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Cosmological perturbations in Bianchi type-I universes

P. G. Miedema
Koninklijke Militaire Academie Kasteelplein 10, NL-4811 XC Breda, The Netherlands

W. A. van Leeuwen
Universiteit van Amsterdam Instituut voor Theoretische Fysica, Valckenierstraat 65, NL-1018 XE Amsterdam, The Netherlands

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The evolution equations for small perturbations in the metric, energy density, and material velocity are derived for an anisotropic viscous Bianchi type-I universe. The equations obtained are the same as those found by Perko, Matzner, and Shepley, and by Tomita and Den. However, the splitting up of these equations is different from the way it is performed by these authors, which results in the fact that, in close analogy with the flat Friedmann-Robertson-Walker universe, the general solution of the perturbation equations can be split up into three noncoupled perturbations: namely, gravitational waves ("tensor perturbations"), vortex motions ("vector perturbations"), and density enhancements ("scalar perturbations"). Moreover, the results are independent of the equation of state of the cosmic fluid and its viscosity. The gravitational waves need not necessarily be transversal in an anisotropically expanding Bianchi type-I universe. It is shown, however, that the longitudinal components of the gravitational waves have no physical significance.

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I. INTRODUCTION

In this paper we study the evolution equations for small perturbations on a background consisting of a homogeneous anisotropic universe of Bianchi type I, filled with a viscous fluid.

The paper is organized as follows. In Sec. II we write down the connection coefficients and the Ricci tensor for Bianchi type-I spaces, the arena of our investigation. We take a viscous fluid, write down the Einstein equations, show that a Bianchi type-I universe is nonrotating, and take the metric diagonal. Furthermore, we define the Hubble parameters in the three principal directions as the eigenvalues of the expansion tensor.

In Sec. III we introduce perturbations on the energy density and the material velocity. The perturbation equations consist of six evolution equations for the perturbed metric, one evolution equation for the perturbed energy density, and three evolution equations for the three components of the perturbed material velocity. Furthermore, there are four constraint equations which must be satisfied by the initial values of the perturbed quantities in the initial hypersurface.

It is well known that the solutions of the Einstein equations are not unique. One can always carry out coordinate transformations to obtain a different, but physically equivalent, solution. Such "gauge transformations" do not change the values of the physical observables such as the energy density ε or the particle density n, but they do change the components of the metric tensor. As indicated above, we solve the Einstein equations step by step. It is a drawback of our iterative procedure that it is not generally covariant. Physical quantities, at a given level of the iteration procedure, may depend on the "gauge." For instance, although the energy density $\varepsilon = \varepsilon_{(0)} + \varepsilon_{(1)} + \varepsilon_{(2)} + \cdots$ is gauge invariant, the first-order correction $\varepsilon_{(1)}$ will turn out to depend on the coordinate system that is chosen. It is the purpose of Sec. IV to fix a suitable gauge and to find thus an unambiguous result for the physical quantities that are solutions of the first-order equations.

The solutions to the perturbation equations can be classified according to their influence on the energy density and the material velocity. In Sec. V we will prove a theorem which leads to this classification. The class of solutions which we call "nonmaterial," and which are defined by the five properties (108), are discussed in Sec. VI. The remaining solutions correspond to density perturbations.

In Sec. VII we give a summary of our results.

Throughout the paper we use the summation convention only with respect to greek indices; i.e., if a greek index occurs twice then it is supposed to be a summation index.

II. BIANCHI TYPE-I SPACES

The expressions for the connection coefficients and the curvature tensor for homogeneous three-dimensional Bianchi type-I spaces are characterized by the structure constants

$$c^i{}_{jk} = 0 \quad (i, j, k = 1, 2, 3). \tag{1}$$

We use a synchronous system of reference [1–3], and signature $(+, -, -, -)$, implying for the components of the metric tensor that $g_{00} = 1$ and $g_{0i} = g_{i0} = 0$. Furthermore, we use the so-called invariant basis [4, 5]. Upon substituting (1) into the expressions for the connection coefficients and the Ricci tensor [4] we arrive at

$$\Gamma^0{}_{00} = \Gamma^i{}_{00} = \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0, \tag{2}$$

$$\Gamma^0{}_{ij} = \theta_{ij}, \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\theta^i{}_j, \tag{3}$$

$$\Gamma^k{}_{ij} = 0, \tag{4}$$

and

$$R^0{}_0 = \dot\theta - \sum_{k=1}^{3}\sum_{l=1}^{3}\theta^k{}_l\theta^l{}_k, \tag{5}$$

$$R^0{}_i = 0, \tag{6}$$

$$R^i{}_j = \dot\theta^i{}_j - \theta\theta^i{}_j + {}^3R^i{}_j, \tag{7}$$

where the curvature tensor of the three-dimensional homogeneous spaces is given by

$${}^3R^i{}_j = 0. \tag{8}$$

Hence, three-dimensional Bianchi type-I spaces have no curvature. The quantities θ_{ij} in (3), (5), and (7) are abbreviations for the derivative with respect to the cosmic time ($x^0 = ct$) of the metric:

$$\theta_{ij} := -\tfrac{1}{2}\dot g_{ij}, \quad \theta^i{}_j := \sum_{k=1}^{3} g^{ik}\theta_{kj}, \quad \theta := \sum_{k=1}^{3}\theta^k{}_k. \tag{9}$$

An overdot denotes differentiation with respect to ct.

