Gravitationally collapsing dust in 2 + 1 dimensions

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We investigate the circumstances under which gravitationally collapsing dust can form a black hole in three-dimensional spacetime.

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General relativity in 2+1 dimensions has been a source of fascination for theorists in recent years, primarily because of the potential insights into quantum gravity that it offers. One has the full structure of the Einstein equations but with an enormous amount of technical simplification due to the smaller number of dimensions. One consequence of this simplicity is that the metric outside of a finite matter source is locally flat [1, 2], and the mass affects the spacetime only globally, seemingly implying that there are no black hole solutions to 2 + 1 gravity.

However, it has recently been pointed out by Bañados *et al.* [3] that if a negative cosmological constant is introduced, so that the field equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{1}$$

where $\Lambda > 0$, then there is a solution for the field around a point source which has an event horizon, i.e., a black hole. For zero angular momentum this is

$$ds^2 = -(\Lambda R^2 - M)dT^2 + \frac{dR^2}{\Lambda R^2 - M} + R^2 d\theta^2. \tag{2}$$

Apart from not being asymptotically flat, this solution exhibits many of the properties of black holes in four dimensions, such as a well-defined temperature and entropy. It is therefore useful in that it provides another opportunity to model classical and quantum dynamics of black holes with a simpler set of field equations.

Here we investigate under what circumstances a disk of pressureless dust [the three-dimensional (3D) analogue of Oppenheimer-Snyder collapse] will collapse to the black hole solution (2). Provided the initial density is sufficiently large, we show that collapse occurs in finite proper time, and external observers see the event horizon form in infinite coordinate time. The other properties of this collapse parallel the results in four dimensions, as well as recent results in two dimensions [4].

Collapsing dust solutions in 3D have been studied before [2]. We therefore extend these results further, reproducing the results of Ref. [2] when $\Lambda = 0$, and finding new solutions for $\Lambda < 0$. In this latter case we find that collapse to a point source is possible, subject to certain conditions.

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Consider a disk of collapsing dust surrounded by a vacuum region, with the metric in the exterior region being given by (2). The dust is falling freely, so we may make it the basis of a comoving coordinate system. We then have a Robertson-Walker metric on the interior region:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right), \tag{3}$$

where r and θ are comoving radial and angular coordinates, t is the proper time of the dust, and $a(t)$ is the scale factor. In these coordinates $T_{\mu\nu} = \rho u_\mu u_\nu$ is the stress energy of the dust, where $\rho(t)$ is the density of the dust and $u_\mu = (1,0,0)$. Conservation of stress-energy $T^{\mu\nu}_{;\nu} = 0$ then implies $\rho a^2 = \rho_0 a_0^2$, where ρ_0 is the initial density of the dust and a_0 is the initial scale factor. The field equations (1) become

$$\ddot{a} = -\Lambda a \tag{4}$$

and

$$\Lambda a^2 + k - 8\pi G\rho_0 a_0^2 + \dot{a}^2 = 0, \tag{5}$$

where the overdot denotes d/dt.

The solution of these equations is

$$a(t) = a_0 \cos(\sqrt{\Lambda}t) + \frac{\dot{a}_0}{\sqrt{\Lambda}}\sin(\sqrt{\Lambda}t), \tag{6}$$

where

$$\dot{a}_0 = \sqrt{8\pi G\rho_0 a_0^2 - k - \Lambda a_0^2} \tag{7}$$

to satisfy the second field equation. As we wish $a(t)$ to be real, we must require

$$8\pi G\rho_0 a_0^2 - k - \Lambda a_0^2 \geq 0. \tag{8}$$

In particular, if we choose the initial conditions $a_0 = 1, \dot{a}_0 = 0$, this relation gives

$$k = 8\pi G\rho_0 - \Lambda. \tag{9}$$

Subject to the condition (8), this solution always collapses to $a(t_c) = 0$ in the finite proper time

$$t_c = \frac{1}{\sqrt{\Lambda}}\arctan\left[\left(\frac{8\pi G\rho_0 a_0^2 - k - \Lambda a_0^2}{\Lambda a_0^2}\right)^{-\frac{1}{2}}\right]. \tag{10}$$

