Gravitation and cosmology in (1+1)-dimensional dilaton gravity

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The properties of a string-inspired two-dimensional theory of gravity are studied. The post-Newtonian and weak-field approximations, "stellar" structure, and cosmological solutions of this theory are developed. Some qualitative similarities to general relativity are found, but there are important differences.

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I. INTRODUCTION

The study of theories of gravity in two spacetime dimensions can provide insight into issues in semiclassical and quantum gravity, as these theories are mathematically much simpler than (3+1)-dimensional general relativity [1–7]. Recently two such theories, that of [1,2], referred to as the "$R=T$" theory, and the string-inspired theory of [5,6], have attracted some interest, due primarily to the fact that their field equations admit black hole solutions, making them an interesting arena for the study of quantum gravitational effects.

The latter theory of gravity arises from a noncritical string theory in two dimensions. Setting to zero the one-loop β function of the bosonic σ model with two target spacetime dimensions gives the effective target space action

$$S=\int d^2x\, e^{-2\Phi}\sqrt{-g}\,[R+4(\nabla\Phi)^2+c] \ . \tag{1}$$

The resultant field equations give rise to a black hole solution asymmetric about the origin:

$$ds^2=-(1-ae^{-Qx})dt^2+\frac{dx^2}{1-ae^{-Qx}} \ , \tag{2}$$

with the dilaton field

$$\Phi=-\frac{Q}{2}x \ , \tag{3}$$

where $Q^2=c$. We have argued [8] that from a gravitational point of view, the asymmetry of (2) about the origin is somewhat objectionable, as it is difficult to see how such a solution could arise from gravitational collapse of clumped matter (for an alternative viewpoint see [9]).

Matter terms may be incorporated into the action as follows:

$$S=\int d^2x\sqrt{-g}\,\{e^{-2\Phi}[R+4(\nabla\Phi)^2+J]+\mathcal{L}_M\} \ , \tag{4}$$

where $\mathcal{L}_M$ is a matter Lagrangian, and J is a source term for the dilaton field. The above action is a general combination of two approaches [9,10] to coupling matter to the string-inspired action studied in [5,6]. From (4) the field equations are

$$e^{-2\Phi}(R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi)=8\pi G T_{\mu\nu} \ , \tag{5}$$

$$R-4(\nabla\Phi)^2+4\nabla^2\Phi+J=0 \ . \tag{6}$$

The stress-energy tensor $T_{\mu\nu}$ introduced by this procedure may be regarded as modeling some unknown higher-order effects in the string theory. It may also be shown that the action (4) is equivalent to that of a massive scalar field $\psi=e^{-\Phi}$ nonminimally coupled to curvature [8], which allows a less ambiguous interpretation of the matter term. If an appropriate point source of matter at the origin is introduced, the solution of (5) and (6) is a symmetric version of the above black hole solution, that is, (2) and (3) with x replaced by $|x|$. We have shown that this solution will result from a collapsing dust if we include appropriate surface stresses and dilaton charges [8]. The latter may be generated by the source J for this dilaton charge; in the dilaton vacuum $J=c$. These sources obey the conservation laws

$$8\pi G\nabla^\mu T_{\mu\nu}=-\tfrac{1}{2}e^{-2\Phi}\nabla_\nu J \ . \tag{7}$$

The properties of this black hole solution have been studied in detail in Refs. [8,11]; they are broadly similar to those found before for the black hole solution of the $R=T$ theory, although there are significant differences.

We regard the theory described by the action (4) as an interesting theory of two-dimensional gravitation in its own right. The purpose of the present paper is to explore the dynamical properties of this theory in more detail, comparing them to similar treatment of the $R=T$ theory in Ref. [2]. We find that the theory has a sensible post-Newtonian expansion and weak-field limit, although in the latter case the notion of a weak field is contingent upon what one considers the vacuum of the theory to be. The equation for stellar equilibrium in this system is obtained. Finally, we consider the cosmological solutions of the field equations, and find that while a dust-filled universe will eventually collapse, the radiation-filled universe cannot collapse at any time. We summarize our results and discuss further areas of interest in a concluding section.

II. POST-NEWTONIAN CALCULATIONS

We wish to demonstrate that this theory has a sensible Newtonian and post-Newtonian limit. We see that in the first approximation Newton's theory holds, and the

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higher-order terms are qualitatively similar to those found before for the $R=T$ theory [2].