The components $R^\mu{}_\nu$ of the Ricci tensor occur in the left-hand side of the Einstein equations

$$R^\mu{}_\nu = \kappa(T^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu T^\alpha{}_\alpha), \tag{10}$$

where $\kappa := 8\pi G/c^4$ with G Newton's gravitation constant and c the speed of light. At the right-hand side of these equations we have the energy-momentum tensor. For a viscous fluid it reads [3]

$$\begin{aligned} T_{\mu\nu} = {} & \left[\varepsilon + p + c(\tfrac{2}{3}\eta - \eta_v)u^\lambda{}_{;\lambda}\right] u_\mu u_\nu \\ & - \left[p + c(\tfrac{2}{3}\eta - \eta_v)u^\lambda{}_{;\lambda}\right] g_{\mu\nu} \\ & + c\eta\left[u_{\mu;\nu} + u_{\nu;\mu} - (u_\mu u_\nu)_{;\lambda}u^\lambda\right], \end{aligned} \tag{11}$$

where ε is the energy density and p the pressure, given by an equation of state $p = p(n, \varepsilon)$ depending on the particle density n and the energy density ε. A semicolon denotes covariant differentiation. Furthermore, $T^\alpha{}_\alpha$ is the trace of the energy-momentum tensor and u^μ is the normalized hydrodynamic four-velocity ($u^\mu u_\mu = 1$). The symbols η and η_v represent the shear- and the volume-viscosity coefficients, respectively. They are certain known functions of the density n and the temperature T:

$$\eta = \eta(n, T), \quad \eta_v = \eta_v(n, T). \tag{12}$$

The actual forms of the transport coefficients can be determined on the basis of relativistic kinetic theory [6, 7].

Nonrotating. We will now show that a Bianchi type-I universe filled with a viscous fluid is nonrotating, i.e.,

$$u^\mu(t) = \delta^\mu{}_0. \tag{13}$$

The conservation laws are

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{14}$$

Writing out the covariant derivative and substituting the connection coefficients (2)–(4) we arrive at the four equations

$$\dot T^{00} + \sum_{k=1}^{3}\sum_{l=1}^{3}\theta_{kl}T^{kl} - \theta T^{00} = 0 \tag{15}$$

and

$$\dot T^{i0} - 2\sum_{k=1}^{3}\theta^i{}_k T^{k0} - \theta T^{i0} = 0 \quad (i = 1, 2, 3). \tag{16}$$

Equation (15) turns out to be a differential equation for the energy density ε, as we shall see later. Equations (16) are three differential equations for the three unknown functions $T^{i0}(t)$ ($i = 1, 2, 3$). The initial conditions at $t = t_0$ can be found from the $(i, 0)$-constraint equations. For a Bianchi type-I universe we have, using (6) and (10), $T^{i0}(t_0) = 0$. From (16) it then follows that

$$T^{i0}(t) = 0 \quad \text{for all } t \quad (i = 1, 2, 3) \tag{17}$$

or, equivalently,

$$\begin{aligned} (\varepsilon + p)u_0u_i - c\eta_v(u^\lambda{}_{,\lambda} - \theta u^0)u_0u_i + c\eta\Bigg\{ & \tfrac{2}{3}(u^\lambda{}_{,\lambda} - \theta u^0)u_0u_i + u_{0,i} + u_{i,0} + 2\sum_{k=1}^{3}\theta^k{}_i u_k \\ & -(u_0u_i)_{,\lambda}u^\lambda - u_i\sum_{k=1}^{3}\sum_{l=1}^{3}\theta^k{}_l u_k u^l\Bigg\} = 0 \quad (i = 1, 2, 3), \end{aligned} \tag{18}$$

where a comma denotes partial differentiation. It may be verified by substitution that $u^\mu(t) = \delta^\mu{}_0$ is a solution of this equation. Hence, a Bianchi type-I universe filled with a viscous fluid is essentially nonrotating.

Metric. From now on, we take the metric diagonal, i.e., $g_{ij}(t) = 0$ ($i \neq j$) for all t. In this way we end up with a system of differential equations, the number of which equals the number of unknown functions. This justifies, *a posteriori*, our choice of a diagonal metric.

The metric tensor has three independent components

g_{ii} $(i = 1, 2, 3)$ on the main diagonal, i.e.,

$$g_{\mu\nu}(t) = \text{diag}(1, -a_1^2(t), -a_2^2(t), -a_3^2(t)). \tag{19}$$

Now define the abbreviations H_i $(i = 1, 2, 3)$ and their mean H by

$$H_i(t) := \frac{\dot{a}_i}{a_i}, \quad H := \tfrac{1}{3}(H_1 + H_2 + H_3). \tag{20}$$

Note that cH_i $(i = 1, 2, 3)$ are the Hubble parameters in the three principal directions.

The derivatives of the metric tensor, (9), can now be expressed as

$$\theta_{ij} = \delta_{ij} a_j^2 H_j, \quad \theta^i{}_j = -\delta^i{}_j H_j, \quad \theta = -3H, \tag{21}$$

where for H_i and H one should read the expressions in terms of a_i following from (20).