In the exterior coordinates, the stress-energy vanishes, and the solution is the black hole (2). The dust edge is taken to be at $r = r_0$ in the interior coordinates, and at $R = \mathcal{R}(t)$ in the exterior coordinates. We now wish to impose conditions to make the dust edge a boundary surface. These are [5, 6]

$$[g_{ij}] = 0 \quad \text{and} \quad [K_{ij}] = 0, \tag{11}$$

where $[\Psi]$ denotes the discontinuity in Ψ across the edge, K_{ij} is the extrinsic curvature of the dust edge, and the subscripts i, j refer to the coordinates on the dust edge. The metric on the edge is

$$ds^2 = -dt^2 + \mathcal{R}^2(t)d\theta^2, \tag{12}$$

so the condition (11) implies

$$-(\Lambda\mathcal{R}^2 - M)\dot{T}^2 + \frac{\dot{\mathcal{R}}^2}{\Lambda\mathcal{R}^2 - M} = -1, \tag{13}$$

where the overdot denotes d/dt, which gives

$$\frac{dT}{dt} = \frac{\sqrt{(\Lambda\mathcal{R}^2 - M) + \dot{\mathcal{R}}^2}}{\Lambda\mathcal{R}^2 - M}, \tag{14}$$

and $\mathcal{R}(t) = r_0 a(t)$, that is, the position of the dust edge in the exterior coordinates is the proper distance from the origin to the dust edge. Note also that the latter condition implies that the boundary conditions $a_0 = 1, \dot{a}_0 = 0$ represent a ball of dust with initial radius $\mathcal{R}_0 = r_0$ initially at rest in the exterior coordinates, and we see that $a = 0$ corresponds to the collapse of the dust to $\mathcal{R} = 0$. We now need to compute

$$K_{ij} = -n_\alpha \frac{\delta e^\alpha_{(i)}}{\delta \xi^j}, \tag{15}$$

where n_α is the normal to the edge, $e^\alpha_{(i)}$ are the basis vectors on the edge, and ξ^i are the coordinates on the edge. In the interior coordinates,

$$e^\alpha_{(0)} = (1,0,0), \quad e^\alpha_{(1)} = (0,0,1/r_0 a(t))$$

and (16)

$$n_\alpha = \left(0, \frac{a(t)}{\sqrt{1 - kr_0^2}}, 0\right).$$

A straightforward calculation gives

$$K_{00} = K_{01} = 0, \quad K_{11} = \frac{\sqrt{1 - kr_0^2}}{r_0 a(t)} \tag{17}$$

in the interior coordinates. In the exterior coordinates, we find

$$e^\alpha_{(0)} = (\dot{T}, \dot{\mathcal{R}}, 0), \quad e^\alpha_{(1)} = (0,0,1/\mathcal{R})$$

and (18)

$$n_\alpha = (-\dot{\mathcal{R}}, \dot{T}, 0).$$

It follows that

$$K_{00} = -\frac{d}{d\mathcal{R}}\sqrt{(\Lambda\mathcal{R}^2 - M) + \dot{\mathcal{R}}^2},$$

(19)

$$K_{01} = 0, \quad K_{11} = \frac{\dot{T}(\Lambda\mathcal{R}^2 - M)}{\mathcal{R}}$$

in the exterior coordinates. We thus find that (11) implies

$$(\Lambda\mathcal{R}^2 - M) + \dot{\mathcal{R}}^2 = 1 - kr_0^2; \tag{20}$$

that is,

$$M = (\Lambda a^2 + k + \dot{a}^2)r_0^2 - 1 = 8\pi G\rho_0 a_0^2 r_0^2 - 1, \tag{21}$$

where (5) has been used.[1]

Thus, the requirement that the dust edge be a boundary surface fixes M in terms of the initial density ρ_0. Note that collapse to a black hole occurs only for ρ_0 sufficiently large (as M must be positive), analogous to the $(1+1)$-dimensional case [4, 7]. For $\rho_0 < \frac{1}{8\pi G a_0^2 r_0^2}$, the end point of collapse is a naked conical singularity in anti–de Sitter space.

So long as $\rho_0 > \frac{1}{8\pi G a_0^2 r_0^2}$, an event horizon will form around the collapsing dust at $R_h = \sqrt{M/\Lambda}$. The comoving time t_h at which the event horizon and the dust edge coincide is found by substituting $ra(t_h) = R_h$, which gives

$$t_h = \frac{2}{\sqrt{\Lambda}}\arctan\left(\frac{\dot{a}_0 r_0 + (\dot{a}_0^2 r_0^2 + a_0^2 r_0^2\Lambda - M)^{1/2}}{\sqrt{\Lambda}r_0 a_0 + \sqrt{M}}\right), \tag{22}$$

and is clearly finite. This is not the case for the coordinate time at which an external observer will observe this formation. A light signal emitted from the surface at time T obeys the null condition

$$\frac{dR}{dT} = \Lambda R^2 - M \tag{23}$$

and arrives at a point $\tilde{R}$ at time

$$\begin{aligned}\tilde{T} &= T + \int_{r_0 a(t)}^{\tilde{R}} \frac{dT}{dR}dR \\ &= T - \frac{1}{\sqrt{M\Lambda}}\text{arctanh}\left(\sqrt{\frac{\Lambda}{M}}R\right)\Bigg|_{r_0 a(t)}^{\tilde{R}}\end{aligned} \tag{24}$$

and thus $\tilde{T} \to \infty$ as $r_0 a(t) \to \sqrt{M/\Lambda}$, so the collapse to the event horizon appears to take an infinite amount of time, and the collapse to $\mathcal{R} = 0$ is unobservable from outside.

