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Gravitational action for spacetimes with nonsmooth boundaries

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In this paper, I examine the gravitational action for spacetimes with nonsmooth boundaries. By two independent techniques, I derive the contribution to the gravitational action of spacelike and timelike two-surfaces on the boundary at which the unit normal changes discontinuously. I discuss the relationship between constraints imposed at such two-surfaces and their contribution to the gravitational action. I derive the form of the action and the juncture conditions appropriate to cases in which a spacetime includes a singular matter distribution whose world history corresponds to a timelike two-dimensional surface.
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I. INTRODUCTION

Over the last two decades, it has become apparent that boundaries and boundary conditions play a pivotal role in many branches of gravitational physics. Much of the importance of this role can be attributed to the direct impact of boundary conditions on the appropriate form of the gravitational action. It is well known, for instance, that the gravitational action appropriate to a manifold with a smooth boundary of fixed intrinsic geometry includes a boundary term [1, 2]

$$\frac{1}{8\pi}\int_{\mathcal{B}} K\sqrt{h}\,d^3x + \mathcal{C}[h_{ij}], \tag{1}$$

where h_{ij} is the intrinsic three-metric of the boundary, K_{ij} is the extrinsic curvature, and $\mathcal{C}[h_{ij}]$ is an arbitrary functional of the fixed boundary three-metric.

Here, I consider the gravitational action for spacetimes with boundaries or internal three-dimensional matter distributions which are nonsmooth in the sense that the unit normal changes direction discontinuously at some two-surface. I refer to two-surfaces where such discontinuities occur as "joints."

Spacetimes whose boundaries have joints arise in a number of different contexts in gravitational physics [3–5]. For instance, they arise naturally in the Hamiltonian treatment of geometrodynamics for spatially bounded spacetimes [6, 7]. Also, certain two-dimensional singularity surfaces, such as those associated with cosmic strings, may be viewed as examples of joints.

Different authors have recognized that joints make a finite contribution to the gravitational action [3–5]. In Sec. II, I derive this contribution by viewing a boundary with a joint as a limiting case of a smooth boundary. This limiting procedure can be applied unambiguously to all timelike joints and to a certain class of spacelike joints. However, for some spacelike joints, smoothing the boundary joint yields a surface at which the normal goes null and this gives rise to an ambiguity.

In Sec. III, I provide an alternate derivation of the contribution of a joint to the gravitational action which proceeds directly from the variational principle. Taking first-order variations of the action, I demonstrate how constraints imposed at the nonsmooth portions of the boundary influence the appropriate form of the gravitational action. This derivation may be applied unambiguously to any spacelike or timelike joint.

In Sec. IV, I discuss joints associated with singular matter distributions inside a spacetime and derive the juncture conditions at such joints.

II. EVALUATING THE CONTRIBUTION OF A JOINT TO THE ACTION BY SMOOTHING

Since the unit normal changes direction discontinuously at a joint, the extrinsic curvature has a divergence there. A consequence is that the joint makes a finite contribution to the boundary term in the gravitational action. To derive what this contribution is, replace the joint by a rounded three-surface, evaluate the extrinsic curvature on this rounded edge, and then take the limit that the rounded edge collapses onto the joint. Versions of this procedure were employed in Refs. [5, 4].

Focus first on the case of a joint which is everywhere timelike. Let $\mathcal{M}$ be a four-manifold with a timelike boundary $\mathcal{B}$. The action appropriate to fixing the intrinsic three-geometry of $\mathcal{B}$ is

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} R\,g^{1/2}\,d^4x + \frac{1}{8\pi}\int_{\mathcal{B}} K\,h^{1/2}\,d^3x + \mathcal{C}. \tag{2}$$

Suppose that a timelike joint $\partial\mathcal{B}$ bifurcates $\mathcal{B}$ into two sub-three-manifolds $\mathcal{B}_0$ and $\mathcal{B}_1$ [see Fig. 1(a)]. Let n_0^μ be the unit normal to the boundary on the $\mathcal{B}_0$ side of $\partial\mathcal{B}$, and similarly, let n_1^μ be the unit normal on the $\mathcal{B}_1$ side of $\partial\mathcal{B}$. The local angle between the two normals at some point on the joint is defined by

