Phase transition in nonlinear viscous cosmology

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A simple example of a phase transition process describing the isotropization of a universe of Bianchi type is outlined. Such a mechanism is induced by a self-gravitating fluid, and it operates as described by Landau's phase transition. The expansion factor (the Hubble constant) plays the part of the control parameter as the temperature does for ordinary matter.

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I. INTRODUCTION

It is well known that the perfect fluid hypothesis (for the behavior of cosmological matter) is adequate for describing some observed properties of our present equilibrium era, although some points are still left unexplained. For example, the presence of horizons in Friedmannian models makes difficult the understanding of the high degree of isotropy of the cosmological background radiation. Such reasons stimulate the quest for more general models which help us to understand periods of evolution of the Universe when its properties were different from those today. The early structure of the Universe (composed of a chaotic mixture of various species of elementary particles and radiation) could be highly anisotropic, which might be described (phenomenologically) as a fluid with viscosity. However, in order to satisfy the actual status of observations, such a model has to exhibit an efficient mechanism of elimination of the anisotropy (as an intrinsic feature of its dynamics). A possible solution to this problem was suggested recently [1, 2]. It is based on Landau's theory of second-order phase transition [3], which involves only ordinary physics (for example, it is used for investigating the behavior of a liquid crystal). Herein, we apply these ideas to a homogeneous and anisotropic universe of Bianchi type I. We show that, indeed, Einstein's equations indicate an evolution toward isotropy, the expansion factor of the fluid characterizing phases of the Universe with varying symmetries. In Sec. II, we review the basics of the gravitationally self-induced phase transition. In Sec. III, we show that, by assuming a planar anisotropy, Einstein's equations reduce to a nonlinear planar autonomous system. Section IV performs a qualitative analysis of such a system (which describes its global behavior).

II. GRAVITATIONALLY SELF-INDUCED PHASE TRANSITION

The source of geometry is a nonperfect fluid of Stokesian type [4] characterized by the stress-energy tensor

$$T_{\mu\nu} = \rho V_\mu V_\nu - p h_{\mu\nu} + \pi_{\mu\nu}, \tag{1}$$

where $h_{\mu\nu} \equiv g_{\mu\nu} - V_\mu V_\nu$, $\pi_{\mu\nu}$ is a functional of the expansion factor $\theta = V^\mu_{;\mu}$ and the shear tensor $\sigma_{\mu\nu} = \frac{1}{2}(V_{\mu;\rho}h^\rho_\nu + V_{\nu;\rho}h^\rho_\mu) - \frac{1}{3}\theta h_{\mu\nu}$ (it satisfies $\sigma_{\mu\nu}V^\mu V^\nu = 0$ and $\sigma^\mu_\mu = 0$). For the sake of simplicity, one limits oneself to the third-order expansion

$$\pi^\mu_\nu[\theta,\sigma] = \left(\alpha_0 - \alpha_1\theta + \beta\sigma^2\right)\sigma^\mu_\nu + \delta\left(\sigma^\mu_\alpha\sigma^\alpha_\nu - \frac{1}{3}\sigma^2 h^\mu_\nu\right), \tag{2}$$

where $\alpha_0, \alpha_1 \geq 0$, β and δ are constants, and $\sigma^2 = \mathrm{Tr}[\hat{\sigma}^2]$. It turns out that the higher-order terms do not influence the topology of the phase diagram of the fluid in the neighborhood of phase transition points (see [3]). Let us recall that the fundamental states are given by the extrema of the free energy. For a self-gravitating fluid, we assume that the increase of free energy is given by

$$(\Delta F)_G = -(m^2/\kappa)\, R_{\mu\nu}\sigma^{\mu\nu} \tag{3}$$

(see [1, 2]). Hence, Einstein's equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}$ yield $(\Delta F)_G = m^2\pi_{\mu\nu}\sigma^{\mu\nu}$. In the case of a planar anisotropy, by using the following parametrization,

$$\sigma^\mu_\nu = \Sigma\begin{pmatrix} 0 & & & \\ & -1/2 & & \\ & & -1/2 & \\ & & & 1 \end{pmatrix}, \tag{4}$$