Ricci tensor. From (2)–(4), (20), and (21) we obtain the expressions for the connection coefficients for a Bianchi type-I metric:

$$\Gamma^0{}_{00} = \Gamma^i{}_{00} = \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0, \tag{22}$$

$$\Gamma^0{}_{ij} = \delta_{ij} a_j^2 H_j, \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta^i{}_j H_j, \tag{23}$$

$$\Gamma^k{}_{ij} = 0. \tag{24}$$

With the help of (21) we find for the components of the Ricci tensor, Eqs. (5)–(7), for a Bianchi type-I universe:

$$R^0{}_0 = -3\dot{H} - \sum_{k=1}^{3} H_k^2, \tag{25}$$

$$R^0{}_i = 0, \tag{26}$$

$$R^i{}_j = -\delta^i{}_j(\dot{H}_i + 3HH_i). \tag{27}$$

Energy-momentum tensor. Using the four-velocity (13) we get for the energy-momentum tensor and its trace for a nonrotating Bianchi type-I universe filled with a viscous fluid

$$T^0{}_0 = \varepsilon, \tag{28}$$

$$T^0{}_i = 0, \tag{29}$$

$$T^i{}_j = -\delta^i{}_j\left[p + 2c\eta(H - H_i) - 3c\eta_v H\right]. \tag{30}$$

Hence, for the trace,

$$T^\mu{}_\mu = \varepsilon - 3p + 9c\eta_v H. \tag{31}$$

Einstein equations. Upon substituting the components of the Ricci tensor (27) and the components of the energy-momentum tensor (30) and its trace (31) into the Einstein equations (10) we find, for the (i,j) components of the Einstein equations for a Bianchi type-I universe filled with a viscous fluid $(i = 1, 2, 3)$,

$$\dot{H}_i = -3HH_i + \tfrac{1}{2}\kappa\left[\varepsilon - p + 4c\eta(H - H_i) + 3c\eta_v H\right]. \tag{32}$$

Substituting the energy-momentum tensor (28) and (30) into Eq. (15) we get an equation resulting from the fact that energy is conserved:

$$\dot{\varepsilon} = -3H(\varepsilon + p) + 9c\eta_v H^2 + 6c\eta A_H H^2, \tag{33}$$

where A_H is the anisotropy in the Hubble parameters defined by

$$A_H(t) := \frac{1}{3}\sum_{k=1}^{3} \frac{(H_k - H)^2}{H^2}. \tag{34}$$

Equations (32) and (33) are four first-order differential equations, for the three Hubble functions H_1, H_2, H_3, and the material energy density ε.

The $(0,\mu)$ components of the Einstein equations yield the constraint equations to be obeyed by the initial values. Eliminating the time derivatives of the Hubble parameters with the help of the Einstein equations (32) we find, for the $(0,0)$ component,

$$9H^2 - \sum_{k=1}^{3} H_k^2 = 2\kappa\varepsilon. \tag{35}$$

A solution of the Einstein equations (32) satisfying this equation initially will necessarily continue to do so for all times, provided the conservation law (33) is satisfied for all times.

III. PERTURBATIONS IN A BIANCHI TYPE-I UNIVERSE WITH VISCOUS FLUID

In Sec. II we considered a homogeneous model of the universe with an imperfect fluid. It is the purpose of this section to arrive at formulas which describe the evolution of gravitational perturbations, vortex perturbations, and small local-density enhancements and the peculiar velocities.

We assume that the local inhomogeneities are "small," so that the departures from a homogeneous Bianchi type-I universe are very small. The metric of such a universe can be written as the sum of a Bianchi type-I metric and a small perturbation to the homogeneous background

$$g_{\mu\nu}(t, x^i) = g_{(0)\mu\nu}(t) + \delta g_{\mu\nu}(t, x^i), \tag{36}$$

where $g_{(0)\mu\nu}(t)$ is the unperturbed background metric (19) of a Bianchi type-I universe, and

$$\delta g_{\mu\nu}(t, x^i) := -h_{\mu\nu}(t, x^i) \tag{37}$$

is a small perturbation on the background Bianchi type-I metric. A universe described by the metric (36) will be referred to as a "perturbed Bianchi type-I universe." All quantities with a subscript (0) refer to the unperturbed Bianchi type-I model, whereas those with a subscript (1) are the first-order perturbations on that quantity, apart from one exception: instead of $-g_{(1)\mu\nu}$ we will write $h_{\mu\nu}$. We use synchronous coordinates [1–3], in the perturbed as well as in the unperturbed situation, implying that

$$h_{0\nu} = h_{\mu 0} = 0 \quad (\mu, \nu = 0, 1, 2, 3). \tag{38}$$

The contravariant components of the perturbation $h^{\mu\nu}(t, x^i)$ on $g_{(0)}^{\mu\nu}$ can be found as follows. Using $\delta g^{\mu\nu} = \delta\left(g^{\mu\tau} g^{\nu\sigma} g_{\tau\sigma}\right)$ we find

$$\delta g^{\mu\nu} = -g^{\mu\tau} g^{\nu\sigma} \delta g_{\tau\sigma} = g^{\mu\tau} g^{\nu\sigma} h_{\tau\sigma}. \tag{39}$$

We consider perturbations in the metric up to first order in h_{ij} only. Hence, raising and lowering of the indices can be done with the help of the unperturbed metric $g^{\mu\nu}_{(0)}$ and $g_{(0)\mu\nu}$, respectively. We thus find, from (39),

$$h^{\mu\nu} := +\delta g^{\mu\nu} = g^{\mu\tau}_{(0)} g^{\nu\sigma}_{(0)} h_{\tau\sigma}. \tag{40}$$

Using (19) we find

$$a_j^2(t) h^j{}_i(t, x^k) = a_i^2(t) h^i{}_j(t, x^k). \tag{41}$$

Equation (41) expresses the fact that the tensor $h^i{}_j$ has at most six independent components. Notice that

$$(g^{\mu\alpha}_{(0)} + h^{\mu\alpha})(g_{(0)\alpha\nu} - h_{\alpha\nu}) = \delta^\mu{}_\nu + O(h^2) \tag{42}$$

as follows from (36) and (40). Hence, $g^{\mu\alpha}_{(0)} + h^{\mu\alpha}$ can be regarded as the inverse of $g_{(0)\alpha\nu} - h_{\alpha\nu}$ up to first order in h_{ij}.