$$\Theta \equiv \arccos(n_0 \cdot n_1). \tag{3}$$

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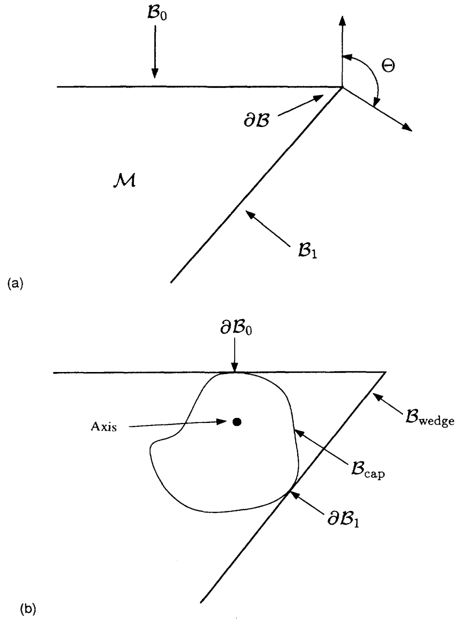

FIG. 1. (a) A timelike joint $\partial\mathcal{B}$ bifurcates a boundary into smooth surfaces $\mathcal{B}_0$ and $\mathcal{B}_1$. (b) A "cylinder" is wedged between $\mathcal{B}_0$ and $\mathcal{B}_1$. The two-surfaces of intersection between the cylinder and $\mathcal{B}_0$ and $\mathcal{B}_1$ are, respectively, $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_1$.

Now wedge a "cylinder" between $\mathcal{B}_0$ and $\mathcal{B}_1$. In fact, this "cylinder" is a just an arbitrary four-manifold which has a boundary with topology $S^1 \times \Sigma^{(2)}$, where $\Sigma^{(2)}$ is the topology of $\partial\mathcal{B}$ (no symmetries of any sort are assumed). Let $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_1$ correspond to the intersection two-surfaces of the cylinder with $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively. Let $\mathcal{B}_{\rm wedge}$ correspond to the portion of $\mathcal{B}$ wedged between $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_1$. Let $\mathcal{B}_{\rm cap}$ correspond to the portion of the cylinder boundary wedged between $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_1$ [see Fig. 1(b)].

Define a new three-surface $\mathcal{B}_\varepsilon = (\mathcal{B} - \mathcal{B}_{\rm wedge}) \bigcup \mathcal{B}_{\rm cap}$. By construction, the three-geometry of $\mathcal{B}_\varepsilon$ is everywhere C^1 as embedded in the four-geometry of $\mathcal{M}$. Therefore,

$$\int_{\mathcal{B}_\varepsilon} K\, h^{1/2}\, d^3x = \int_{\mathcal{B}-\mathcal{B}_{\rm wedge}} K\, h^{1/2}\, d^3x + \int_{\mathcal{B}_{\rm cap}} K\, h^{1/2}\, d^3x. \tag{4}$$

[Since the extrinsic curvature has at most jump discontinuities at $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_1$, these two-surfaces make no finite contribution to (4).] We wish to evaluate (4) in the limit that the proper length of the cylinder's S^1 goes to zero.

Consider first the problem of evaluating the extrinsic curvature on the boundary of the cylinder. Let a coordinate $x^0 \equiv r$ foliate nested three-surfaces which extend inward from the boundary of the cylinder and converge on an arbitrary coordinate singularity two-locus which we will denote as the "axis." Without loss of generality, let the surface $r = \varepsilon$ correspond to the boundary and the surface $r = 0$ correspond to the axis. The four-metric may be expressed in terms of a $1+3$ radial foliation:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (N^2 + h_{ij}N^iN^j)\, dr^2 + 2h_{ij}N^i\, dx^j\, dr + h_{ij}\, dx^i\, dx^j,$$

where N and N^i are the radial lapse and shift and where h_{ij} is the three-metric of a surface of constant r. Without loss of generality, choose the foliation such that $\bar{N} \equiv N|_{r=0}$ is bounded and nonzero everywhere on the axis.

Now perform a $1+2$ foliation of the three-geometry of the boundary of the cylinder. Let a coordinate $x^1 \equiv \theta$ with period 2π parametrize the S^1 of the cylinder's boundary. The three-line element on the cylinder's boundary is then

$$\begin{aligned} {}^{(3)}ds^2 &= h_{ij}\, dx^i\, dx^j \\ &= (M^2 + \sigma_{AB}M^AM^B)\, d\theta^2 + 2\sigma_{AB}M^A\, dx^B\, d\theta \\ &\quad + \sigma_{AB}\, dx^A\, dx^B, \end{aligned}$$

where M and M^A correspond to the lapse and shift associated with the θ foliation and where σ_{AB} is the intrinsic pseudo-Riemannian two-metric of a surface of constant θ and r. (Indices A, B range from 2 to 3.)

