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Numerical study of cosmic no-hair conjecture. II. Analysis of initial data

Ken-ichi Nakao
Department of Physics, Kyoto University, Sakyo-ku, Kyoto 606, Japan

Kei-ichi Maeda
Department of Physics, Waseda University, Shinjuku-ku, Tokyo 169, Japan

Takashi Nakamura
Yukawa Institute for Theoretical Physics, Kyoto University, Sakyo-ku, Kyoto 606, Japan

Ken-ichi Oohara
National Laboratory for High Energy Physics (KEK), Tsukuba, Ibaraki 305, Japan
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We investigate the initial data for localized gravitational waves in space-time with a cosmological constant Λ. Using a conformal transformation, we find that the Hamiltonian and momentum constraints in the conformal frame turn out to be the same as those for a vacuum space-time without Λ. As initial data, we consider Brill's waves in the conformal frame and discuss the trapped and antitrapped surfaces in the physical frame. Just as Brill's wave in asymptotically flat space-time, the gravitational "mass" of our case is positive; however, the waves with a large gravitational mass do not always provide trapped surfaces in contrast with the case of $\Lambda=0$. The large amount of gravitational waves, hence, does not seem to be an obstacle to the cosmic no-hair conjecture.

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I. INTRODUCTION

The present isotropy and homogeneity of the Universe are almost confirmed from observations. They are among the mysteries within the framework of the standard big bang scenario. The inflationary universe scenario was proposed to resolve this isotropy-homogeneity problem [1]. As we discussed in our previous paper (paper I) [2], however, since most inflationary models have worked within Friedmann-Robertson-Walker space-time, there still remains some doubt of a homogenization process due to inflation: Even if the initial inhomogeneities are very large before the onset of inflation, is de Sitter–like rapid cosmic expansion always realized as long as a vacuum energy exists? Does inflation really homogenize any space-time? The "cosmic no-hair conjecture" states that all space-times approach the de Sitter space-time if a positive cosmological constant exists [3]. If this conjecture is true as it is, there is no doubt to the above question. However, in general, the inhomogeneities have energy and produce the gravitational field by itself. Then, we can imagine that when the inhomogeneities are very large, those inhomogeneities are not homogenized by the cosmic expansion, but rather collapse into black holes or naked singularities.

As for the origin of inhomogeneities, we can classify them into the two parts: One is due to an inflaton scalar field itself and the other is due to known fields. Once inflation starts, the energy of known fields will drop much faster than that of the inflaton, and then only inhomogeneities due to the inflaton field will survive. But, when we ask about the onset of inflation and the homogenization process, we should take into account both fields as sources of initial inhomogeneities. The inhomogeneities of known fields may be further divided into two classes: One is the inhomogeneity of the space-time itself, i.e., due to the gravitational waves, and the other is the inhomogeneity by ordinary matter. Inhomogeneities of an inflaton field have been investigated by several authors both in analytic and in numerical approaches, [4] and [5]. As for a dust fluid in space-time with Λ, we have some analytic approaches, showing that some inhomogeneities collapse into black-hole space-time, [6] and [7]. Because there has been little research in inhomogeneities due to gravitational waves, this is our present topic.

In paper I we presented a new formalism to solve the Einstein equations by using a computer as numerical cosmology and applying it to the case of linear perturbations, giving an analytic solution and clarifying the homogenization mechanism in de Sitter background space-time. In the present paper, our attention is focused on the initial data of localized gravitational waves in vacuum space-times with a positive cosmological constant Λ. Here "localized" means that the spacelike asymptotic region is the Schwarzschild–de Sitter space [8]. In such a situation, it is convenient to perform a conformal transformation and work in conformal space-time [2]. Because we consider the initial value problem, through an appropriate conformal transformation, the Hamiltonian and momentum constraints in the conformal space-time reduce to the same form as those of vacuum space-times without Λ. The asymptotic Schwarzschild–de Sitter boundary condition turns out to be the ordinary asymptotic flatness condition in the conformal space-time. It is a big advantage because there have been many works in the initial value problem with asymptotically flat space-time.

In particular, as a localized axisymmetric gravitational wave solution of the time-symmetric initial value problem for vacuum space-times, we have Brill's wave and know its physical property well [9]. In this paper, hence, we shall focus on Brill's waves in the conformal space-time, since it is easy to study the effect of Λ by comparing it with the original Brill waves.

Brill discussed the positivity of the mass of those gravitational waves and showed that when the amplitude of the gravitational wave exceeds some critical value, the three-dimensional spacelike hypersurface could not be asymptotically flat but is closed by the energy of the gravitational waves. These are the features common to our Brill waves in the conformal space-time.

However, in contrast with the original Brill waves, there is a cosmic expansion effect in our circumstance. The cosmic expansion effect causes the antitrapped surfaces and the cosmological apparent horizon, which will be mentioned in Sec. IV. We find that a too large gravitational mass prevents the formation of the trapped region in contrast with the original Brill waves without Λ. These are the same features as the spherically symmetric dust collapse with the Schwarzschild–de Sitter asymptotic region [7]. Thus it is likely that the large number of gravitational waves is not always an obstacle to the cosmic no-hair conjecture.

This paper is organized as follows: In Sec. II, we derive the basic equations for the initial value problem of vacuum space-times with Λ by use of a conformal transformation. In Sec. III, we consider Brill waves in the conformal space-time and solve these numerically. In order to gain some insight from those data, we investigate the trapped and antitrapped surfaces in Sec. IV. Some discussion and remarks are given in Sec. V. In this paper, we adopt units of $c=G=1$. Our conventions are the same as those in paper I.

II. INITIAL VALUE PROBLEM OF VACUUM SPACE-TIMES WITH A COSMOLOGICAL CONSTANT

We consider vacuum space-times with a cosmological constant Λ, which is governed by the Einstein equations

$$G_{\mu\nu}=-3H_0^2 g_{\mu\nu} , \tag{2.1}$$

where

$$H_0 \equiv \left[\frac{\Lambda}{3}\right]^{1/2} . \tag{2.2}$$

When we deal with the above equations, it is convenient to use the 3+1 formalism for the conformally transformed Einstein equations [2]. In this section, we shall focus on the Hamiltonian and momentum constraints of the conformally transformed Einstein equations.