We consider a system of particles experiencing mutual gravitational attraction, and let $\overline{M}$, $\overline{r}$, and $\overline{v}$ be typical values of their masses, separations, and speeds. Comparison with Newton's theory of gravity in one spatial dimension yields the Newtonian potential

$$\xi=2\pi G\overline{M}|x| \ , \tag{8}$$

where a constant of integration has been ignored. If we consider a test particle falling into this potential from $|x|=\overline{r}$ initially at rest, its maximum speed will be

$$\overline{v}^{\,2}\sim G\overline{M}\overline{r} \ , \tag{9}$$

which gives an approximate relation between $\overline{M}$, $\overline{r}$, and $\overline{v}$ for a system of particles. The Newtonian approximation gives the first-order terms in the small parameter $\overline{v}^{\,2}$, so the objective of the post-Newtonian approximation is to supply the higher-order terms in the expansion of physical quantities.

We expect the metric to be approximately Minkowskian where gravity is weak, but we do not assume that it has any particular form. The expansion of the metric is

$$g_{00}=-1+{}^2g_{00}+{}^4g_{00}+\cdots \ , \tag{10}$$

$$g_{01}={}^3g_{01}+{}^5g_{01}+\cdots \ , \tag{11}$$

$$g_{11}=1+{}^2g_{00}+{}^2g_{00}+{}^4g_{00}+\cdots \ , \tag{12}$$

where ${}^Ng_{\mu\nu}$ denotes the term of order $\overline{v}^{\,N}$ in the expansion. We can then calculate the Christoffel symbols and thus the required components of the Ricci tensor:

$${}^2R_{00}=-\tfrac{1}{2}\partial_1\partial_1{}^2g_{00} \ , \tag{13}$$

$${}^4R_{00}=-\tfrac{1}{2}[\partial_1\partial_1{}^4g_{00}-\tfrac{1}{2}\partial_1{}^2g_{00}\partial_1{}^2g_{11} -\partial_0\partial_0{}^2g_{00}+\partial_1{}^2g_{00}\partial_1{}^2g_{00}] \ , \tag{14}$$

$${}^3R_{01}=-\tfrac{1}{2}\partial_1\partial_1{}^3g_{01} \ , \tag{15}$$

$${}^2R_{11}=-\tfrac{1}{2}\partial_1\partial_1{}^2g_{11} \ , \tag{16}$$

where ${}^NR_{\mu\nu}$ denotes the term of order $\overline{v}^{\,N}/\overline{r}^{\,2}$ in the expansion of $R_{\mu\nu}$. We interpret T^{00}, T^{01}, and T^{11} as the energy density, momentum density, and momentum flux, which leads us to make the following expansions:

$$T_{00}={}^0T_{00}+{}^2T_{00}+\cdots \ , \tag{17}$$

$$T_{01}={}^1T_{01}+{}^3T_{01}+\cdots \ , \tag{18}$$

$$T_{11}={}^2T_{11}+{}^4T_{11}+\cdots \ . \tag{19}$$

From our experience with the black hole solutions, we also expect

$$\Phi={}^2\Phi+{}^4\Phi+\cdots \ , \tag{20}$$

$$J=c+{}^2J+{}^4J+\cdots \ , \tag{21}$$

where Φ is the dilaton field and J is the dilaton current.

The field equations can be expanded in powers of our small parameter, giving us the forms

$${}^NR_{\mu\nu}+2(\nabla_\mu{}^N\nabla_\nu\Phi)=8\pi G^{(N-2)}T_{\mu\nu} \ , \tag{22}$$

$${}^NR-4[{}^N(\nabla\Phi)^2]+4({}^N\nabla^2\Phi)+{}^NJ=0 \ , \tag{23}$$

where N is the order in $\overline{v}^{\,2}/\overline{r}$. For the Newtonian approximation, we only need to determine ${}^2g_{00}$ so we will only need the $\mu=0$, $\nu=0$ component of (22) to order $N=2$. For the post-Newtonian expansion, we will need the 00 component of (22) to order 4, the 01 component to order 3, and the 11 component to order 2, as well as (23) to order 2.