If the axis is to correspond to a regular two-locus as embedded in the four-geometry of $\mathcal{M}$, certain local smoothness conditions must be satisfied. In particular, it is necessary that, as $r \to 0$,

$$M \to r\,(M_{,r} - M^AM_{,A}) \tag{5}$$

and

$$1 = \frac{1}{\bar{N}}\,(M_{,r} - M^AM_{,A})\Big|_{r=0}. \tag{6}$$

The easiest way to derive these conditions is to go into a locally Gaussian coordinate frame on a small four-patch which includes a thin slice of the axis. Thus, let $\Delta\mathcal{M}$ be a thin slice of the cylinder which extends from $x_0^A < x^A < x_0^A + \delta x^A$. Perform the coordinate transformation

$$\begin{aligned} \tilde{r} &= r, \\ \tilde{\theta} &= \theta + \bar{N}^r r, \\ \tilde{x}^A &= x^A + (\bar{N}^A + \bar{M}^A\bar{N}^r)r + \bar{M}^A\theta, \end{aligned}$$

where $\bar{N}^i, \bar{M}^A$ are the limiting values of N^i, M^A as $r \to 0$ on ΔM. Then, the four-element on $\Delta\mathcal{M}$ has the limiting form, as $r \to 0$,

$$ds^2 = N^2\, dr^2 + M^2\, d\tilde{\theta}^2 + \sigma_{AB}\, d\tilde{x}^A\, d\tilde{x}^B. \tag{7}$$

In this coordinate frame, it is easy to see that the necessary conditions for the axis, $r = 0$, to be a regular two-surface in $\mathcal{M}$ include

$$\begin{aligned} M|_{r\to 0} &= M(r), \\ M|_{r=0} &= 0, \\ \frac{\partial N}{\partial\tilde{\theta}}\Big|_{r=0} &= \frac{\partial\sigma_{AB}}{\partial\tilde{\theta}}\Big|_{r=0} = 0. \end{aligned} \tag{8}$$

Expand $M(r, \tilde{\theta}, \tilde{x}^A)$ around the axis $r = 0$ to obtain

$$M(r, \tilde{\theta}, \tilde{x}^A) = r\left(\frac{\partial M}{\partial r}\Big|_{r=0}\right) + O(r^2). \tag{9}$$

Substitute this limiting form back into (7), to find that a conical singularity exists at $r = 0$ unless

$$\frac{1}{N}\frac{\partial M}{\partial r}\bigg|_{r=0} = 1. \tag{10}$$

When expressed in terms of the general coordinate frame (x^μ), conditions (9) and (10) become (5) and (6).

Now, note that $\sqrt{h} = M\sqrt{\sigma}$ and that

$$K\,h^{1/2} = \frac{1}{N}\left(\left(h^{1/2}\right)_{,r} - \left(N^i h^{1/2}\right)_{,i}\right).$$

Make use of (8), (5), and (6), to obtain that, in the limit $\varepsilon \to 0$,

$$\int_{\mathcal{B}_{\rm cap}} K\,h^{1/2}\,d^3x = \int_{\partial\mathcal{B}} \Theta\,\sigma^{1/2}\,d^2x. \tag{11}$$

Define $\mathcal{B}_{\rm smooth} \equiv \lim_{\varepsilon\to 0}(\mathcal{B} - \mathcal{B}_{\rm wedge})$ and substitute (4) into (2) to obtain

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} R\,g^{1/2}\,d^4x + \frac{1}{8\pi}\int_{\mathcal{B}_{\rm smooth}} K\,h^{1/2}\,d^3x + \frac{1}{8\pi}\int_{\partial\mathcal{B}} \Theta\,\sigma^{1/2}\,d^3x + \mathcal{C}.$$

This is the desired result.

It is possible to extend this smoothing procedure in an unambiguous fashion to derive the contribution of certain spacelike joints.

