultaneously with me [10]. While their study is from a considerably different point of view, with very different goals, being chiefly concerned with deriving master equations and treating the thermodynamics of these generalized systems, their results overlap mine to a certain extent.

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