From (19) and (36) we find, for the determinant g of $g_{\mu\nu}$,

$$g(t, x^i) = g_{(0)}(t) \left(1 - \sum_{k=1}^{3} h^k{}_k(t, x^i) \right). \tag{43}$$

Thus, the perturbation on the determinant of the metric tensor is, up to first order in h_{ij}, $\delta g = -g_{(0)} \sum_{k=1}^{3} h^k{}_k$.

With the help of the definitions given above we will now derive expressions for the perturbations to the connection coefficients and the Ricci tensor.

The connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are given in terms of the metric tensor $g_{\mu\nu}$ by

$$\Gamma^\lambda{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right). \tag{44}$$

With the help of (39) we can write the change $\delta\Gamma^\lambda{}_{\mu\nu}$ of the connection coefficients in terms of $\delta g_{\mu\nu}$:

$$\delta\Gamma^\lambda{}_{\mu\nu} = -g^{\lambda\rho}(\delta g_{\rho\sigma})\Gamma^\sigma{}_{\mu\nu} + \tfrac{1}{2} g^{\lambda\rho} \left(\frac{\partial \delta g_{\rho\mu}}{\partial x^\nu} + \frac{\partial \delta g_{\rho\nu}}{\partial x^\mu} - \frac{\partial \delta g_{\mu\nu}}{\partial x^\rho} \right). \tag{45}$$

Substituting $\delta g_{ij} = -h_{ij}$ and using $\delta g_{00} = \delta g_{0i} = 0$ we get the perturbations on the connection coefficients for a Bianchi type-I metric:

$$\delta\Gamma^0{}_{00} = \delta\Gamma^0{}_{0i} = \delta\Gamma^0{}_{i0} = \delta\Gamma^i{}_{00} = 0, \tag{46}$$

$$\delta\Gamma^0{}_{ij} = \delta\Gamma^0{}_{ji} = -\tfrac{1}{2} a_i^2 (\dot{h}^i{}_j + 2H_i h^i{}_j), \tag{47}$$

$$\delta\Gamma^i{}_{0j} = -\tfrac{1}{2} [\dot{h}^i{}_j + 2h^i{}_j(H_i - H_j)], \tag{48}$$

$$\delta\Gamma^i{}_{jk} = \delta\Gamma^i{}_{kj} = -\frac{1}{2} \left(h^i{}_{j,k} + h^i{}_{k,j} - \frac{a_j^2}{a_i^2} h^j{}_{k,i} \right). \tag{49}$$

The Ricci tensor $R_{\mu\nu} := R^\lambda{}_{\mu\lambda\nu}$ is given by

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda{}_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda{}_{\mu\lambda}}{\partial x^\nu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\lambda{}_{\lambda\sigma} - \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma}. \tag{50}$$

The perturbations $\delta R_{\mu\nu}$ of the Ricci tensor in terms of the perturbations in the connection coefficients thus follow:

$$\delta R_{\mu\nu} = \frac{\partial \delta\Gamma^\lambda{}_{\mu\nu}}{\partial x^\lambda} - \frac{\partial \delta\Gamma^\lambda{}_{\mu\lambda}}{\partial x^\nu} + (\delta\Gamma^\sigma{}_{\mu\nu})\Gamma^\lambda{}_{\lambda\sigma} + \Gamma^\sigma{}_{\mu\nu}(\delta\Gamma^\lambda{}_{\lambda\sigma}) - (\delta\Gamma^\sigma{}_{\mu\lambda})\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\sigma{}_{\mu\lambda}(\delta\Gamma^\lambda{}_{\nu\sigma}). \tag{51}$$

The perturbations $\delta X^\mu{}_\nu$ of an arbitrary tensor $X^\mu{}_\nu$ can be calculated from the perturbations $\delta X_{\mu\nu}$ using the expression

$$\delta X^\mu{}_\nu = h^{\mu\sigma} X_{\sigma\nu} + g^{\mu\sigma}_{(0)}\, \delta X_{\sigma\nu}. \tag{52}$$

Substituting the connection coefficients (22)–(24) and their perturbations (46)–(49) into (51) and raising the index with the help of (52) we find the expressions for the perturbations on the Ricci tensor for a Bianchi type-I metric:

$$\delta R^0{}_0 = \sum_{k=1}^{3} \left(\tfrac{1}{2}\ddot{h}^k{}_k + H_k \dot{h}^k{}_k \right), \tag{53}$$

$$\delta R^0{}_i = \frac{1}{2} \sum_{k=1}^{3} [\dot{h}^k{}_{k,i} - \dot{h}^k{}_{i,k} + (H_k - H_i)(h^k{}_{k,i} - 2h^k{}_{i,k})], \tag{54}$$