we obtain

$$(\Delta F)_G = \frac{3}{2}m^2\Sigma^2\left(\alpha_1(\theta^\star - \theta) + \frac{1}{2}\delta\Sigma + \frac{3}{2}\beta\Sigma^2\right), \tag{5}$$

where $\theta^\star = \alpha_0/\alpha_1$. By understanding the amplitude of the shear Σ as an order parameter (where $\Sigma = 0$ characterizes the isotropic phase), the quantity given in Eq. (5) provides us with the behavior of the free energy in the neighborhood of a phase transition point, if it is interpreted as an expansion in term of Σ. Now, let us define the following θ values,

$$\theta_c = \theta^\star - \frac{3}{64}\frac{\delta^2}{\alpha_1\beta}, \quad \theta_T = \theta_c + \frac{1}{192}\frac{\delta^2}{\alpha_1\beta}, \tag{6}$$

and follow closely the analysis of a second-order phase transition given by [2]. The equilibrium states are determined by the maxima of the function given by Eq. (3). Thus, according to [1], it turns out that the most likely state corresponds to

(i) an isotropic phase, when $\theta < \theta_c$;

(ii) an isotropic phase with a local minimum corresponding to a small anisotropy, when $\theta_c < \theta < \theta_T$;

(iii) an anisotropic phase with a local minimum corresponding to an isotropic phase, when $\theta_T < \theta < \theta^\star$;

(iv) an anisotropic phase, when $\theta > \theta^\star$.

The above mechanism of phase transition can be understood as a direct consequence of the following conditions:

(a) the validity of Einstein's equations,

(b) the pressure defined by Eq. (2),

(c) the free energy of the fluid defined by Eq. (3).

III. EINSTEIN'S EQUATIONS

Let us investigate the modifications of metrical properties of the space-time which are generated by the Stokesian fluid described above. We set up the system of Einstein's equations in the Bianchi type-I geometry where the anisotropic phases and the isotropic ones are easily characterized. Let us use the following variables,

$$\theta = \sum_{i=1}^{3} \frac{\dot a_i}{a_i}, \qquad \epsilon = \sum_{i=1}^{3}\left(\frac{\dot a_i}{a_i}\right)^2. \tag{7}$$

The metric is given by

$$ds^2 = dt^2 - a_1^2(t)dx^2 - a_2^2(t)dy^2 - a_3^2(t)dz^2. \tag{8}$$

Hence, (according to its definition) the shear tensor reads

$$\sigma^\mu_\nu = \left(\frac{\theta}{3} - \frac{\dot a_\nu}{a_\nu}\right) h^\mu_\nu \quad (\nu = 1,2,3 \text{ not summed}). \tag{9}$$

The nonzero connection coefficients are given by $\Gamma^j_{0k} = \delta^j_k \dot a_k / a_k$ and $\Gamma^0_{jk} = \delta_{jk}\dot a_k a_k$, which leads to the Ricci tensor coefficients $R_{00} = \sum_{j=1}^3 \ddot a_j/a_j$, $R_{jk} = -[\ddot a_j a_j + \dot a_j a_j \sum_{i\neq j}(\dot a_i/a_i)]\delta_{jk}$ (which shows that $T_{\mu\nu}$ is diagonal), and to a scalar curvature $R = 2\sum_{j=1}^3 \ddot a_j/a_j + \sum_{j\neq k}(\dot a_j/a_j)(\dot a_k/a_k)$. Hence, the Einstein's equations transform into a system defined by

$$\frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} + \frac{\dot a_1}{a_1}\frac{\dot a_3}{a_3} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} = \kappa\rho, \tag{10}$$

in addition to three equations obtained by a circular permutation of indices of

$$\frac{\ddot a_2}{a_2} + \frac{\ddot a_3}{a_3} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} = -\kappa\left\{ \epsilon\left(\frac{\delta}{3} - \beta\frac{\dot a_1}{a_1} + \frac{\beta}{3}\theta\right) + \theta^2\left(-\frac{2}{9}\delta + \frac{\beta}{3}\frac{\dot a_1}{a_1} - \frac{\beta}{9}\theta - \frac{\alpha_1}{3}\right) + \theta\left(\frac{\alpha_0}{3} + \alpha_1\frac{\dot a_1}{a_1} + \frac{2}{3}\delta\frac{\dot a_1}{a_1}\right) + p - \alpha_0\left(\frac{\dot a_1}{a_1}\right) - \delta\left(\frac{\dot a_1}{a_1}\right)^2\right\}. \tag{11}$$