The comoving time interval dt between emissions of wave crests is equal to the natural wavelength λ that would be emitted in the absence of gravitation, and the interval $d\tilde{T}$ between arrivals of wave crests is the observed wavelength $\tilde{\lambda}$. Thus the redshift of light from the dust edge is

$$z = \frac{d\tilde{T}}{dt} - 1 = \frac{1}{\sqrt{1 - kr_0^2} + r_0\dot{a}(t)} - 1, \tag{25}$$

[1] Note that this identification of the parameter M agrees with that used in [3].

and $r_0\dot{a}(t_h) = -\sqrt{1-kr_0^2}$, so $z \to \infty$ as $t \to t_h$. Thus the collapsing fluid will fade from sight, as the redshift of light from its surface diverges. These properties entirely parallel those of Oppenheimer-Snyder collapse in four dimensions.

If we now consider the case where the cosmological constant is set to zero, we see that the event horizon is no longer present, and collapse occurs to a naked source. This is the scenario investigated in [2]; writing the interior metric in the form (3), the solution of the field equations is

$$a(t) = a_0 + \dot{a}_0 t, \text{ where } \dot{a}_0 = \pm\sqrt{8\pi G\rho_0 a_0^2 - k}, \tag{26}$$

where we must require

$$8\pi G\rho_0 a_0^2 \geq k \tag{27}$$

for a to be real [Eq. (104) of [2]]. This condition is equivalent to the requirement that $\alpha^2 > 0$ in [2]. As pointed out in [2], pressureless dust need not collapse; only if $\dot{a}_0$ is negative will collapse occur in time $t_c = (8\pi G\rho_0 - k/a_0^2)^{-1/2}$. The exterior metric (3) for $\Lambda = 0$ has the appropriate signature only if M is negative, in which case we can convert it into the form in [2],

$$ds^2 = -dT'^2 + dR'^2 + cR'^2 d\theta^2, \tag{28}$$

by taking $T'^2 = -MT^2, R'^2 = -R^2/M$ and $c = -M$. If we choose the dust edge to be at $r = r_0$, $R' = \mathcal{R}(t)$, we may apply the matching conditions (11) to the metrics (26) and (28). The metric on the dust edge is (12), so (11) gives $\sqrt{c}\mathcal{R} = r_0 a(t)$ and

$$\frac{dT'}{dt} = \sqrt{1 + \frac{r_0^2\dot{a}_0^2}{c}}. \tag{29}$$

Imposing the condition that the dust edge move at subluminal speed[2] ($|\frac{dR'}{dT'}| < 1$) yields

$$\frac{r_0^2\dot{a}_0^2}{1 - kr_0^2} < 1, \tag{30}$$

which is easily seen to be equivalent to Eq. (103) of [2] upon using the relationship

$$r = \frac{\sin(\sqrt{k}\chi_0)}{\sqrt{k}} \tag{31}$$

between the two interior metrics. The extrinsic curvature is calculated as before, and (11) implies

$$c = 1 - 8\pi G\rho_0 a_0^2 r_0^2. \tag{32}$$

The condition (30) implies $c < 1 - kr_0^2$ in our coordinates, which is the same as (27).

Finally, we consider the case $\Lambda < 0$ in the field equation (1). This corresponds to a positive cosmological constant in the usual terminology, and we write (1) as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda' g_{\mu\nu}, \tag{33}$$

$\Lambda' > 0$. In the interior metric (3), the solution now becomes

$$a(t) = a_0\cosh(\sqrt{\Lambda'}t) + \dot{a}_0\sinh(\sqrt{\Lambda'}t), \tag{34}$$

where

$$\dot{a}_0 = -\sqrt{8\pi G\rho_0 a_0^2 - k + a_0^2\Lambda'} \tag{35}$$

to satisfy the other field equation. We now see that, for $a(t)$ to be real, we must have

$$8\pi G\rho_0 a_0^2 - k + a_0^2\Lambda' \geq 0. \tag{36}$$

We also see that we must have $\dot{a}_0 < -a_0\sqrt{\Lambda'}$ to achieve collapse to a point. This is not surprising, as the positive cosmological constant is causing an overall expansion of spacetime. When collapse is possible, $a(t_c) = 0$ at

$$t_c = \frac{1}{\sqrt{\Lambda'}}\text{arctanh}\left[\left(\frac{8\pi G\rho_0 a_0^2 - k + \Lambda' a_0^2}{\Lambda' a_0^2}\right)^{-\frac{1}{2}}\right]. \tag{37}$$