Hereafter we work in the conformal space-time with the four-dimensional metric $\tilde{g}_{\mu\nu}$, which is related with the physical metric $g_{\mu\nu}$ by

$$g_{\mu\nu}=\Omega^2\tilde{g}_{\mu\nu} , \tag{2.3}$$

where a conformal factor $\Omega(x)$ will be fixed later. The Einstein equation in this conformal space-time is rewritten as

$$\tilde{G}_{\mu\nu}=-3H_0^2\Omega^2\tilde{g}_{\mu\nu}+8\pi\tilde{T}_{\mu\nu} , \tag{2.4}$$

where

$$\tilde{T}_{\mu\nu}\equiv\frac{1}{4\pi}[\Omega^{-1}(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\Omega-\tilde{g}_{\mu\nu}\tilde{\nabla}_\rho\tilde{\nabla}^\rho\Omega) -2\Omega^{-2}(\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega-\tfrac{1}{4}\tilde{g}_{\mu\nu}\tilde{\nabla}_\rho\Omega\tilde{\nabla}^\rho\Omega)] , \tag{2.5}$$

and $\tilde{G}_{\mu\nu}$ and $\tilde{\nabla}_\mu$ are the Einstein tensor and the covariant derivative with respect to the conformal metric $\tilde{g}_{\mu\nu}$, respectively.

Following the well-known procedure, we decompose the above conformal Einstein equations into the 3+1 form. The Hamiltonian and momentum constraints in the conformal space-time are given by

$${}^{(3)}\tilde{R}-\tilde{K}_a^b\tilde{K}_b^a+\mathrm{Tr}\tilde{K}^2=16\pi\tilde{\rho}_H+6H_0^2\Omega^2 , \tag{2.6}$$

$$\tilde{D}_6(\tilde{K}_a^b-\delta_a^b\,\mathrm{Tr}\tilde{K})=8\pi\tilde{J}_a , \tag{2.7}$$

where ${}^{(3)}\tilde{R}$, $\tilde{K}_a^b$ and $\mathrm{Tr}\tilde{K}$ are the scalar curvature, the extrinsic curvature and its trace part, respectively. These quantities are defined on the three-dimensional spacelike hypersurface embedded in the conformal space-time. $\tilde{D}_b$ is the covariant derivative with respect to the intrinsic metric $\tilde{\gamma}_{ab}$ of the hypersurface. The "apparent" energy and momentum densities $\tilde{\rho}_H$ and $\tilde{J}_a$, which appeared from a factorization of a conformal factor Ω, are given by

$$\tilde{\rho}_H=\frac{1}{8\pi}(2\Omega^{-1}\tilde{D}_a\tilde{D}^a\Omega+2\Omega^{-1}\chi\,\mathrm{Tr}\tilde{K} -\Omega^{-2}\tilde{D}_a\Omega\tilde{D}^a\Omega-3\Omega^{-2}\chi^2) , \tag{2.8}$$

$$\tilde{J}_a=-\frac{1}{4\pi}[\Omega^{-1}\tilde{D}_a(\chi\Omega^{-2})+\Omega^{-1}\tilde{K}_a^b\tilde{D}_b\Omega] , \tag{2.9}$$

where

$$\chi\equiv\mathcal{L}_{\tilde{n}}\Omega . \tag{2.10}$$

Here $\mathcal{L}_{\tilde{n}}$ is the Lie derivative along the hypersurface unit normal vector $\tilde{n}^\mu$ in the conformal space-time, which satisfies $\tilde{g}_{\mu\nu}\tilde{n}^\mu\tilde{n}^\nu=-1$.

In order to solve Eqs. (2.6) and (2.7), we have to specify a conformal factor, i.e., to fix Ω and χ. Here, we first impose the condition

$$\frac{\chi}{\Omega^2}=H_0 . \tag{2.11}$$

In isotropic and homogeneous space-time, the above condition turns out to be the Friedmann equation in terms of the conformal time $\eta, \Omega^{-2}(d\Omega/d\eta)=H_0$, resulting in the de Sitter solution, $\Omega=-1/(H_0\eta)$. Thus, Ω naively corresponds to the scale factor of the de Sitter background in our specification.

In the next section, we consider localized gravitational waves. Here "localized" means that there is an asymptotic Schwarzschild–de Sitter region. We, hence, expect that the asymptotic Schwarzschild–de Sitter boundary condition may be reduced to the ordinary asymptotic flatness condition in the conformal space-time. Because if the conformal space-time is flat, the condition (2.11) guarantees that the physical space-time is de Sitter. However, for arbitrary Ω, even though the condition (2.11) is satisfied, the boundary condition is not always asymptotically flat in the conformal space-time. We have, furthermore, to fix either Ω or χ. In order that the asymptotic flatness condition is imposed in the conformal space-time, at least the integral of the right-hand side (RHS) of Eq. (2.6) on our hypersurface should be finite, i.e.,

$$\left| \int (16\pi\tilde{\rho}_H + 2\Lambda\Omega^2) d\tilde{v} \right| < \infty \ . \tag{2.12}$$

Therefore, considering circumstances with the asymptotic Schwarzschild–de Sitter region, if we would like to impose an asymptotically flat condition in a conformal frame, we have to demand the asymptotic behavior of a conformal factor as

$$\Omega \rightarrow \text{const} + O(r^{-1}) \quad \text{for } r \rightarrow +\infty \ . \tag{2.13}$$

Conversely, the above behavior will be preserved in the course of the time evolution as long as the conformal space-time is asymptotically flat. Here we adopt the simplest choice such that Ω is spatially constant. In that case, we can always rescale the spatial coordinates to set $\Omega = 1$. However, here we remain the parameter Ω to be free in order to obtain a one-parameter family of solutions from one numerical solution. Handling Ω, we can easily see the effect of the scale of inhomogeneities.