$$\delta R^i{}_j = h^i{}_j[\dot{H}_i - \dot{H}_j + 3H(H_i - H_j)] + \tfrac{1}{2}\delta^i{}_j H_i \sum_{k=1}^{3} \dot{h}^k{}_k + \dot{h}^i{}_j \left(H_i - H_j + \tfrac{3}{2}H \right) + \tfrac{1}{2}\ddot{h}^i{}_j + {}^3R^i_{(1)j}, \tag{55}$$

where ${}^3R^i_{(1)j}$ is the curvature tensor of the three-dimensional space sections of constant time t. Notice that ${}^3R^i{}_j = {}^3R^i_{(0)j} + {}^3R^i_{(1)j}$ is of first order in h_{ij} since ${}^3R^i_{(0)j} = 0$ for a Bianchi type-I metric. We have

$${}^3R^i_{(1)j} = \frac{1}{2} \sum_{k=1}^{3} \left(\frac{1}{a_i^2} \left(h^k{}_{i,j,k} + h^k{}_{j,i,k} - h^k{}_{k,i,j} \right) - \frac{1}{a_k^2} h^i{}_{j,k,k} \right). \tag{56}$$

For future reference we calculate the curvature scalar ${}^3R_{(1)} := \sum_{k=1}^{3} {}^3R^k_{(1)k}$ of the perturbed metric:

$${}^3R_{(1)} = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{1}{a_l^2} \left(h^k{}_{l,k,l} - h^k{}_{k,l,l} \right). \tag{57}$$

So far we only have considered perturbations in the metric components. In order to write down the perturbed Einstein equations

$$\delta R^\mu{}_\nu = \kappa\delta \left(T^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu T^\alpha{}_\alpha \right), \tag{58}$$

we need the perturbations $\delta T^\mu{}_\nu$ of the energy-momentum tensor $T_{\mu\nu}$ for a viscous fluid, given by (11). In the perturbations on the energy-momentum tensor, $\delta T_{\mu\nu}$, the total values of the energy density ε, the pressure p and the hydrodynamic velocity u^μ consist of two parts:

$$\begin{aligned} \varepsilon(t,x^i) &= \varepsilon_{(0)}(t) + \varepsilon_{(1)}(t,x^i), \\ p(t,x^i) &= p_{(0)}(t) + p_{(1)}(t,x^i), \\ u^\alpha(t,x^i) &= \delta^\alpha{}_0 + u^\alpha_{(1)}(t,x^i), \end{aligned} \tag{59}$$

where the quantities with a subscript (0) are position independent, since they refer to the homogeneous background Bianchi type-I model, and those with a subscript (1) are small first-order perturbations. From the normalization condition $u^\mu u_\mu = 1$ it follows that, up to first order in the perturbation $u^\mu_{(1)}$,

$$u^0_{(1)} = u_{(1)0} = 0. \tag{60}$$

From the fact that an unperturbed Bianchi type-I universe is nonrotating and Eq. (60) it follows that up to first order in h_{ij} and $u_{(1)k}$ we have

$$u^i_{(1)} = -\frac{1}{a_i^2} u_{(1)i}. \tag{61}$$

The phenomenon of viscosity is considered to be a first-order effect. Hence we will not take into account the perturbations on the viscosity coefficients due to the perturbations (59), i.e.,

$$\eta_{(1)}(t,x^i) = 0, \quad \eta_{v(1)}(t,x^i) = 0. \tag{62}$$

So, $\eta(t,x^i) = \eta_{(0)}(t)$ and $\eta_v(t,x^i) = \eta_{v(0)}(t)$. We will omit the index (0) at the viscosity coefficients.

We now calculate $\delta T_{\mu\nu}$ starting from (11). Using the connection coefficients (22)–(24), the expressions for their perturbations, (46)–(49), and the expressions (59) for the perturbations on the physical quantities we find, for the perturbations on the energy-momentum tensor for a Bianchi type-I universe,

$$\delta T^0{}_0 = \varepsilon_{(1)}, \tag{63}$$

$$\delta T^0{}_i = -a_i^2 u^i_{(1)} \left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right], \tag{64}$$

$$\begin{aligned} \delta T^i{}_j = & -2c\eta(H_i - H_j)h^i{}_j - c\eta \dot{h}^i{}_j \\ & -\delta^i{}_j \left\{ p_{(1)} + c(\tfrac{2}{3}\eta - \eta_v) \sum_{k=1}^{3} (u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k) \right\} \\ & + c\eta \left(u^i_{(1),j} + \frac{a_j^2}{a_i^2} u^j_{(1),i} \right), \end{aligned} \tag{65}$$

$$\delta T^\alpha{}_\alpha = \varepsilon_{(1)} - 3p_{(1)} + 3c\eta_v \sum_{k=1}^{3} (u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k), \tag{66}$$

where we also used Eq. (52).

Upon substituting Eqs. (55), (65), and (66) into Eq. (58), using Eqs. (32) to eliminate the time derivatives of the Hubble parameters, we obtain the (i,j) components of the perturbation equations:

$$\begin{aligned} & \tfrac{1}{2}\ddot{h}^i{}_j + \dot{h}^i{}_j \left(H_i - H_j + \tfrac{3}{2}H + \kappa c\eta\right) + {}^3R^i_{(1)j} + \tfrac{1}{2}\delta^i{}_j H_i \sum_{k=1}^{3} \dot{h}^k{}_k \\ & = -\tfrac{1}{2}\kappa\delta^i{}_j \left\{ \varepsilon_{(1)} - p_{(1)} + c(\tfrac{4}{3}\eta + \eta_v) \sum_{k=1}^{3} (u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k) \right\} + \kappa c\eta \left(u^i_{(1),j} + \frac{a_j^2}{a_i^2} u^j_{(1),i} \right), \end{aligned} \tag{67}$$

where the curvature ${}^3R^i_{(1)j}$, induced by the perturbations is given by Eq. (56).