To make this claim precise, let $\partial\mathcal{B}$ be any joint (spacelike or timelike) on a boundary $\mathcal{B}$. Wedge a "cylinder" against the boundary and across the joint as was done above. Now let "class I" joints be those for which it is possible to choose the cylinder such that the normal on $\mathcal{B}_{\rm cap}$ nowhere goes null (see Fig. 2). "Class II" joints are those for which this is not possible (see Fig. 3). A smoothing procedure directly analogous to that outlined above may be extended unambiguously to any class I spacelike joint.

For instance, consider a spacelike joint embedded in a timelike boundary [with unit normals on either side of the joint oriented as in Fig. 2(a)]. One may apply the limiting procedure to obtain that

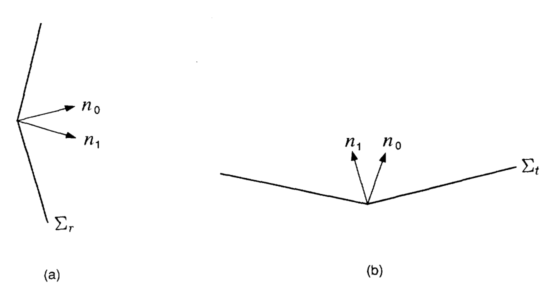

FIG. 2. (a) Class I spacelike joint in a timelike boundary with both normals spacelike and $\eta \equiv \text{arccosh}(n_0 \cdot n_1)$. (b) Class I spacelike joint in spacelike boundary with both normals timelike and $\eta \equiv \text{arccosh}(-n_0 \cdot n_1)$.

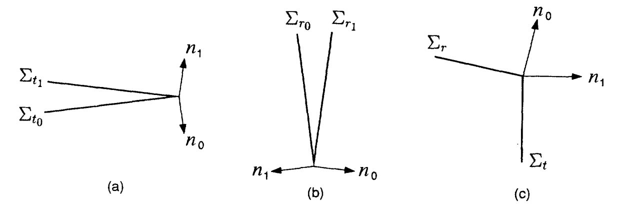

FIG. 3. Examples of class II joints. (a) $\eta \equiv \text{arccosh}(n_0 \cdot n_1)$. (b) $\eta \equiv \text{arccosh}(-n_0 \cdot n_1)$. (c) $\eta \equiv \text{arcsinh}(n_0 \cdot n_1)$.

$$\frac{1}{8\pi}\int_{\mathcal{B}} K\,h^{1/2}\,d^3x = \frac{1}{8\pi}\int_{\mathcal{B}_{\rm smooth}} K\,h^{1/2}\,d^3x + \frac{1}{8\pi}\int_{\partial\mathcal{B}} \eta\,\sigma^{1/2}\,d^2x, \tag{12}$$

where $\eta \equiv \text{arccosh}(n_0 \cdot n_1)$ is the local rapidity ("boost parameter") associated with a given point on the joint. In the case of a spacelike joint embedded in a spacelike boundary [with normals oriented as in Fig. 2(b)], one also obtains Eq. (12), but this time the rapidity is defined by $\eta \equiv \text{arccosh}(-n_0 \cdot n_1)$.

For class II joints (see Fig. 3), the smoothing procedure gives rise to an ambiguity at those surfaces where the normal goes null. To assess the contribution to the gravitational action of joints in this class, it is valuable to seek a derivation which proceeds directly from the variational principle.

III. USING VARIATIONAL METHODS TO EVALUATE THE CONTRIBUTION OF A JOINT TO THE ACTION

Recall that if an action functional is to have a well-defined variational principle subject to a given set of boundary constraints, its first-order variations must vanish identically. This condition prescribes the appropriate boundary term to be included in an action functional. In this section, I consider the gravitational action for a spacetime with class II spacelike joints and examine how the choice of constraints to be imposed at the joints determines their contribution to the gravitational action.

Let $\{\mathcal{M}, g_{\mu\nu}\}$ be a spacetime whose connected boundary $\mathcal{B}$ consists of two finite spacelike surfaces each connected by a spacelike class II joint to a smooth timelike tube (see Fig. 4). Let Σ_{t_0} and Σ_{t_1} be the spacelike surfaces, let $\mathcal{J}_0$ and $\mathcal{J}_1$ be the joints, and let Σ_{r_1} be the timelike tube. Spacetimes with this boundary structure arise naturally in the treatment of Hamiltonian dynamics for spatially bounded systems [6, 7].