Assuming $\mathrm{Tr}\tilde{K} = 0$ initially, the constraint equations coincide with those of vacuum space-times without Λ, i.e.,

$$^{(3)}\tilde{R} - \tilde{K}_a^{\ b}\tilde{K}_b^{\ a} = 0 \ , \tag{2.14}$$

$$\tilde{D}_b \tilde{K}_a^{\ b} = 0 \ . \tag{2.15}$$

We can solve the above equations by the procedure to obtain the initial data of the asymptotically flat space-time. This approach has a big advantage because we know many works for Eqs. (2.14) and (2.15).

III. BRILL WAVE IN THE CONFORMAL SPACE-TIME

In the preceding section we reduced the constraint equations to the same form as those of vacuum space-time without Λ by the appropriate choice of Ω and χ. In this section, we construct time symmetric initial data of localized gravitational waves with axial symmetry in the conformal space-time. Here, the time symmetry in the conformal space means that the extrinsic curvature $\tilde{K}_{ab}$ vanishes. Therefore, the momentum constraint (2.15) is trivially satisfied and the Hamiltonian constraint becomes the same form as that of the ordinary time symmetric initial value problem for a vacuum space-time without Λ [10]:

$$^{(3)}\tilde{R} = 0 \ . \tag{3.1}$$

Hereafter, we assume that the topology of a three-dimensional spacelike hypersurface is $\mathbf{R}^3$.

The time-symmetric initial data of localized gravitational waves without Λ were discussed by Brill [9] and treated numerically by Eppley [11]. The Brill wave can be interpreted as a snapshot of axisymmetric gravitational waves as a moment on maximum expansion (or minimum contraction), while our wave solution corresponds to a snapshot of the waves in the uniformly expanding background because of the condition (2.11), although their procedure and discussion are almost valid in our case as well.

Following Brill, we write the intrinsic metric of the conformal three-dimensional spacelike hypersurface in the form

$$\begin{aligned} d\tilde{l}^2 &\equiv \tilde{\gamma}_{ab} dx^a dx^b \\ &= \psi^4(R,z)[e^{Aq(R,z)}(dR^2+dz^2)+R^2 d\varphi^2] \ , \end{aligned} \tag{3.2}$$

where A is a constant, which corresponds to the amplitude of the gravitational waves, and $q(R,z)$ is an arbitrary function that satisfies the boundary conditions

$$\begin{aligned} &q = 0 = \partial_R q \quad \text{at } R = 0 \ , \\ &q \rightarrow O(1/r^2) \quad \text{or faster for } r \rightarrow +\infty \ , \end{aligned} \tag{3.3}$$

where $r = \sqrt{R^2+z^2}$. q provides a localization of the waves. We then adopt the following function $q(R,z)$:

$$q(R,z) = \left[\frac{R}{\tilde{r}_0} \right]^2 \exp\left[-\frac{r^2}{2\tilde{r}_0^2} \right] , \tag{3.4}$$

where $\tilde{r}_0$ is a constant, which corresponds to a width of the waves in the conformal frame. It should be noticed that the width of the gravitational waves in the physical space is determined by $r_0 \equiv \Omega\tilde{r}_0$ rather than $\tilde{r}_0$ itself.

With (3.2) and (3.4), the Hamiltonian constraint (3.1) is written as

$$\partial_R^2\psi + \frac{1}{R}\partial_R\psi + \partial_z^2\psi + \frac{A}{8}U(R,z)\psi = 0 \ , \tag{3.5}$$

where

$$U(R,z) = \left[\frac{2}{\tilde{r}_0^2} - \frac{6R^2}{\tilde{r}_0^4} + \frac{R^2 r^2}{\tilde{r}_0^6} \right] \exp\left[-\frac{r^2}{2\tilde{r}_0^2} \right] . \tag{3.6}$$

As mentioned in the preceding section, in order to impose the asymptotic Schwarzschild–de Sitter boundary condition, we demand the ordinary asymptotically flat behavior for the conformal factor ψ as

$$\psi \rightarrow 1 + \frac{\tilde{M}}{2r} \quad \text{for } r \rightarrow +\infty \ , \tag{3.7}$$

where $\tilde{M}$ is constant. It is again worthy of notice that $M \equiv \Omega\tilde{M}$ corresponds to the gravitational mass in the asymptotic Schwarzschild–de Sitter space-time. Hence, we regard it as a gravitational mass of the wave and, as shown by Brill, it is always positive for nonvanishing A.

From a regularity on the symmetric axis, $R = 0$, and from a reflection symmetry with respect to the equatorial plane, $z = 0$, the conformal factor ψ should satisfy the boundary condition

$$\partial_R\psi|_{R=0} = \partial_z\psi|_{z=0} = 0 \ . \tag{3.8}$$

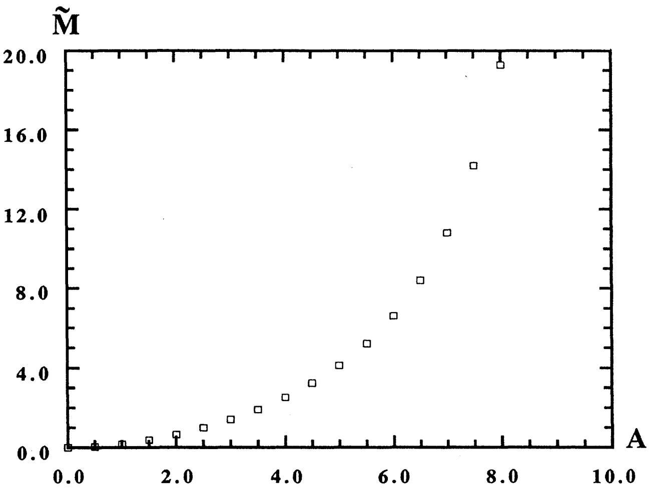

FIG. 1. The relation between $\tilde{M}$ and A is depicted. $\tilde{M}$ monotonically increases with respect to A. For $A=10$, $\tilde{M}$ becomes infinite, which corresponds to a "closed" universe.

As pointed out by Nakamura, Oohara, and Kojima [12], it is an even function of R and of z, since the conformal factor ψ should be regular everywhere. From the point of view of numerical accuracy, $X \equiv R^2$ and $Y \equiv z^2$ are better independent variables than R and z themselves. Using X and Y, hence, we perform the numerical calculation with 250×250 spatial lattices. The widths of the gravitational waves $\tilde{r}_0$ are set to unity in our calculation. The physical width of the waves is, then, described by $r_0 = \Omega$.