For future reference we calculate the evolution equation for the trace $\sum_{k=1}^{3} h^k{}_k$ of the perturbation tensor. Taking in Eq. (67) $i = j$ and summing over the repeated index, one finds

$$\begin{aligned} & \frac{1}{2} \sum_{k=1}^{3} \ddot{h}^k{}_k + (3H - \tfrac{3}{4}\kappa c\eta_v) \sum_{k=1}^{3} \dot{h}^k{}_k + {}^3R_{(1)} \\ & = -\tfrac{3}{2}\kappa \left\{ \varepsilon_{(1)} - p_{(1)} + c\eta_v \sum_{k=1}^{3} u^k_{(1),k} \right\}, \end{aligned} \tag{68}$$

where ${}^3R_{(1)}$ is given by (57).

Finally, we derive the conservation laws for the perturbations. Writing out the equations $(T_\mu{}^\nu)_{;\nu} = 0$ we find

$$(T_\mu{}^\nu)_{,\nu} + \Gamma^\nu{}_{\tau\nu} T_\mu{}^\tau - \Gamma^\tau{}_{\mu\nu} T_\tau{}^\nu = 0. \tag{69}$$

Letting δ operate on these equations we get

$$\begin{aligned} & (\delta T_\mu{}^\nu)_{,\nu} + (\delta\Gamma^\nu{}_{\tau\nu}) T_\mu{}^\tau + \Gamma^\nu{}_{\tau\nu} (\delta T_\mu{}^\tau) - (\delta\Gamma^\tau{}_{\mu\nu}) T_\tau{}^\nu \\ & - \Gamma^\tau{}_{\mu\nu} (\delta T_\tau{}^\nu) = 0. \end{aligned} \tag{70}$$

Upon substituting the connection coefficients (22)–(24), the components of the energy-momentum tensor (11), and the perturbations (46)–(49) and (63)–(65), respectively, we get, for the perturbed conservation laws, using Eq. (52),

$$\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + \sum_{k=1}^{3} \left\{ (u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k) \left[\varepsilon_{(0)} + p_{(0)} + 4c\eta(H - H_k) - 6c\eta_v H\right] \right\} = 0 \tag{71}$$

and

$$\frac{1}{c}\frac{d}{dt}\{u^i_{(1)}\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right]\} + u^i_{(1)}\left(3H + 2H_i\right)\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right]$$

$$-c\eta\sum_{k=1}^{3}\frac{1}{a_i^2}\left[(H_k - H_i)(h^k{}_{k,i} - 2h^k{}_{i,k}) - \dot{h}^k{}_{i,k}\right] + \frac{1}{a_i^2}p_{(1),i} + c(\tfrac{2}{3}\eta - \eta_v)\sum_{k=1}^{3}\frac{1}{a_i^2}(u^k_{(1),k,i} - \tfrac{1}{2}\dot{h}^k{}_{k,i})$$

$$-c\eta\sum_{k=1}^{3}\left(\frac{1}{a_i^2}u^k_{(1),i,k} + \frac{1}{a_k^2}u^i_{(1),k,k}\right) = 0. \quad (72)$$

Equation (71) describes the evolution of local-density enhancements (or rarefactions). Equations (72) describe the evolution of the peculiar velocity of matter.

The $(0,\nu)$-perturbation equations constitute the constraint equations to be obeyed by the initial values. They can be found by substituting the perturbations on the $(0,\nu)$ components (53) and (54) of the Ricci tensor and the perturbations on the $(0,\nu)$ components (63), (64), and (66) into the perturbation equations (58). The constraint equations read

$$\sum_{k=1}^{3}(\tfrac{1}{2}\ddot{h}^k{}_k + H_k\dot{h}^k{}_k) = \tfrac{1}{2}\kappa\left\{\varepsilon_{(1)} + 3p_{(1)} - 3c\eta_v\sum_{k=1}^{3}(u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k)\right\} \quad (73)$$

and

$$\sum_{k=1}^{3}\frac{1}{a_i^2}[\dot{h}^k{}_{k,i} - \dot{h}^k{}_{i,k} + (H_k - H_i)(h^k{}_{k,i} - 2h^k{}_{i,k})]$$
$$= -2\kappa u^i_{(1)}\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right]. \quad (74)$$

Just as in the unperturbed case these equations are satisfied automatically for all times if the initial values do so and if the conservation laws (71) and (72) are satisfied for all times.

Equations (67), (71), and (72) are ten coupled linear homogeneous differential equations for ten unknown functions: namely, the six independent [see Eq. (41)] components $h^i{}_j(t,x^k)$ of the perturbation of the background metric, the perturbation $\varepsilon_{(1)}(t,x^k)$ of the energy density, and the three nonzero components $u^i_{(1)}(t,x^k)$ of the perturbation on the four-velocity.

In Sec. VI we will show that we can distinguish three different types of solutions: namely, gravitational waves, vortex perturbations, and density perturbations. Moreover, it will be shown that these solutions do not influence one another. But first, we must fix the coordinate system, i.e., the "gauge."