First, we compute the Newtonian approximation. The 00 component of (22) to order 2 gives

$$\partial_1\partial_1{}^2g_{00}=-16\pi G{}^0T_{00} \ , \tag{24}$$

which has solution ${}^2g_{00}=-4\xi$, where

$$\xi(x,t)=2\pi G\int dx'{}^0T_{00}(x',t)|x-x'| \tag{25}$$

is the Newtonian potential. Note that this differs from the result ${}^2g_{00}=-2\xi$ of [2] by a factor of 2.

We now compute the post-Newtonian terms, ${}^2g_{11}$, ${}^3g_{01}$, ${}^4g_{00}$, and ${}^2\Phi$. The 11 component of (22) to order 2 gives

$$\partial_1\partial_1{}^2g_{11}=4\partial_1\partial_1{}^2\Phi \ , \tag{26}$$

which has solution ${}^2g_{11}=4{}^2\Phi$. If we now consider the dilaton equation (23) to order 2 we get

$$\tfrac{1}{2}\partial_1\partial_1{}^2g_{00}-\tfrac{1}{2}\partial_1\partial_1{}^2g_{11}+4\partial_1\partial_1{}^2\Phi+{}^2J=0 \ , \tag{27}$$

and substituting ${}^2g_{11}=4{}^2\Phi$ gives

$$\partial_1\partial_1{}^2g_{11}=-\partial_1\partial_1{}^2g_{00}-{}^2J=4\xi''-{}^2J \ , \tag{28}$$

so the solution is ${}^2g_{11}=4\beta$, and thus ${}^2\Phi=\beta$, where β is a new field defined by

$$\beta(x,t)=\int dx'|x-x'|(\tfrac{1}{2}\xi''-\tfrac{1}{8}{}^2J) \ . \tag{29}$$

Note in particular that if ${}^2J=0$, $\beta=\xi$.

We now take the 01 component of (22) to order 3 to determine ${}^3g_{01}$. This gives

$$\begin{aligned}\partial_1\partial_1{}^3g_{01}&=-16\pi G{}^1T_{01}+4\partial_1\partial_0{}^2\Phi\\ &=-16\pi G{}^1T_{01}+4\partial_0\partial_1\beta \ ,\end{aligned} \tag{30}$$

which gives ${}^3g_{01}=\eta$, where η is a new field defined by

$$\eta(x,t)=\int dx'|x-x'|[-8\pi G{}^1T_{01}(x',t)+2\dot{\beta}'] \ . \tag{31}$$

Finally, we compute ${}^4g_{00}$ from the 00 component of (22) to order 4, substituting for ${}^2g_{11}$ and ${}^2\Phi$ from above:

$$\partial_1\partial_1{}^4g_{00}=-16\pi G{}^2T_{00}+4(\ddot{\beta}-\ddot{\xi})-16\xi^2 \ , \tag{32}$$

which gives ${}^4g_{00}=\psi$, if we define the new field ψ by

$$\psi=\int dx'|x-x'|[2(\ddot{\beta}-\ddot{\xi})-8\xi'^2-8\pi G{}^2T_{00}] \ . \tag{33}$$

It is also perhaps worth noting that

$$\ddot{\beta}-\ddot{\xi}=-\tfrac{1}{8}\int dx'|x-x'|\frac{\partial^2{}^2J}{\partial t^2} \ , \tag{34}$$

which will vanish if ${}^2J$ is linear in time.

This completes the calculation of the post-Newtonian approximation. It should also be noted that because this is a truncated series in powers of the distance r, it will be a better approximation near the system, even though we expect the spacetime to be asymptotically flat for constant J and $T_{\mu\nu}=0$ [8]. The main relevance of this calculation is that it shows that one can carry out an expansion in this theory about its Newtonian limit. Expansion at large distances (i.e., about the asymptotically flat solution) was considered in [6]; we shall not pursue this issue any further here.

III. WEAK-FIELD APPROXIMATION

The weak-field expansion of this theory is somewhat more complicated than that considered in Ref. [2] because the notion of a vacuum depends upon whether or not one considers the vacuum to be that region of spacetime for which $T_{\mu\nu}=0$ and $J=0$ or for which $T_{\mu\nu}=0$ and $J=c$.