We show the relation between $\tilde{M}$ and A in Fig. 1. The gravitational mass M is a monotonically increasing function with respect to the wave amplitude, fixing the width r_0.

When A exceeds the critical value obtained by Brill, the three-dimensional spacelike hypersurface is eventually closed by the self-energy of the gravitational waves. In our case, the critical wave amplitude is $A \approx 10$. In this paper, however, we have assumed the existence of the asymptotic Schwarzschild–de Sitter region; then we consider only the cases of $A \leq 10$.

IV. THE TRAPPED AND ANTITRAPPED SURFACES

Once we set the initial data, we can search for horizons for the localized gravitational waves. This study may give us some physical insight into the formation mechanism of horizons. In the following discussions, we shall adopt the definitions of a trapped surface and an apparent horizon of Hawking [13]. Hawking defined the trapped surface as the closed spacelike two-surface such that the family of future-directed *outgoing* null geodesics orthogonal to the surface is converging at each point on it. Hence the expansion of outgoing null geodesic congruence orthogonal to a trapped surface is negative. Since we do not discuss the formation of singularities, we need not consider the expansion of the *ingoing* null geodesic congruence. Here *ingoing* means the direction toward the regions enclosed by a surface and *outgoing* is the opposite direction. The apparent horizon is defined by the outermost closed spacelike two-surface with vanishing expansion of the future-directed outgoing null geodesic congruence orthogonal to it. Hereafter, we call it a black-hole apparent horizon.

When gravitational waves are localized enough and have a large amplitude, the trapped surfaces are formed in the asymptotically flat space. Also in our case, an *appropriately* large amount of gravitational waves form a trapped surface. As will be mentioned later, however, when the cosmic expansion effect exists, the gravitational waves with a too large gravitational mass prevent the formation of a trapped region.

The cosmic expansion effect causes another kind of surface. We call it an antitrapped surface. The antitrapped surface is a closed spacelike two-surface such that both expansions of the future-directed ingoing and outgoing null geodesic congruence orthogonal to the surface are positive. So the area of a wave front of light, which is emitted inward and orthogonal to the antitrapped surface, does not decrease but rather does increase at that moment. We shall also define the cosmological apparent horizon by the inner boundary of the antitrapped surfaces, i.e., the outermost closed spacelike two-surface with vanishing expansion of the future-directed ingoing null geodesic congruence orthogonal to the surface.

It should be noticed that such surfaces are observer dependent, as can be seen in the example of the de Sitter universe. However, since we are interested in the cosmic expansion effect on the localized gravitational waves, the observer is the gravitational waves themselves, which are localized near the origin. Thus we focus on the antitrapped surfaces and the cosmological apparent horizon surrounding the origin.

Suppose we have some closed spacelike two-surface S's in the three-dimensional spacelike hypersurface. Then the expansions $\theta_{\rm in}$ and $\theta_{\rm out}$ of the ingoing and outgoing null orthogonal to S are given by

$$\theta_{\rm in} = \frac{1}{\Omega}\tilde{\theta}_{\rm in} + 2\frac{\chi}{\Omega^2} = \frac{1}{\Omega}\tilde{\theta}_{\rm in} + 2H_0 , \tag{4.1}$$

$$\theta_{\rm out} = \frac{1}{\Omega}\tilde{\theta}_{\rm out} + 2\frac{\chi}{\Omega^2} = \frac{1}{\Omega}\tilde{\theta}_{\rm out} + 2H_0 , \tag{4.2}$$

with

$$\tilde{\theta}_{\rm in} = -\tilde{D}_a \tilde{s}^a + \tilde{K}_{ab}\tilde{s}^a\tilde{s}^b - {\rm Tr}\tilde{K} , \tag{4.3}$$

$$\tilde{\theta}_{\rm out} = +\tilde{D}_a \tilde{s}^a + \tilde{K}_{ab}\tilde{s}^a\tilde{s}^b - {\rm Tr}\tilde{K} , \tag{4.4}$$

where $\tilde{s}^a$ is the outward spacelike unit-normal vector of S in the conformal space-time, which satisfies $\tilde{\gamma}_{ab}\tilde{s}^a\tilde{s}^b = 1$. The second equalities in Eqs. (4.1) and (4.2) are due to the condition (2.11). It should be noticed that $\theta_{\rm in}$ and $\theta_{\rm out}$ are not expansions $\theta_{\rm in}^{(g)}$ and $\theta_{\rm out}^{(g)}$ of the ingoing and outgoing null geodesic congruences. However, $\theta_{\rm in}$ and $\theta_{\rm out}$ are related with $\theta_{\rm in}^{(g)}$ and $\theta_{\rm out}^{(g)}$ by

$$\theta_{\rm in}^{(g)} = f(x)\theta_{\rm in} \quad \text{and} \quad \theta_{\rm out}^{(g)} = f(x)\theta_{\rm out} , \tag{4.5}$$

where $f(x)$ is some positive-definite function, which is determined by the geodesic equation. Hence, the trapped surfaces are found by $\theta_{\rm out}<0$, and the black-hole apparent horizon is obtained by setting $\theta_{\rm out}=0$. On the other hand, the antitrapped surfaces are found by $\theta_{\rm in}>0$ and $\theta_{\rm out}>0$ and the cosmological apparent horizon corresponds to $\theta_{\rm in}=0$.

The last terms on the RHS of Eqs. (4.1) and (4.2) correspond to the cosmic expansion effect. As expected, we can see that the cosmic expansion effect makes both expansions of ingoing and outgoing null larger. As a result, the antitrapped region can be formed, while the cosmic expansion makes a trapped region hard to be formed.

By rescaling of Ω, we can easily see the effect of the width of the waves or of the gravitational mass M on a formation of trapped regions. The value of Ω itself determines the physical scale length; hence, the physical width of the gravitational wave is $r_0=\Omega$. The gravitational mass M of the waves also increases linearly with respect to $\Omega, M=\Omega\widetilde{M}$. The effect of Ω in the expansions $\theta_{\rm in}$ and $\theta_{\rm out}$ appears such that the absolute values of $\widetilde{\theta}_{\rm in}/\Omega$ and $\widetilde{\theta}_{\rm out}/\Omega$ in Eqs. (4.1) and (4.2) become smaller as Ω larger. Thus, when the cosmic expansion effect exists, the large M or r_0 prevents a formation of the trapped region.