Let us assume a (quite general) equation of state given by

$$p = (\gamma - 1)\rho, \tag{12}$$

with

$$0 < \gamma < 2. \tag{13}$$

According to Eq. (10), it is interesting to note that a relation of the form $p = p(\rho)$ allows us to express the pressure p in terms of variables $x_k = \dot a_k/a_k$. After a little algebra, Eqs. (10) and (11) transform into a nonlinear autonomous system $[\dot x_k = F_k(x_1,x_2,x_3),\ k = 1,3]$. By assuming a planar anisotropy, e.g., $x_2 = x_3$, the Einstein's equations can be written in terms of the variables

$$x = \frac{\dot a_1}{a_1}, \qquad y = \frac{\dot a_2}{a_2} = \frac{\dot a_3}{a_3}, \tag{14}$$

as follows,

$$\begin{aligned} dx/dt = P(x,y) &= -2ax + 2ay + bx^2 + cy^2 + dxy \\ &\quad -4fx^3 + 4fy^3 - 12fxy^2 + 12fx^2y, \end{aligned} \tag{15}$$

$$\begin{aligned} dy/dt = Q(x,y) &= ax - ay + gx^2 + hy^2 + kxy \\ &\quad +2fx^3 - 2fy^3 + 6fxy^2 - 6fx^2y \end{aligned}$$

where

$$\begin{aligned} a &= \frac{1}{3}\kappa\alpha_0, \\ b &= -\frac{2}{9}\kappa\delta + \frac{2}{3}\kappa\alpha_1 - 1, \\ c &= -\frac{2}{9}\kappa\delta - \frac{4}{3}\kappa\alpha_1 - \frac{\gamma}{2} + 1, \\ d &= \frac{4}{9}\kappa\delta + \frac{2}{3}\kappa\alpha_1 - \gamma, \\ f &= \frac{1}{9}\kappa\beta, \\ g &= \frac{1}{9}\kappa\delta - \frac{1}{3}\kappa\alpha_1, \\ h &= \frac{1}{9}\kappa\delta + \frac{2}{3}\kappa\alpha_1 - \frac{\gamma}{2} - 1, \\ k &= -\frac{2}{9}\kappa\delta - \frac{1}{3}\kappa\alpha_1 - \gamma + 1. \end{aligned} \tag{16}$$

IV. QUALITATIVE ANALYSIS

Section II shows that an evolving nonlinear Stokesian fluid is able to self-induce a transition between the anisotropic and isotropic phases. Section III shows that the related dynamics is described (in Bianchi-I geometry) by the nonlinear planar autonomous system given by Eq. (15). The purpose of this section is to analyze such a mechanism by means of qualitative analysis (see [5, 6]). Namely, we investigate the global behavior of integral curves in the neighborhood of critical points in the phase space (including those at infinity). The finite critical points, which are defined by

$$P(u,z)=0, \qquad Q(u,z)=0 \tag{17}$$

[see Eq. (15)], correspond to equilibrium states. Note that the origin, $p_c=(0,0)$, is actually the only one [since $2Q(x,y)+P(x,y)$ is a second-degree polynomial which has a negative discriminant], as ensured by the condition given by Eq. (13). Let $\widehat{\Omega}$ denote the following matrix

$$\widehat{\Omega}=\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}, \tag{18}$$

see Eq. (15), which reads

$$\widehat{\Omega}_c=a\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, \tag{19}$$

at the origin p_c. The nature of this equilibrium state is determined by means of invariants of $\widehat{\Omega}_c$ (there are the determinant and the trace). It turns out that the relevant values, which are $\det\widehat{\Omega}_c=0$ and $\mathrm{Tr}\widehat{\Omega}_c=-3a\neq 0$, indicate that p_c is a multiple equilibrium point. Further analysis [5] enables us to classify it as a saddle node [i.e., an equilibrium state whose canonical neighborhood is the union of one parabolic and two hyperbolic sectors (see Fig. 1)].