We consider the metric to be a perturbation on a Minkowski background and the dilaton field to be a perturbation about a solution ϕ of the vacuum equations:

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ , \quad \Phi=\phi+\varphi \tag{35}$$

and the source J to be

$$J=c\epsilon+\mathcal{J}\ , \tag{36}$$

where $\epsilon=0$ or 1 depending upon the choice of vacuum.

Consider first $\epsilon=0$. The field equations are

$$R^{(1)}=16\pi GT+\mathcal{J}\ , \tag{37}$$

$$2\nabla_\mu\nabla_\nu\varphi=8\pi G(T_{\mu\nu}-g_{\mu\nu}T)-\tfrac{1}{2}g_{\mu\nu}\mathcal{J}\ , \tag{38}$$

where $R^{(1)}$ is the linear order Ricci scalar, the nonlinear terms have been neglected, and the zeroth-order field equations yield the solution $\phi=0$.

In two dimensions the identity $R_{\mu\nu}\equiv\frac{1}{2}g_{\mu\nu}R$ forces the relationship

$$\partial^2h^{\mu\nu}=\tfrac{1}{2}\eta_{\mu\nu}\partial^2h\ , \tag{39}$$

and so coordinates may be chosen so that $h_{\mu\nu}=\frac{1}{2}\eta_{\mu\nu}h$. With this, (37) becomes

$$\partial^2h=-32\pi GT-2J\ , \tag{40}$$

which is a wave equation with source term in the trace $h=h^\mu_\mu$ of the metric perturbation. Note that this differs from the analogous equation in [2] by the ubiquitous factor of 2, and by the fact that the dilaton source may act as a source of metric perturbations in the absence of matter. The particular integral solutions of this equation are given in terms of retarded and advanced potentials

$$h(x,t)=\pm16\pi G\int dx'\int^t dt'[\mathcal{T}(x',t'\mp|x-x'|)]\ , \tag{41}$$

to which any solution of the corresponding homogeneous system [(40) with $T=0=J$] may be added. Here $\mathcal{T}(x,t)=T(x,t)+(1/16\pi G)J(x,t)$.

Spacetime is flat outside any distribution of matter (i.e., any region where $T_{\mu\nu}=0=J$). Consider a system of oscillating matter which has an energy-momentum tensor trace and dilaton current representable as a Fourier integral or sum over frequencies ω. A single Fourier component is

$$\mathcal{T}(x,t)=\mathcal{T}(x,\omega)\exp(-i\omega t)+\text{c.c.}\ , \tag{42}$$

and the retarded potential solution (41) giving gravitational waves is then

$$h(x,t)=8\pi G\int dx'\int^t dt'\mathcal{T}(x',\omega)\times\exp[-i\omega(t'-|x-x'|)]+\text{c.c.}\ , \tag{43}$$

where c.c. denotes the complex conjugate of the preceding term. If the source is finite, with maximum extension $R=|x'|$, and we are situated in a region of space outside of the source with $r\equiv|x|>R$, then the metric perturbation takes the form of a plane wave

$$h=H\exp(ik_\mu x^\mu)+\text{c.c.}\ , \tag{44}$$

with amplitude and wave vector

$$H=8\pi Gi\omega^{-1}\int dx'\mathcal{T}(x',\omega)\exp(-ik_1x') \equiv 8\pi Gi\omega^{-1}\mathcal{T}(k_1,\omega)\ , \tag{45}$$

$$k_0=-\omega\ , \quad k_1=\omega\hat{x}\ , \tag{46}$$

where the complete Fourier transform of the energy-momentum tensor trace is defined. Here we have used $|x-x'|=r-x'\hat{x}$ with $\hat{x}\equiv x/r$.

As for the theory considered in [2] it is the global nature of the gravitational "wave" which contains the nontrivial physics. Such waves are coordinate waves locally: they may be set to zero by performing a coordinate transformation which "travels with the wave." However, while this may be carried out on either side of the source, outside of the source the wave is of the form

$$h=H\exp[-i\omega(t-|x|)]\ , \tag{47}$$

and so such a coordinate transformation cannot be applied globally. An observer crossing the source will see a flip in the wave's direction of propagation. In this sense there is gravitational radiation in the weak-field approximation.