In order to obtain black-hole and cosmological horizons, we adopt the prescription proposed by Sasaki *et al.* [14]. We assume that the topology of the equiexpansion surfaces is S^2. As long as the deviation of the shape of such a surface from a sphere is not so large, these surfaces are expressed as

$$\begin{aligned} R&=r(\vartheta)\sin\vartheta\ ,\\ z&=r(\vartheta)\cos\vartheta\ , \end{aligned} \tag{4.6}$$

where $r=r(\vartheta)$ is assumed to be single-valued function.

The equations for the equiexpansion surfaces with $\theta_{\rm in}$ and $\theta_{\rm out}$ for our initial data discussed in Sec. III are given by

$$\frac{d^2r}{d\vartheta^2}=A_3\left[\frac{dr}{d\vartheta}\right]^3+A_2\left[\frac{dr}{d\vartheta}\right]^2+A_1\left[\frac{dr}{d\vartheta}\right]+A_0\ , \tag{4.7}$$

where

$$\begin{aligned} A_0=r[2&+R(4\psi^{-1}\partial_R\psi+A\partial_Rq)\\ &+z(4\psi^{-1}\partial_z\psi+A\partial_zq)]-N\widetilde{\theta}_{\rm out}\ , \end{aligned} \tag{4.8}$$

$$\begin{aligned} A_1=-[z/R&+z(4\psi^{-1}\partial_R\psi+A\partial_Rq)\\ &-R(4\psi^{-1}\partial_z\psi+A\partial_zq)]\ , \end{aligned} \tag{4.9}$$

$$\begin{aligned} A_2=r^{-1}[3&+R(4\psi^{-1}\partial_R\psi+A\partial_Rq)\\ &+z(4\psi^{-1}\partial_z\psi+A\partial_zq)]\ , \end{aligned} \tag{4.10}$$

$$\begin{aligned} A_3=-r^{-2}[z/R&+z(4\psi^{-1}\partial_R\psi+A\partial_Rq)\\ &-R(4\psi^{-1}\partial_z\psi+A\partial_zq)]\ , \end{aligned} \tag{4.11}$$

with

$$N=r\psi^{-1}\left[r^2+\left[\frac{dr}{d\vartheta}\right]^2\right]^{-3/2}\ , \tag{4.12}$$

and

$$\widetilde{\theta}_{\rm out}=\begin{cases} r_0(-\theta_{\rm in}+2H_0)\\ \text{for the case of } \theta_{\rm in}=\text{const}\ ,\\ r_0(+\theta_{\rm out}-2H_0)\\ \text{for the case of } \theta_{\rm out}=\text{const}\ . \end{cases} \tag{4.13}$$

In Eq. (4.13) we have used $\Omega=r_0/\widetilde{r}_0=r_0$ and $\widetilde{\theta}_{\rm in}=-\widetilde{\theta}_{\rm out}$.

We solve the above equation numerically and obtain equiexpansion surfaces for initial data with $A=3$ and 6. In Figs. 2 and 3, $\widetilde{\theta}_{\rm out}$ for $A=3$ and for $A=6$ are plotted with respect to r, respectively. The gravitational mass M is $1.40r_0$ for $A=3$ and $6.61r_0$ for $A=6$. The dashed lines in these figures are $\widetilde{\theta}_{\rm out}$ of the Schwarzschild–de Sitter space-time with same M as those of the gravitational waves. As expected, these values of the gravitational waves and of the Schwarzschild–de Sitter space-time agree well with each other in the far-out region.

Since the cosmic expansion effect can be discussed just by adding H_0 to $\widetilde{\theta}_{\rm out}$, we first consider $\widetilde{\theta}_{\rm out}$ itself, which is the same as in an asymptotically flat case. For the case of $A=3$, there is no trapped region, which is essentially the same as the Minkowski space. On the other hand, the trapped region exists in the case of $A=6$. In the trapped region, the ingoing expansion $\widetilde{\theta}_{\rm in}$ is positive, since that is the opposite sign of the outgoing expansion $\widetilde{\theta}_{\rm out}$. Since $\widetilde{\theta}_{\rm out}$ is not positive, however, it is not the antitrapped region. So its boundary with $\widetilde{\theta}_{\rm in}=0$ is not the cosmological apparent horizon. We call this outer boundary the white-hole apparent horizon. Then the black-hole apparent horizon coincides with the white-hole apparent horizon in the present asymptotically flat case.

Next, we consider the effect of cosmic expansion, which changes the positions of the above horizons. We can obtain $\theta_{\rm in}$ and $\theta_{\rm out}$ from $\widetilde{\theta}_{\rm out}$ by the use of Eq. (4.13). In this case, the antitrapped region and cosmological apparent horizon appear for both $A=3$ and 6. The case of $A=3$ is qualitatively the same as the de Sitter space-time. Only a cosmological horizon appears. On the other hand, in the case of $A=6$ the situation is rather complicated. When $0<r_0\le 4\times10^{-3}l_H$, both the black-hole and white-hole apparent horizons appear, but these do not coincide with each other. The black-hole apparent horizon is located inside the white-hole apparent horizon in this case. Here, we have introduced the cosmological horizon scale, $l_H\equiv H_0^{-1}$, which is a typical scale in a space-time with cosmological constant. If r_0 is larger than $4\times10^{-3}l_H$, the trapped region vanishes and then the black-hole apparent horizon also vanishes. However, as long as $r_0\le 2.9\times10^{-2}l_H$, there still exists a white-hole apparent horizon. When r_0 is equal to about $2.9\times10^{-2}l_H$, the white-hole apparent horizon coincides with the cosmological apparent horizon. In the case of $r_0>2.9\times10^{-2}l_H$, the white-hole apparent horizon also disappears, and only a cosmological apparent horizon ex-