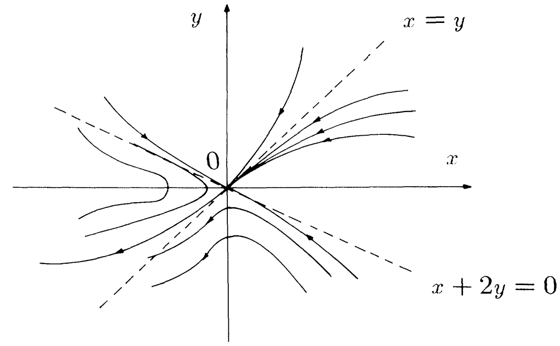

FIG. 1. The phase space: the neighborhood of the critical point located at the origin of coordinates. The lines represent integral curves.

The behavior of the dynamical system at infinity is investigated on the Poincaré sphere (obtained by compactifying the phase space of the system, completed by the elements located at infinity). The related transformation of variables (called Poincaré transformation) are given by

$$x=\frac{1}{z}, \qquad y=\frac{u}{z}. \tag{20}$$

Hence, by using the variable τ, which is defined by $d\tau=dt/z^2$, Eq. (15) transforms as follows:

$$du/d\tau=P^*(u,z)=2f+gz-2fu+az^2+(k-b)uz-6fu^2+10fu^3+auz^2+(h-d)u^2z-4fu^4-2au^2z^2-cu^3z,$$
$$dz/d\tau=Q^*(u,z)=4fz-bz^2-12fuz+2az^3-duz^2+12fu^2z-2auz^3-cu^2z^2-4fu^3z. \tag{21}$$

These equations have all the necessary information about the behavior of the system at infinity. Thus let us proceed to analyze this system in the same way as we did before. The critical points are the solutions of the equations

$$P^*(u,z)=0, \qquad Q^*(u,z)=0. \tag{22}$$

One obtains $p_{c_1^*}=\left(-\frac{1}{2},0\right)$ and $p_{c_2^*}=(1,0)$. The invariants of $\widehat{\Omega}$ at the neighborhood of $p_{c_1^*}$ are given by $\det\widehat{\Omega}_{c_1^*}=\left(\frac{27}{2}\right)^2 f^2\neq 0$ and $\mathrm{Tr}\,\widehat{\Omega}_{c_1^*}=27f\neq 0$. These values, together with the study of its characteristic equation and roots [5], classify the point $p_{c_1^*}$ as a simple equilibrium state called *unstable node* (see Fig. 2). At the neighborhood of $p_{c_2^*}$, it turns out that $\widehat{\Omega}_{c_1^*}=0$, which forces us to use the second-order expansion of Eq. (15). A great simplification is introduced by assuming $\alpha_1=\frac{3}{2}\frac{1}{k}$. According to [7], which provides us with a general procedure for analyzing the configuration of integral curves,

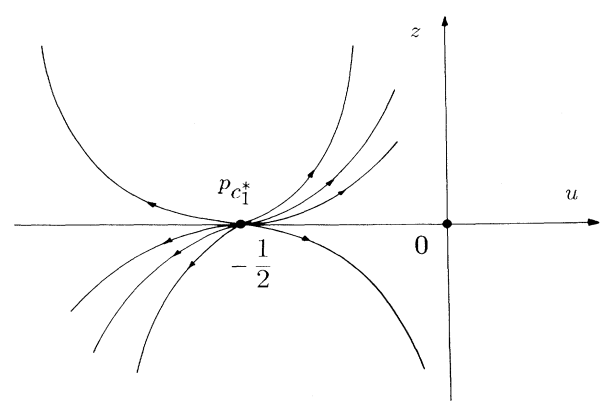

FIG. 2. The Poincaré sphere: behavior of integral curves in the neighborhood of the equilibrium point $p_{c_1^*}$.