For the case $T_{\mu\nu}=0$ and $J=c$ (the "dilaton vacuum" where $\epsilon=1$), the full system of equations yields in general the unique solution (2) and (3). In this case it is useful to manipulate the field equations and redefine variables before proceeding with the weak-field expansion. Writing $g_{\mu\nu}=e^{2\Phi}\hat{g}_{\mu\nu}$, the field equation (5) becomes

$$-\hat{\nabla}_\mu\hat{\nabla}_\nu(e^{-2\Phi})+\hat{g}_{\mu\nu}\hat{\nabla}^2(e^{-2\Phi})=8\pi GT_{\mu\nu}+\tfrac{1}{2}J\ , \tag{48}$$

where (6) and the relationship

$$R[e^{2\Phi}g]=e^{-2\Phi}(\hat{R}-2\hat{\nabla}^2\Phi) \tag{49}$$

have been used, quantities with carets being defined with respect to the metric $\hat{g}_{\mu\nu}$. Equation (6) becomes, after some manipulation,

$$\hat{R} = 8\pi G e^{2\Phi} \hat{g}^{\mu\nu} T_{\mu\nu} . \tag{50}$$

Defining $\sigma \equiv e^{-2\Phi}$, the weak-field approximation is then defined via

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu} , \quad \sigma = \sigma_0 + \varphi \tag{51}$$

and the field equations become

$$-\partial_\mu \partial_\nu \sigma + \eta_{\mu\nu}(\partial^2 \sigma_0 - \tfrac{1}{2}c) = 0 \tag{52}$$

at zeroth order, which has the solution

$$\sigma_0 = \frac{c}{4}(x - x_0)^2 - M , \tag{53}$$

where x_0 and M are constants of integration. x_0 merely sets the origin of coordinates. In the dilatonic vacuum, the zeroth-order equations are the full field equations (as $\hat{R}$ vanishes), and the "physical" metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{(c/4)(x - x_0)^2 - M} , \tag{54}$$

which for nonzero M is equivalent to the black hole solution (2) under a change of coordinates.

The higher-order field equations then yield perturbations about this solution. To first order we have, from (50),

$$\partial^2 \hat{h} = -16\pi G \frac{T}{\sigma_0} , \tag{55}$$

which has the solution (41), with $\mathcal{T} = \frac{1}{2}T$. The dilaton perturbation equation which follows from (48) is

$$-\partial_\mu \partial_\nu \varphi + \eta_{\mu\nu} \partial^2 \varphi + \frac{c}{4}(x_{(\mu} \partial_{\nu)} \hat{h} - \tfrac{1}{2}\eta_{\mu\nu} x \cdot \partial \hat{h}) - \frac{c}{4}\eta_{\mu\nu}\hat{h} = 8\pi G T_{\mu\nu} + \tfrac{1}{2}\eta_{\mu\nu}\mathcal{J} \tag{56}$$

taking $J = c + \mathcal{J}$. The trace of this equation is

$$\partial^2 \varphi = \frac{c}{2}\hat{h} + 8\pi G T + \tfrac{1}{2}\mathcal{J} \tag{57}$$

and has the solution

$$\varphi(x,t) = \pm 4\pi G \int dx' \int^t dt' [\mathcal{G}(x', t' \mp |x - x'|)] , \tag{58}$$

as all quantities on the right-hand side of (57) are known. Here $\mathcal{G} \equiv T + (1/4\pi G)[J + (c/2)\hat{h}]$. Homogeneous terms may be added to φ; these will be set by the full equation (56) in a manner consistent with the linearization of the conservation laws (7):

$$8\pi G \partial^\mu T_{\mu\nu} + \tfrac{1}{2}\partial_\nu J + 4\pi G \partial_\nu \sigma_0 \frac{T}{\sigma_0} = 0 , \tag{59}$$

which also follow easily upon taking the partial derivative of (56).

Finally, the linearized metric takes the form

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{(c/4)(x - x_0)^2 - M} \times \left[1 - \frac{1}{2}\hat{h} - \frac{\varphi}{(c/4)(x - x_0)^2 - M} \right] , \tag{60}$$

where φ and $\hat{h}$ are given in terms of the sources above. This represents a first-order perturbation about the black hole solution (2).