Consider the action functional

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} R\sqrt{g}\,d^4x - \frac{1}{8\pi}\int_{\Sigma_{t_1}} K\sqrt{h}\,d^3x + \frac{1}{8\pi}\int_{\Sigma_{t_0}} K\sqrt{h}\,d^3x + \frac{1}{8\pi}\int_{\Sigma_{r_1}} \tilde{K}\sqrt{\gamma}\,d^3x. \tag{13}$$

In the above, a surface of constant t has an intrinsic three-metric h_{ij} and extrinsic curvature K_{ij}, while a surface of constant r has an intrinsic three-metric γ_{ab} and extrinsic curvature $\widetilde{K}_{ab}$. Note that this action includes a boundary correction term (appropriate for fixed boundary geometry) at each of the smooth boundary faces, but that no correction term is added at either of the joints.

Let $n^{\mu}_{(t)}$ be the future pointing normal to a surface of constant t, and let $n^{\mu}_{(r)}$ be the outward pointing unit normal to a surface of constant r. With the convention that a negative radial shift from Σ_t to $\Sigma_{t+\delta t}$ corresponds to a positive radial velocity, the local boost parameter at a given point along the intersection two-surface of Σ_t and Σ_r is defined by

$$\eta \equiv \operatorname{arcsinh}(n_{(t)} \cdot n_{(r)}). \tag{14}$$

In order to evaluate the first-order variations of the action (13), it is useful to employ foliations of the four-metric with respect to t and r. For more detail on these foliations, see Ref. [6]. Taking first-order variations of the action (13), we obtain

$$\delta I = \frac{1}{16\pi}\int_{\mathcal{M}} G_{\mu\nu}\,\delta g^{\mu\nu}\,d^4x + \int_{\Sigma_{t_1}} p^{ij}\,\delta h_{ij}\,d^3x - \int_{\Sigma_{t_0}} p^{ij}\,\delta h_{ij}\,d^3x + \int_{\Sigma_{r_1}} \widetilde{p}^{ab}\,\delta\gamma_{ab}\,d^3x - \frac{1}{8\pi}\int_{\mathcal{J}_1}\sqrt{\sigma}\,\delta\eta\,d^2x + \frac{1}{8\pi}\int_{\mathcal{J}_0}\sqrt{\sigma}\,\delta\eta\,d^2x, \tag{15}$$

where p^{ij} and $\widetilde{p}^{ab}$ are, respectively, the momentum fields conjugate to h_{ij} and γ_{ab}, and σ_{AB} denotes the intrinsic two-metric of surfaces $\Sigma_r \bigcap \Sigma_t$.

If the action is to be extremized, each variational term in (15) must vanish independently. The vanishing of the variations inside $\mathcal{M}$ yields Einstein's equations $G_{\mu\nu} = 0$. If we fix the intrinsic three-metric along the smooth portions of the boundary, the variational terms at these surfaces will also vanish. By virtue of continuity, fixing the three-metric on the smooth portions of the boundary implies fixing the intrinsic two-metrics of the joints. Note, however, that in order for the variational terms to vanish at the joints, we would also require that η be held fixed there. From the point of view of a classical variational problem, this is an unsavory situation. One expects that, in general, there will be no solution to Einstein's equations satisfying constraints on both the intrinsic two-metric and the boost parameter at a given joint.

On the other hand, consider the action

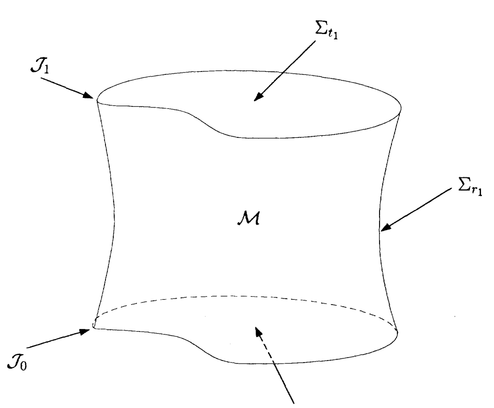

FIG. 4. A spacetime $\{\mathcal{M}, g_{\mu\nu}\}$, which extends from an initial spacelike hypersurface Σ_{t_0} to a final spacelike hypersurface Σ_{t_1} and out to a timelike hypersurface Σ_{r_1}. These boundary surfaces are connected by class II spacelike joints $\mathcal{J}_0$ and $\mathcal{J}_1$.