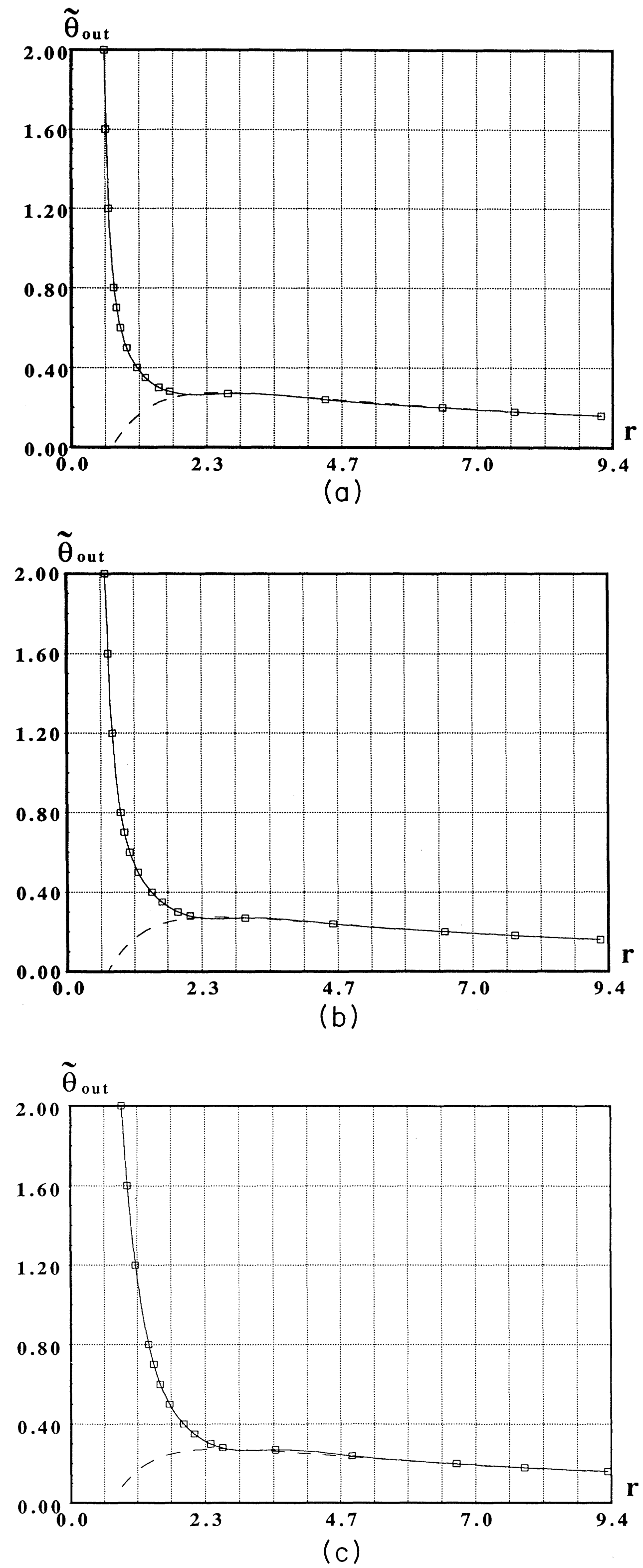

FIG. 2. $\tilde{\theta}_{out}$ of the equiexpansion surfaces of the gravitational waves with $A=3$ is depicted by squares with respect to r on the $R=0$ axis (a), $R=z$ (b), and $z=0$ (c). The dashed line is $\tilde{\theta}$ of the Schwarzschild–de Sitter space-time with the same mass of the gravitational wave with $A=3$.