IV. STELLAR STRUCTURE

We now consider the equations of fluid equilibrium, which govern the existence of "stars" (clumps of matter in one spatial dimension) in the theory. We use the static metric of the form

$$ds^2 = -B^2(x) dt^2 + dx^2 \tag{61}$$

together with the field equations

$$R + 2\nabla^2 \Phi = 8\pi G T e^{2\Phi} , \tag{62}$$

$$2\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla^2 \Phi = 8\pi G (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) e^{2\Phi} , \tag{63}$$

$$R - 4(\nabla\Phi)^2 + 4\nabla^2 \Phi + J = 0 . \tag{64}$$

The metric gives

$$R = -2\frac{B''}{B}$$

and (65)

$$\nabla^2 \Phi = \Phi'' + \frac{B'}{B}\Phi'$$

if the stress energy is the perfect fluid energy-momentum tensor. The field equation (63) gives

$$\Phi'' - \frac{B'}{B}\Phi' = 4\pi G (p + \rho) e^{2\Phi} , \tag{66}$$

where p is the pressure, ρ is the density, and $' = d/dx$. Also, the field equations (62) and (64) give

$$\Phi'^2 = 4\pi G (p - \rho) e^{2\Phi} + \frac{J}{4} + \frac{1}{2}\frac{B''}{B} , \tag{67}$$

so we may obtain an expression for $\nabla^2 \Phi$. Substitution of this expression in (62) gives an equation quadratic in B''/B, whose roots are

$$\frac{B''}{B} = 8\pi G \rho e^{2\Phi} + \left[\frac{p'}{p + \rho} \right]^2 \pm \left[\frac{p'}{p + \rho} \right] \left[\left[\frac{p'}{p + \rho} \right]^2 + 16\pi G p e^{2\Phi} + J \right]^{1/2} . \tag{68}$$

We also have the equation of hydrostatic equilibrium,

$$-p' = (p + \rho)[\ln(-g_{00})^{1/2}]' , \tag{69}$$

which in our case reduces to

$$\frac{B'}{B} = -\frac{p'}{p + \rho} . \tag{70}$$

Thus, the general equation for hydrostatic equilibrium in our system is

$$-p'' = (p + \rho)\frac{B''}{B} - \frac{p'(2p' + \rho')}{p + \rho} , \tag{71}$$

where B''/B is given by (68). Given an equation of state $p=p(\rho)$, the solution of (71) will give $\rho=\rho(r)$ and hence the metric.

If we compare this to Newton's equation of stellar equilibrium,

$$-p''=4\pi G\rho^2-\frac{p'\rho'}{\rho}\ , \tag{72}$$

we see that in the Newtonian limit $p\to 0$, $J\to 0$, $\Phi\to 0$ (68) reduces to

$$\frac{B''}{B}=8\pi G\rho\ . \tag{73}$$

Thus, in this limit our equation (71) will only differ from the Newtonian equation (72) by a factor of 2 multiplying G.

V. COSMOLOGY

Consider the two-dimensional Robertson-Walker metric

$$ds^2=-dt^2+a^2(t)\frac{dx^2}{1-kx^2}\ , \tag{74}$$

with the field equations (62)–(64). As previously noted in Ref. [2], in two dimensions we do not have three different cosmological models corresponding to open, closed, and flat spacetimes since the denominator in the second term in (74) can be absorbed into a definition of the spatial coordinate.

If we assume the perfect fluid stress energy, (62) and (63) become

$$e^{-2\Phi}\left[-\ddot{\Phi}+\frac{\dot{a}}{a}\dot{\Phi}\right]=-4\pi G(p+\rho)\ , \tag{75}$$

$$e^{-2\Phi}(2\ddot{a}-2\dot{\Phi}\dot{a}-2\ddot{\Phi}a)=8\pi G(p-\rho)a\ . \tag{76}$$

If we now take the equation of state to be $p=(\gamma-1)\rho$, then conservation of energy $T^{\mu\nu}_{;\nu}=0$ will give $a\rho^{\gamma}=a_0\rho_0^{\gamma}$, and substitution in (75) and (76) will give

$$\ddot{a}=\frac{2}{\gamma}\dot{a}\dot{\Phi}+2\frac{\gamma-1}{\gamma}a\ddot{\Phi}\ , \tag{77}$$

and

$$e^{-2\Phi}\left[-\frac{\ddot{a}}{a}+2\ddot{\Phi}\right]=8\pi G\rho\ . \tag{78}$$

We also find that (64) becomes

$$\ddot{a}+2a\dot{\Phi}^2-2a\ddot{\Phi}-2\dot{a}\dot{\Phi}+\tfrac{1}{2}Ja=0\ . \tag{79}$$

The solution of these equations for general γ is quite difficult. However, a solution may easily be obtained for two special cases, $\gamma=1$ and 2, corresponding to dust- and radiation-filled spacetimes, respectively. For simplicity we shall henceforth consider only $J=c$ solutions.