$$J = I + \frac{1}{8\pi}\int_{\mathcal{J}_1}\eta\sqrt{\sigma}\,d^2x - \frac{1}{8\pi}\int_{\mathcal{J}_0}\eta\sqrt{\sigma}\,d^2x. \tag{16}$$

(The relative sign difference between the two terms arises because of the convention that the timelike normal is taken to be future pointing in both cases.) The variations of J at the joints are

$$\frac{1}{8\pi}\int_{\mathcal{J}_1}\eta\,\delta\sqrt{\sigma}\,d^2x - \frac{1}{8\pi}\int_{\mathcal{J}_0}\eta\,\delta\sqrt{\sigma}\,d^2x. \tag{17}$$

Since we assume that the three-metrics on the smooth portions of the boundary are held fixed, the two-metrics at the joints are also fixed by virtue of continuity. Thus, the variational terms (17) automatically vanish and the action J has a well-defined variational principle subject to fixing the geometry of its boundary.

By appealing to the variational principle, we have found that it is appropriate to include the same joint correction term for a class II spacelike joint as we derived by the limiting procedure of Sec. II for a class I spacelike joint. In fact, it is straightforward to extend the variational treatment outlined above to apply to all spacelike and timelike joints. It is worth reiterating, however, that for a spacelike joint, the definition of the boost parameter in terms of the unit normals on either side of the joint depends on the orientations of these normals (see Figs. 2 and 3).

IV. JUNCTURE CONDITIONS AND JOINTS INSIDE A SPACETIME

To this point, we have focused entirely on joints which occur on the boundaries of spacetimes. In this section, we consider joints which occur within spacetimes.

Suppose, for instance, we have an infinitely thin shell of matter which has a sharp one-dimensional "corner" or "edge." Let Σ_r be the timelike three-surface corresponding to the world history of the shell, and let $\mathcal{J}_r$ be the timelike joint corresponding to the world history of the edge. We suppose that the energy per unit surface area is finite everywhere except at the edge where we suppose a finite energy per unit length. Now let us derive the juncture conditions at the shell and, in particular, at the joint.

Let $g_{\mu\nu}$ be the four-metric and assume that it is smooth everywhere except possibly at Σ_r. Let γ_{ab} be the intrinsic three-metric of Σ_r, and let $\widetilde{K}_{ab}$ be its extrinsic curvature. We take γ_{ab} to be continuous from Σ_{r_-} to Σ_{r_+}, but allow for a possible jump discontinuity in $\widetilde{K}_{ab}$ between these neighboring surfaces. We also allow for a jump discontinuity in the edge angle $\Theta \equiv \arccos(n_0 \cdot n_1)$ from $\mathcal{J}_{r_-}$ to $\mathcal{J}_{r_+}$.

It is easy to derive (see, for instance, Ref. [8]) that the total action for this spacetime is

$$I = \frac{1}{16\pi}\int_{\mathcal{M}_{\rm smooth}} R\sqrt{g}\, d^4x - \frac{1}{8\pi}\int_{\Sigma_r} \widetilde{K}\Big|_-^+ \sqrt{\gamma}\, d^3x + \int_{\mathcal{M}} \mathcal{L}_{\rm m}\, d^4x, \tag{18}$$

where $\mathcal{L}_{\rm m}$ is the Lagrangian scalar density associated with the matter distribution. Further, with $S_r \equiv \Sigma_r - \mathcal{J}$, we have

$$\frac{1}{8\pi}\int_{\Sigma_r} \widetilde{K}\Big|_-^+ \sqrt{\gamma}\, d^3x = \frac{1}{8\pi}\int_{S_r} \widetilde{K}\Big|_-^+ \sqrt{\gamma}\, d^3x + \frac{1}{8\pi}\int_{\mathcal{J}_r} \Theta|_-^+ \sqrt{\sigma}\, d^2x. \tag{19}$$

Taking first-order variations of the action, we obtain the usual variational terms which yield Einstein's equations in the regions where $g_{\mu\nu}$ is smooth, plus a variational term at Σ_r:

$$\int_{S_r} \left\{ - \widetilde{p}^{ab}|_-^+ - \frac{\sqrt{\gamma}}{2} S^{ab} \right\} \delta\gamma_{ab}\, d^3x + \frac{1}{16\pi}\int_{\mathcal{J}_r} \{- \Theta|_-^+ \sigma^{AB} - 8\pi T^{AB}\}\, \delta\sigma_{AB}\sqrt{\sigma}\, d^2x, \tag{20}$$

where $S^{ab} \equiv -\frac{2}{\sqrt{\gamma}}\frac{\delta\mathcal{L}_{\rm m}}{\delta\gamma_{ab}}\Big|_{S_r}$ is the surface stress energy tensor for S_r and where $T^{AB} \equiv -\frac{2}{\sqrt{\sigma}}\frac{\delta\mathcal{L}_{\rm m}}{\delta\sigma^{AB}}\Big|_{\mathcal{J}_r}$ is the stress tensor associated with the matter on the edge.