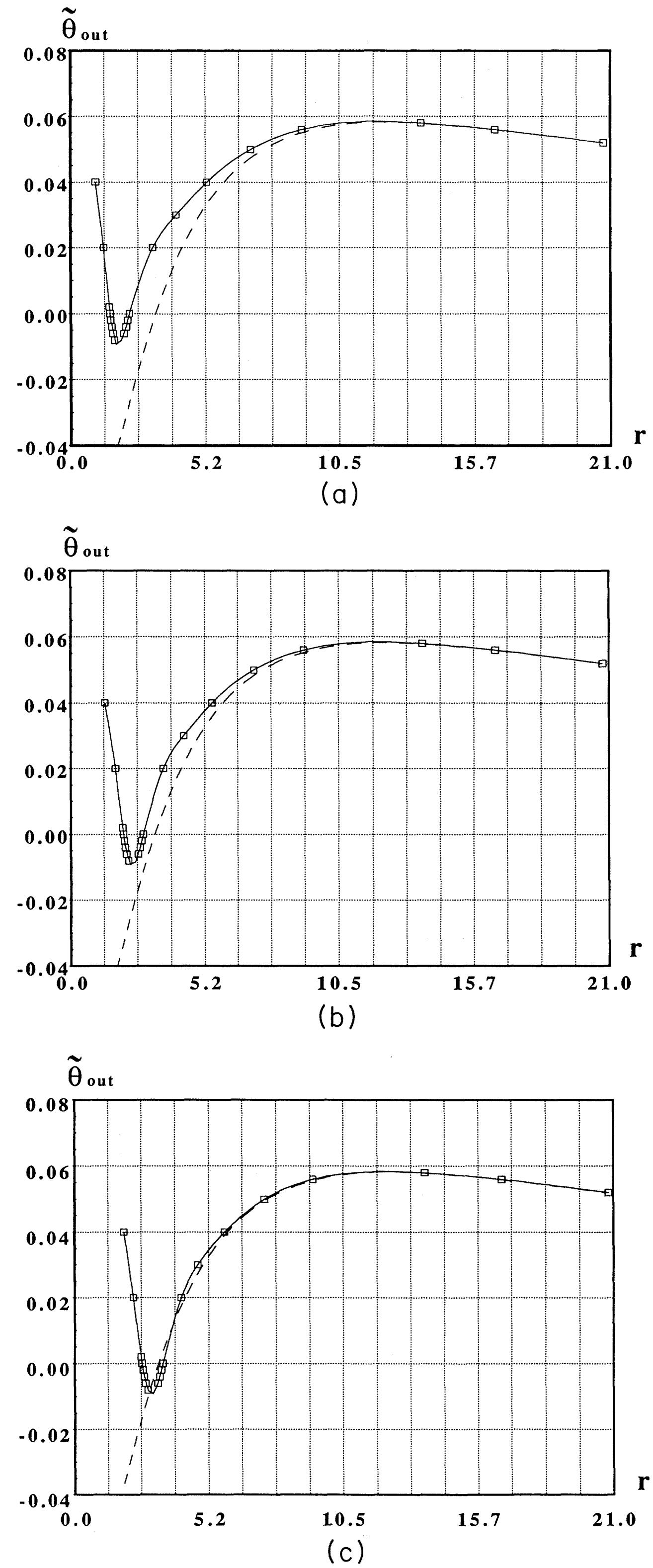

FIG. 3. $\tilde{\theta}_{out}$ of the equiexpansion surfaces of the gravitational waves with $A=6$ is depicted by squares with respect to r on the $R=0$ axis (a), $R=z$ (b), and $z=0$ (c). The dashed line is $\tilde{\theta}$ of the Schwarzschild–de Sitter space-time with the same mass of the gravitational wave with $A=6$.

ists, which appears from the inside region. Hence, the area of cosmological horizon shrinks to a smaller one when r_0 becomes larger than $2.9\times10^{-2}l_H$.

We should note that, in the Schwarzschild–de Sitter space-time, the critical value of H_0 for horizons to exist is given by $H_0=1/\sqrt{9\Lambda}$ and, in our notation, by $r_0/l_H=1/\sqrt{27\tilde{M}^2}$, which is equal to about 2.9×10^{-2} for $A=6$ ($\tilde{M}=6.61$). In the case of the Schwarzschild–de Sitter space-time, if r_0/l_H is larger than this critical value, the white-hole and cosmological apparent horizons do not exist. However, in our case, the cosmological apparent horizon exists, although the white-hole apparent horizon disappears beyond the critical value. When r_0/l_H passes through this critical value, the position of cosmological apparent horizon shifts discontinuously into the inside region.

Since we can freely choose the scale of r_0, i.e., the width of the gravitational waves, the above discussion can be understood in two ways. For fixed r_0, we contend that the above behavior is due to the cosmological constant. The large cosmological constant prevents a formation of the black-hole apparent horizon and makes the region enclosed by the cosmological horizon smaller. On the other hand, for a fixed cosmological constant, r_0 causes the same effect as the cosmological constant. In other words, the gravitational waves, if localized enough, easily form the trapped region even though the gravitational mass is small, and this tendency corresponds to the fact that, as already mentioned, large Ω ($=r_0$) forms a trapped region and a black-hole apparent horizon with difficulty. This reason is simple because if r_0 is smaller than the cosmological horizon scale l_H the background expansion effect can be ignored, resulting in the same conclusion as in the asymptotically flat case.

The gravitational mass M increases linearly with respect to r_0. It seems that the waves with large gravitational mass always cause a trapped region to form easily. However, when r_0 increases, the physical scale also increases. The width of localized waves also becomes large. So, just by the rescaling of r_0, we cannot say anything definite about a formation of a trapped surface. In order to clarify what happens, then, we should investigate the effect of the gravitational mass M on the formation of the trapped region by fixing r_0. For various values of A, or equivalently for those of M, with fixed r_0, hence, we find the critical value $(r_0/l_H)_{\rm crit}$, where the trapped region disappears. From such a study, we can conclude that if the critical value $(r_0/l_H)_{\rm crit}$ is large, the trapped region is easily formed even if the cosmic expansion is fast.

In Fig. 4, we show $(r_0/l_H)_{\rm crit}$ as a function of A. The squares are $(r_0/l_H)_{\rm crit}$ of the gravitational waves, while the circles are those of the Schwarzschild–de Sitter space-time with the same gravitational mass M. In the case of the Schwarzschild–de Sitter space-time, $(r_0/l_H)_{\rm crit}$ monotonically decreases with respect to the gravitational mass. This means that the large gravitational mass prevents the formation of the trapped region. On the other hand, in the case of the gravitational waves, for $A<7$, i.e., $M<10.8r_0$, $(r_0/l_H)_{\rm crit}$ increases with respect to the gravitational mass. However, for $A\geq7$ ($M\geq10.8r_0$), $(r_0/l_H)_{\rm crit}$ decreases with respect to the gravitational mass M similar to the case of the Schwarzschild–de Sitter space-time. Hence, although the trapped region is more easily formed for the larger gravitational mass if $M<10.8r_0$, too large a gravitational mass of the waves also prevents the formation of the trapped region in our case.

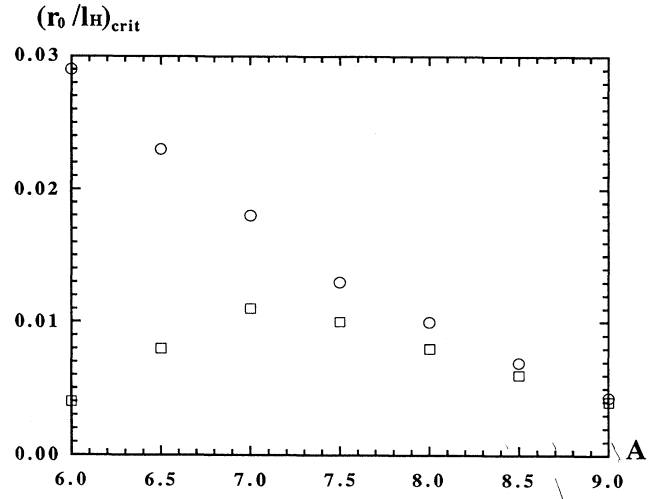

FIG. 4. The critical values $(r_0/l_H)_{\rm crit}$, when the black-hole apparent horizon of the localized gravitational waves disappears, are depicted with respect to A by squares. Those of the Schwarzschild–de Sitter space-time are also depicted by circles.