IV. GAUGE TRANSFORMATIONS

If we wish to study small perturbations on a background Bianchi type-I metric we are faced with the problem that some of these perturbations can be achieved by an infinitesimal coordinate transformation. Perturbations found in this way are a solution of the perturbation equations in the new coordinates. However, these solutions describe the same physical situation and cannot be observed by experiment. In order to be able to distinguish between solutions that change measurable quantities (such as the energy density) and solutions due to a coordinate transformation, we first study the latter. Consistent with the perturbation analysis, only first-order effects of the coordinate transformations need to be taken into account.

Let us consider an unperturbed Bianchi type-I metric described in the usual coordinates x^μ. Now perform an infinitesimal coordinate transformation to new coordinates $\hat{x}^\mu$ by

$$\hat{x}^\mu = x^\mu + \xi^\mu(x^\nu), \quad (75)$$

where ξ^μ is an arbitrary, small four-vector. Then an arbitrary tensor X transforms under (75) as

$$\hat{X} = X + \mathcal{L}_\xi X, \quad (76)$$

where $\mathcal{L}_\xi X$ is the Lie derivative of the tensor X in the direction of the four-vector ξ. From the definition of the Lie-derivative (see Wald [2], Appendix C)

$$(\mathcal{L}_\xi X)^\alpha{}_\beta := X^\alpha{}_{\beta;\sigma}\xi^\sigma - X^\sigma{}_\beta\xi^\alpha{}_{;\sigma} + X^\alpha{}_\sigma\xi^\sigma{}_{;\beta}, \quad (77)$$

we can see that the metric tensor $g_{\mu\nu}$ transforms as

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}. \quad (78)$$

Note that $h_{ij} := -\delta g_{ij}$ [see (37)]; hence the perturbations on the metric transform under a gauge transformation according to

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu}). \quad (79)$$

The four-velocity u^μ transforms as

$$\hat{u}^\mu = u^\mu + u^\mu{}_{;\sigma}\xi^\sigma - u^\sigma\xi^\mu{}_{;\sigma}. \quad (80)$$

The transformation of the energy density ε can be found from its definition

$$\varepsilon := T^{\mu\nu}u_\mu u_\nu. \quad (81)$$

In a comoving system of reference ($u^\mu = \delta^\mu{}_0$) we get

$$\varepsilon = T^0{}_0. \quad (82)$$

Upon substituting Eq. (82) into Eq. (77), we find

$$\hat{\varepsilon} = \varepsilon + \varepsilon_{,\sigma}\xi^{\sigma}, \tag{83}$$

where we have used that $T^0{}_i = 0$ in a comoving coordinate system.

So far we have considered general coordinate transformations. Now we additionally require that the synchronous reference system will be transformed again into a synchronous reference system; i.e., the metric tensor $g_{\mu\nu}$ must have the form, before and after the transformation,

$$g_{00} = 1, \quad g_{0i} = 0, \quad \hat{g}_{00} = 1, \quad \hat{g}_{0i} = 0. \tag{84}$$

From (78) it follows that

$$\xi_{0;0} = 0, \quad \xi_{i;0} + \xi_{0;i} = 0. \tag{85}$$

Writing out the covariant derivatives in (85) and using the unperturbed connection coefficients for a Bianchi type-I metric (22)–(24) we arrive at

$$\xi^0 = \psi(x^k), \quad \xi^i = \psi_{,i}(x^k) \int^{ct} \frac{d\tau}{a_i^2(\tau)} + \chi^i(x^k), \tag{86}$$

where ψ and χ^i are arbitrary functions of the space coordinates. Coordinate transformations (75) satisfying the property (86) will be called *gauge transformations* from now on, as is usual in this context.

We finally write down the transformation rules for the perturbation on the metric $h^i{}_j$, the hydrodynamic four-velocity u^μ, and the energy density ε,

$$\hat{h}^i{}_j = h^i{}_j - \left(\psi_{,i,j} \int^{ct} \frac{d\tau}{a_i^2(\tau)} + \chi^i{}_{,j}\right) - \frac{a_j^2}{a_i^2}\left(\psi_{,j,i} \int^{ct} \frac{d\tau}{a_j^2(\tau)} + \chi^j{}_{,i}\right) - 2\delta^i{}_j H_i \psi. \tag{87}$$

The four-velocity u^μ transform as

$$\hat{u}^0 = u^0 = 1, \quad \hat{u}^i_{(1)} = u^i_{(1)} - \frac{\psi_{,i}}{a_i^2}, \quad \hat{u}_{(1)i} = u_{(1)i} + \psi_{,i}. \tag{88}$$

Finally, combining Eqs. (83) and (86) and using that $\varepsilon_{(0)} = \varepsilon_{(0)}(t)$ we find that the perturbations on the energy density ε transform as

$$\hat{\varepsilon}_{(1)} = \varepsilon_{(1)} + \dot{\varepsilon}_{(0)}\psi. \tag{89}$$

The transformation rule for the pressure p can be derived from Eq. (89), once the equation of state $p = p(n, \varepsilon)$ is known. Equations (87) and (88) constitute a generalization of the formulas given in Peebles [8], Sec. 81, and Weinberg [3], Chap. 15, whereas Eq. (89) holds true in the anisotropic as well as the isotropic case. Setting $a_1 = a_2 = a_3 = a$ and $H_1 = H_2 = H_3 = H$ one recovers the results for the flat Friedmann-Robertson-Walker (FRW) model given by these authors.