Since all variational terms must vanish independently if the action is to be extremized, we obtain the usual Israel juncture conditions on S_r,

$$\widetilde{p}^{ab}|_-^+ = -\frac{\sqrt{\gamma}}{2} S^{ab}, \tag{21}$$

plus the joint juncture condition

$$\Theta|_-^+ \sigma^{AB} = -8\pi T^{AB}. \tag{22}$$

Equation (22) implies

$$T^A_B = \mu\delta^A_B, \tag{23}$$

where $\mu = -\frac{1}{8\pi}\,\Theta|_-^+$ is the local energy per unit length of the edge.

A corollary of Eq. (22) is that if there is no singular matter source at $\mathcal{J}_r$, the joint angle is continuous from $\mathcal{J}_{r_-}$ to $\mathcal{J}_{r_+}$.

We have until now examined only joints which are embedded in three-dimensional surfaces. Let us now extend our discussion to include joints which are not embedded in three-dimensional surfaces. Cosmic strings and event horizons (or "bolts" [9]) are examples of such joints.

From the analysis above, it is clear that when one wishes to fix the intrinsic two-metric of an isolated timelike joint, one must correct the gravitational action by a term

$$\frac{1}{8\pi}\int_{\mathcal{J}} \Theta\sqrt{\sigma}\, d^2x. \tag{24}$$

If one wishes to fix the intrinsic two-metric of an isolated spacelike joint, one must correct the gravitational action by a term [10]

$$\frac{1}{8\pi}\int_{\mathcal{J}} \eta\sqrt{\sigma}\, d^2x. \tag{25}$$

If one wishes to fix either Θ for a timelike joint or η for a spacelike one, no correction to the action is necessary.

Also, when a spacetime contains a singular matter distribution which gives rise to an isolated conical singularity two-surface, this surface makes a finite contribution to the gravitational action. We can model this situation by taking a thin tube of matter and collapsing it onto a one-dimensional axis. We do this so that the world history of the axis corresponds to $\mathcal{J}$.

The tube's contribution to the gravitational action is

$$-\frac{1}{8\pi}\int_{\Sigma_r} \widetilde{K}\Big|_-^+ \sqrt{\gamma}\, d^3x, \tag{26}$$

where $\widetilde{K}_{ab}\Big|_-$ is the extrinsic curvature of Σ_r as embedded in the flat four-geometry inside the tube. In the limit that the tube collapses on the axis, the contribution to the gravitational action becomes

$$-\frac{1}{8\pi}\int_{\mathcal{J}} \Theta|_-^+ \sqrt{\sigma}\, d^2x = \frac{1}{8\pi}\int_{\mathcal{J}} \alpha\sqrt{\sigma}\, d^2x, \tag{27}$$

where $\Theta|_- = 2\pi$ and $\alpha = 2\pi - \Theta|_+$ is the local deficit angle at some point of $\mathcal{J}$. When α is constant over $\mathcal{J}$, note that the gravitational action associated with the conical singularity reduces to the Nambu action.

V. SUMMARY

In summary, we have derived the contribution of spacelike and timelike joints to the gravitational action by two independent techniques. We have discussed the relationship between constraints imposed at a joint and its contribution to the gravitational action. We have also derived contributions to the gravitational action and juncture conditions at joints which occur as a result of singular matter distributions. We have found that the gravitational action for an isolated timelike joint with constant deficit angle reduces to the Nambu action.

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[10] Fixing the intrinsic two-metric of an isolated joint can be viewed as a special case of fixing the three-metric on a boundary in the limit that this boundary collapses on the joint. In this case, the locus of points associated with the joint are strictly not included in the manifold over which the action is defined. Thus, even though fixing the two-metric at the joint generally leads to a conical singularity there, it is not appropriate to include the contribution of this conical singularity to the gravitational action.