The case $\gamma=1$ was studied in [8], but we include the results here in the interest of completeness. We find (77) becomes

$$\ddot{a}=2\dot{a}\dot{\Phi}\ , \tag{80}$$

and substitution of this in (79) gives

$$\ddot{\Phi}-\dot{\Phi}^2=\frac{c}{4}\ , \tag{81}$$

with the solution

$$\Phi(t)=-\ln\left|\cos\left[\frac{Q}{2}t+\beta\right]\right|+\Phi_0\ , \tag{82}$$

where $Q^2=c$, and β and Φ_0 are constants of integration. If we substitute this solution into (80), we find that a is given by

$$a(t)=-\lambda\tan\left[\frac{Q}{2}t+\beta\right]+\alpha\ , \tag{83}$$

where λ and α are constants, and thus (78) implies ρ_0 is given by

$$\rho_0=\frac{Q^2\alpha}{16\pi G}e^{-2\Phi_0}\ . \tag{84}$$

Note that we chose $\alpha=1$ and $\beta=0$ for convenience in the previous paper [8]. We see that this solution will collapse at

$$t_c=\frac{2}{Q}\arctan\left[\frac{\alpha}{\lambda}\right]-\beta\ , \tag{85}$$

and as the density varies inversely with a, the density will diverge at the collapse. The curvature

$$R=-\frac{Q^2\lambda\sec^2[(Q/2)t+\beta]\tan^2[(Q/2)t+\beta]}{\alpha-\lambda\tan[(Q/2)t+\beta]} \tag{86}$$

also diverges at t_c. This collapse is similar that of the cosmological solution of the $R=T$ theory [2]. However, in contrast to [2], one cannot construct a two-dimensional analogue of the Friedmann-Robertson-Walker (FRW) cosmology in general relativity using (83), since a vanishes at only one time t_c, and diverges at finite times both before and after t_c.

In the case $\gamma=2$ (radiation), (77) becomes

$$\ddot{a}=\dot{a}\dot{\Phi}+a\ddot{\Phi}\ , \tag{87}$$

which may be integrated to give

$$\dot{a}=a\dot{\Phi}+B\ , \tag{88}$$

B a constant of integration. If we substitute this in (79), we may obtain

$$\ddot{a}=\frac{2(\dot{a}-B)^2}{a}+\frac{1}{2}ca\ . \tag{89}$$

Considering first solutions with $B=0$, we find

$$\Phi(t)=-\ln\left|\cos\left[\frac{Q}{\sqrt{2}}t+\beta\right]\right|+\Phi_0\ , \tag{90}$$

$$a(t)=A\sec\left[\frac{Q}{\sqrt{2}}t+\beta\right]\ , \tag{91}$$

where Φ_0, β, and A are arbitrary constants. This solution obviously does not collapse, and corresponds to a universe which is expanding.

Turning now to the more general case, we may obtain a first integral of (89) by treating it as a differential equation for $\dot{a}$ as a function of a. The implicit solution is

$$\ln(ca^2+4B^2-4By)+\frac{4By}{ca^2+4B^2-4By}=\ln(a^2)+\kappa \ , \tag{92}$$

where $y=\dot{a}(a)$, and κ is a constant. We can rewrite this in a simpler form by scaling $\hat{a}=\sqrt{c}\,a/2B$, $\hat{y}=y/B$, which gives

$$\ln(\hat{a}^2+1-\hat{y})+\frac{\hat{y}}{\hat{a}^2+1-\hat{y}}=\ln(\hat{a}^2)+k \ , \tag{93}$$

k a constant.