In Fig. 5, the cosmological horizons are depicted. The dashed line is the case of $A=0$, i.e., the de Sitter space-time, and the solid line is the case of $A=4.3$ with the gravitational mass $M=2.93r_0$. It is difficult to find the cosmological apparent horizons for the gravitational waves with $A>4.3$, since the shape of the cosmological

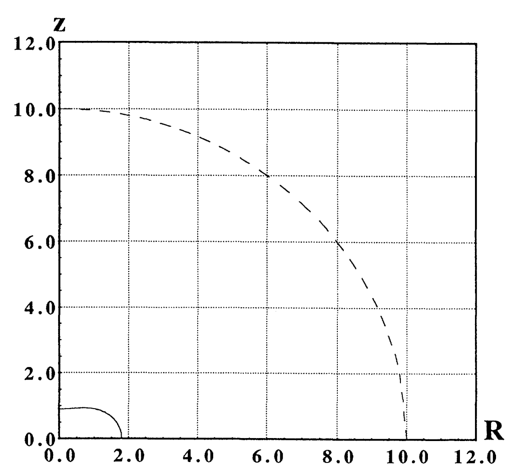

FIG. 5. The cosmological apparent horizons with $r_0=0.1l_H$ are depicted in the cylindrical coordinate (R,z). The dashed line is the cosmological horizon with $A=0$, which corresponds to the de Sitter space-time. The solid line is that with $A=4.3$ ($M=2.93r_0$).

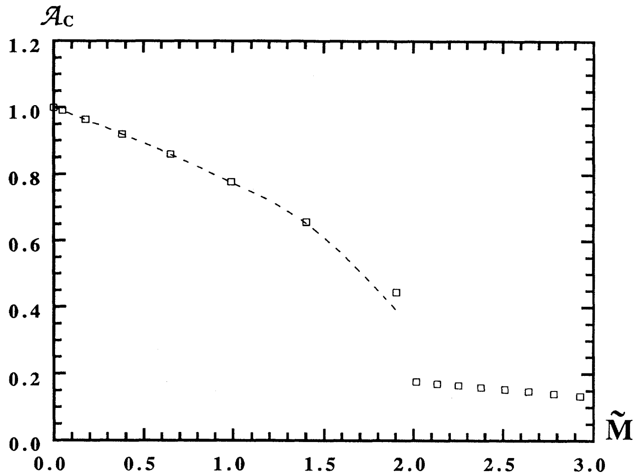

FIG. 6. $\mathcal{A}_C$ is depicted as a function of $\tilde{M}=m/r_0$. The dashed line is $\mathcal{A}_C$ for the Schwarzschild–de Sitter space-time, while $\mathcal{A}_C$ of the gravitational waves are depicted by squares. In the case with small gravitational mass, $\mathcal{A}_C$ agree with that of the Schwarzschild–de Sitter space-time for the gravitational waves.

horizons are highly deformed, and $r=r(\vartheta)$ may not be single valued.

Here we investigate the relation between the cosmological apparent horizon and the gravitational mass M for fixed r_0. Let $\mathcal{A}_C$ be the area of the cosmological apparent horizon normalized by $4\pi l_H^{-2}$, i.e., the area of the cosmological horizon of the de Sitter space-time. Hence, if there is no gravitational wave, $\mathcal{A}_C$ is unity. In Fig. 6, we depict the relation between $\mathcal{A}_C$ and $\tilde{M}$ with $r_0=0.1 l_H$. The dashed line is the $\mathcal{A}_C$ of the Schwarzschild–de Sitter space-time in which the cosmological apparent horizon vanishes for $M<M_{\rm crit} \equiv 1/\sqrt{27H_0^2} \simeq 1.92 r_0$. The squares are those of our initial data. As expected, when the gravitational mass is small, $\mathcal{A}_C$ of these two cases agree well with each other. At $M=M_{\rm crit}$, $\mathcal{A}_C$ varies discontinuously for the same reason, in the case of the gravitational waves, that the position of cosmological apparent horizon changes discontinuously at the critical value for M of the Schwarzschild–de Sitter space-time.

These features almost agree with a spherically symmetric dust collapse with the asymptotic Schwarzschild–de Sitter region. The simplest example is the Oppenheimer-Snyder space-time with Λ, which describes the motion of a homogeneous dust sphere. This example shows that the dust sphere with large gravitational mass exceeding the critical value $M_{\rm crit}=1/\sqrt{9\Lambda}$ cannot recollapse but does expand and approaches the de Sitter space-time asymptotically, if the asymptotic background de Sitter space is initially expanding [7]. Hence, we show that the gravitational waves with large gravitational mass do not seem to be an obstacle to the cosmic no-hair conjecture either.

V. CONCLUSION

We present the formalism to solve the initial value problem for vacuum space-times with a cosmological constant Λ by the use of a conformal transformation and obtain time-symmetric initial data of localized gravitational waves with axial symmetry. These initial data are just Brill waves in a conformal frame and are regarded to be snapshots of gravitational waves in a uniformly expanding background space-time. The three-dimensional metric of these gravitational waves is the same form as the original Brill waves; hence, as Brill showed, the gravitational mass is always positive in our case too.

In contrast with an asymptotic flat case without Λ, a cosmic expansion exists in the present case. The effect of the cosmic expansion causes the antitrapped region and the cosmological apparent horizon, which is the inner boundary of the antitrapped region. In order to see the relation between a cosmological horizon and a gravitational mass of the waves, we calculate the surface area of the cosmological apparent horizon $\mathcal{A}_C$.

We show that $\mathcal{A}_C$ is a monotonically decreasing function with respect to the gravitational mass of the gravitational waves below the critical mass, $M_{\rm crit}=1/\sqrt{9\Lambda}$, and then beyond the critical value the area $\mathcal{A}_C$ eventually shrinks discontinuously to a smaller value but does not vanish, in contrast with the Schwarzschild–de Sitter case.

In the asymptotic flat cases, the large gravitational waves, localized enough, form the trapped region and then the black-hole apparent horizon. On the other hand, in our case with the cosmic expansion effect, the gravitational waves with gravitational mass too large prevent formation of the trapped region.

These features are essentially the same as the spherically symmetric dust collapse with the asymptotic Schwarzschild–de Sitter region [7]. Hence the gravitational waves with large mass are not always an obstacle to the cosmic no-hair conjecture.

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