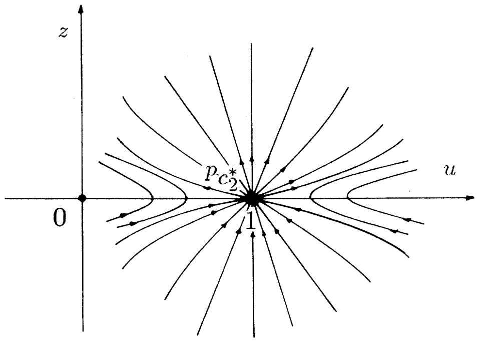

FIG. 3. The Poincaré sphere: behavior of integral curves in the neighborhood of the equilibrium point $p_{c_2^*}$.

$p_{c_2^*}$ can be classified as a *dicritical point.* A detailed study of this case furnishes the following topological structure for the point $p_{c_2^*}$ (see Fig. 3). All the necessary data for describing the system as a whole (including its behavior for arbitrary large values of its coordinates) is available in Fig. 4, on which all the equilibrium states are represented. This diagram results from the orthogonal projection of the lower half of the Poincaré sphere in the phase plane and has the form of a disk. The border represents infinity in such a way that antipodal points describe the same point. Our next step is to identify in this diagram the mechanism of phase transition between the anisotropic and isotropic phases of the fluid. First we ask whether all the trajectories in this diagram may represent the evolving Universe. If we restrict the acceptable solutions to those with a positive energy density ρ, which is given by

$$\rho = y^2 + 2xy, \tag{23}$$

then it turns out that a large number of them are ruled out. The reason is easy to understand since the first of Einstein's equations provides us with the energy density ρ in terms of the coordinates (x, y), as shown in Fig. 4. The positivity of ρ forces us to consider only the trajectories outwards of $\rho < 0$ sectors (see Fig. 4). We are also interested in characterizing the isotropy properties of solutions. A solution is called isotropic when it lies on the line with equation $x = y$, while any other location on the plane describes an anisotropic phase. These considerations help us to exhibit the trajectories that we are looking for, which are those that represent anisotropic phases of the Universe evolving toward the isotropic phase. The expansion factor is represented on this diagram by the family of lines defined by equation $x+2y =$ const. Hence, we understand that the permitted behaviors show in the past (at infinity) an anisotropic phase with huge expansion (i.e., a large positive value of the expansion parameter). As the evolution proceeds the expansion factor gets smaller and the trajectories tend to be tangent to the line $x = y$, which characterizes the isotropization. The expansion ends at the origin ($\theta = 0$) which represents the Minkowski space (see Fig. 4).

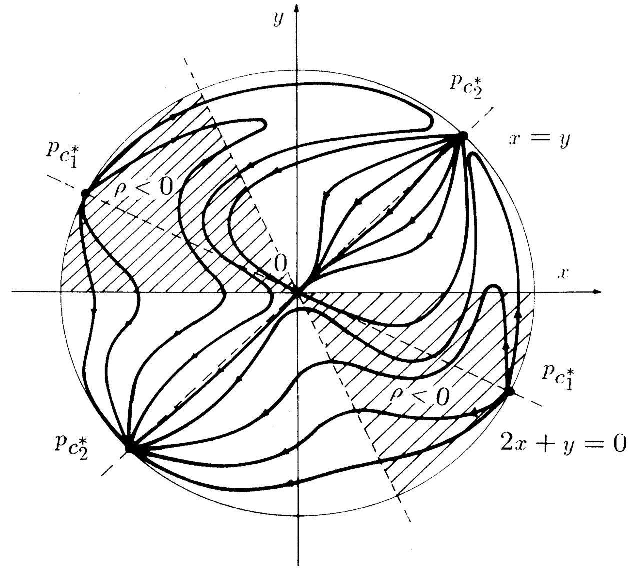

FIG. 4. The Poincaré sphere: global view.

V. CONCLUSION

The choice of the Bianchi-I geometry for studying the evolution of the nonlinear Stokesian fluid provides us with an explicit confirmation of results obtained by [1], these are summarized in Sec. II. The Universe is assumed to be described by an anisotropic viscous fluid. During its evolution (expansion), it reaches a state where the expansion factor θ has a value which suffices for giving rise to a phase transition toward an isotropic era.

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