We check now for collapsing solutions. If the solution given implicitly by (93) collapses, it follows that this equation must be satisfied as $a\to 0$. Thus, we let $\hat{a}^2=\epsilon$, $\hat{y}=1+\epsilon-\delta$, and consider (93) as $\epsilon\to 0$. We find (93) may be rewritten as

$$\ln(\delta)+\frac{1}{\delta}=\ln(\epsilon)+k \ , \tag{94}$$

which implies

$$-\frac{1}{\epsilon}e^{-k}=-\frac{1}{\delta}e^{-1/\delta}\geq -\frac{1}{e} \ , \tag{95}$$

since $xe^x\geq -1/e$ for all x. Thus,

$$\epsilon e^k \geq e \ , \tag{96}$$

which cannot be satisfied for finite k as $\epsilon\to 0$. Thus, (93) cannot be satisfied as $a\to 0$, so the solution described by (93) cannot collapse.

Alternatively, since (89) implies that $\ddot{a}$ is always positive, collapse is impossible if $\dot{a}$ is initially positive. If $\dot{a}$ is initially negative but vanishes before the collapse occurs (i.e., for $a\in(0,1]$), collapse is also impossible. Thus, if the initial condition is $\hat{y}(\hat{a}^2=1)=y_0$, then

$$k=\ln(2+y_0)+\frac{2}{2+y_0}-1 \ , \tag{97}$$

and collapse is possible only if $\hat{y}$ does not vanish for $\hat{a}^2\in(0,1]$ (note that $\hat{y}$ and $\dot{a}$ do not necessarily have the same sign). When $\hat{y}=0$, $\hat{a}$ is given by

$$\hat{a}^2=\frac{1}{e^k-1} \ , \tag{98}$$

and thus $\hat{y}=0$ for $\hat{a}^2\in(0,1]$ if $e^k\geq 2$, which implies

$$-\frac{2}{2+y_0}e^{-2/(2+y_0)}\geq -\frac{1}{e} \ . \tag{99}$$

This is true for all y_0; therefore collapse is impossible.

Hence although we are unable to obtain the full solution $a(t)$ for a radiation-filled spacetime, we have been able to demonstrate such a universe, in general, cannot collapse. Time-reversal invariance therefore implies that the scale factor reaches a minimal value at some time $t=t_{\min}$.

VI. CONCLUSIONS

The above considerations, when combined with the results in Refs. [8,11] on gravitational collapse, indicate that the two-dimensional theory of gravity given by the action (4) yields a classical theory of gravity which is as rich in structure as the $R=T$ theory proposed earlier [2]. Of course from a classical relativist's viewpoint, the presence of the dilaton introduces features which are markedly different from the $R=T$ theory. In the latter case, curvature is generated solely by stress energy which is prescribed from the matter Lagrangian. In the string-motivated theory studied here, the dilaton field cannot be decoupled from the remaining gravity-matter system: even a vanishing dilaton field imposes constraints on the stress energy in addition to those which follow from the conservation laws. Indeed, upon reparametrizing the dilaton field so that $e^{-2\Phi}=\varphi$, it is easily seen that the action (4) (with $J=0$) is two-dimensional Brans-Dicke theory with $\omega=-1$.

The post-Newtonian expansion is similar to that of general relativity, although the presence of the dilaton field introduces some extra complications. In the weak-field approximation the trace of the metric perturbation obeys a wave equation, although the form differs depending upon whether or not one perturbs about the flat vacuum ($T_{\mu\nu}=0=J$) or the dilaton vacuum ($T_{\mu\nu}=0$, $J=c$). The equation of stellar equilibrium was obtained, and we saw that in the Newtonian limit, it reduced to a form similar to the Newtonian equation. The Newtonian approximations to this theory were identical to those of [2] apart from a factor of 2 multiplying G.

The development of the cosmological solution in [8] was included. This study shows a dust-filled spacetime will collapse in finite proper time for suitable initial conditions, with divergent density and curvature, although a solution which collapses cannot have developed from an initial singularity, and vice versa. This solution is the basis for the demonstration that the symmetric black hole solution arises from gravitational collapse [8,11]. In the case of a radiation-filled spacetime, we were unable to solve the equations in general, although we were able to show that the spacetime never collapses. The collapsing dust solution has similarities to the collapsing dust in the $R=T$ theory [2], but the general properties of the cosmological solutions are quite different.

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