The exterior metric in this case is

$$ds^2 = -(-\Lambda' R^2 - M)dT^2 + \frac{dR^2}{-\Lambda' R^2 - M} + R^2 d\theta^2, \tag{38}$$

where M must be negative in order to get the right signature (yielding an exterior de Sitter space metric), so we will write $c = -M$ from here on. There is a cosmological event horizon at $R_h = \sqrt{c/\Lambda'}$. Taking the dust edge to be at $r = r_0$, $R = \mathcal{R}(t) < R_h$, and applying the conditions (11) to the metrics (34,38), we find $\mathcal{R}(t) = r_0 a(t)$,

$$\frac{dT}{dt} = \frac{\sqrt{(\Lambda'\mathcal{R}^2 + c) + \dot{\mathcal{R}}^2}}{\Lambda'\mathcal{R}^2 + c}, \tag{39}$$

and

$$c = 1 - (-\Lambda' a^2 + k + \dot{a}^2)r_0^2 = 1 - 8\pi G\rho_0 a_0^2 r_0^2 \tag{40}$$

as before. From (36), we see that $c < 1 - (-\Lambda' a_0^2 + k)r_0^2$ to satisfy this condition. The collapse condition $\dot{a}_0 < -a_0\sqrt{\Lambda'}$ implies $c < 1 - kr_0^2$, which is more restrictive.

As the collapse in this case is to a naked conical singularity, we should again require

$$\left|\frac{dR'}{dT'}\right| < 1; \tag{41}$$

that is, the velocity of the dust edge in the exterior coordinates of [2] should be less than the speed of light. Thus, we require that

$$\left|\frac{r_0\dot{a}(1 - kr_0^2 - r_0^2\dot{a}^2)}{1 - kr_0^2}\right| < c. \tag{42}$$

As $\dot{a}(t)$ is a monotonically increasing function of t in $(0, t_c)$, we may treat the left-hand side as a function of $\dot{a}$, and we then need only impose the condition at the point

[2]For $\Lambda > 0$, we need not impose this condition, as the dust edge asymptotically comes to rest at the horizon from the point of view of an external observer.

in the interval $(\dot{a}_0, \dot{a}_c)$ where it is a maximum.

The situation then divides into three cases, depending on the relative size of c. If $c < 2/3(1-kr_0^2)$, the maximum is at $\dot{a}_c$, and we must impose

$$c > (1 - kr_0^2) - (1 - kr_0^2)^2, \tag{43}$$

which requires as a corollary $1-kr_0^2 > 1/3$. If $c > 2/3(1-kr_0^2)$, but $c < 2/3(1 - kr_0^2) + \Lambda' a_0^2 r_0^2$, then the maximum is at an intermediate point, and we must impose

$$c > \frac{2}{3\sqrt{3}}\sqrt{1 - kr_0^2}, \tag{44}$$

which requires as a corollary $1 - kr_0^2 > 4/27$. If $c > 2/3(1 - kr_0^2) + \Lambda' a_0^2 r_0^2$, and $\Lambda' a_0^2 r_0^2 < 1/3(1 - kr_0^2)$, then the maximum is at $\dot{a}_0$, and we must require

$$c > \left| r_0 \dot{a}_0 \left(1 - \frac{r_0^2 \dot{a}_0^2}{1 - kr_0^2}\right)\right|. \tag{45}$$

In summary, pressureless dust in $2 + 1$ dimensions can undergo a variety of collapse scenarios, depending upon the relative signs and magnitudes of its initial density, initial collapse speed, and the cosmological constant $(-\Lambda)$. The stationary black hole solution found in [3] arises naturally from gravitational collapse of pressureless dust for a negative cosmological constant. Its properties are similar to those of the higher-dimensional Oppenheimer-Snyder case in that collapse to a point singularity occurs in finite proper time, and the event horizon forms in infinite coordinate time, with an infinite redshift. Requiring that the dust edge be a boundary surface gives a relation between the parameter M in the exterior coordinates and the initial density of the dust.

For $\Lambda \leq 0$, collapse, if it occurs at all, is to a naked conical singularity. The solution for $\Lambda = 0$ in our coordinate system is equivalent to that found in [2]. The condition that the dust edge move subluminally in the exterior coordinates may be imposed as an additional condition, although this does not provide any new information. The results for the case $\Lambda < 0$ follow the same pattern, although here collapse requires that the initial velocity be great enough to overcome the overall expansion of the spacetime caused by the cosmological constant. Imposing the condition that the velocity of the dust edge be less than the speed of light yields a somewhat complicated relationship between this quantity and the initial density. Collapse to a naked conical singularity is also possible for $\Lambda > 0$ if the initial density is sufficiently small.

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