The use of a synchronous reference system does not determine the functions $\psi(x^k)$ and $\chi^i(x^k)$ completely. There remains some freedom in the choice of a coordinate system. Hence, if we tried now to solve the perturbation equations given in Sec. III, we would not succeed since the solutions would still depend on the unspecified functions $\psi(x^k)$ and $\chi^i(x^k)$.

V. NONMATERIAL PERTURBATIONS

In Sec. III we derived the linear perturbation equations for the metric, (67), for the energy density, (71), and for the spatial components of the four-velocity, (72). The initial values for this system of equations must obey the constraint equations (73) and (74). It is well known [1, 3, 8–10] that in a flat FRW universe (i.e., an isotropic Bianchi type-I universe) we can distinguish three independent types of solutions: namely, gravitational waves, vortex perturbations, and density perturbations. In Sec. VI we will prove that in an anisotropic Bianchi type-I universe we can make the same classification of independent solutions.

We first consider solutions which have no material consequences, i.e., $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$. The classification of the material as well as the nonmaterial perturbations is most convenient once one has at one's disposition the theorem stated in Sec. V B. It shows the consequences for the perturbation on the metric tensor $\mathbf{h}$ and the perturbation on the material velocity $\mathbf{u}_{(1)}$ if the perturbation on the energy density $\varepsilon_{(1)}$ and the pressure $p_{(1)}$ vanish.

The proof of the theorem uses two lemmas, proved in Secs. V A and V B.

A. Flat FRW universes

This section on the isotropic Bianchi type-I universe (or flat FRW universe), is a preparation for the anisotropic Bianchi type-I universe. We study solutions which have no material consequences. In particular, we derive the integrability conditions for the perturbed Einstein equations in the case $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$ for an isotropic Bianchi type-I universe. Finally, we define what we shall call nonmaterial perturbations, making use of these integrability conditions.

Lemma 1. *Consider a flat FRW universe filled with a viscous or nonviscous fluid. The conditions on the perturbations on the fluid velocity $u^i_{(1)}$ and the perturbations on the metric $h^i{}_j$ given by*

$$\sum_{k=1}^{3} u^k_{(1),k}(t, x^i) = 0, \quad \sum_{k=1}^{3} h^k{}_k(t, x^i) = 0, \quad \sum_{k=1}^{3}\sum_{l=1}^{3} h^{kl}{}_{,k,l}(t, x^i) = 0, \tag{90}$$

are necessary and sufficient conditions for the perturbed Einstein equations (67) to be integrable for perturbations for which $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$.

To prove the lemma we make use of the perturbed Einstein equations (67), the conservation laws (71) and (72), the constraint equations (73) and (74), and the gauge transformations (87)–(89), with $H_1 = H_2 = H_3 = H$ and $a_1 = a_2 = a_3 = a$.

In Sec. IV we limited the possible reference systems to those that correspond to a synchronous reference system. This implies that the coordinate transformations (75) can only be of the form (86). Now, we require additionally that before and after a gauge transformation the perturbations on the energy density and the pressure vanish. From the transformation rule of the energy density (89) it then follows that

$$\psi(x^k) = 0. \tag{91}$$

The only free functions left over are the three functions $\chi^i(x^k)$ occurring in the gauge transformations. The gauge transformation (87) now reads

$$\hat{h}^i{}_j = h^i{}_j - \chi^i{}_{,j} - \chi^j{}_{,i}. \tag{92}$$

The functions $h^i{}_j$ and their spatial derivatives, being components of the metric, can be specified arbitrarily in the initial hypersurface $t = t_0$ without affecting the physics.

Proof. We first prove the necessity. Upon substituting $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$ into Eq. (71) we get, for the perturbation on the expansion scalar $\delta u^\mu{}_{;\mu}$,

$$\sum_{k=1}^{3}(u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k) = 0. \tag{93}$$

From the constraint equation (73) we find for an isotropic Bianchi type-I universe filled with a viscous fluid, using (93),

$$\frac{1}{2}\sum_{k=1}^{3}\ddot{h}^k{}_k + H\sum_{k=1}^{3}\dot{h}^k{}_k = 0. \tag{94}$$

Combining Eqs. (93) and (94) we get

$$\frac{1}{c}\frac{d}{dt}\left(\sum_{k=1}^{3}u^k_{(1),k}\right) + 2H\sum_{k=1}^{3}u^k_{(1),k} = 0. \tag{95}$$

Differentiating the conservation law (72) with respect to x^i and summing over the repeated index i we find

$$\frac{1}{c}\frac{d}{dt}\left((\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\sum_{i=1}^{3}u^i_{(1),i}\right) + 5H(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\sum_{i=1}^{3}u^i_{(1),i} + c\eta\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\frac{1}{a_i^2}\dot{h}^k{}_{i,k,i} - 2\sum_{i=1}^{3}\sum_{k=1}^{3}\frac{1}{a_i^2}u^k_{(1),i,k,i}\right) = 0, \tag{96}$$

where we have used Eq. (93). Differentiating the constraint equation (74) with respect to x^i and summing over the repeated index i we get

$$\sum_{i=1}^{3}\sum_{k=1}^{3}\frac{1}{a_i^2}(\dot{h}^k{}_{k,i,i} - \dot{h}^k{}_{i,k,i}) = -2\kappa\left(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H\right)\sum_{i=1}^{3}u^i_{(1),i}. \tag{97}$$

Eliminating $\sum_{i=1}^{3}\sum_{k=1}^{3}a_i^{-2}\dot{h}^k{}_{i,k,i}$ from Eq. (96) with the help of Eq. (97) we obtain, using (93),

$$\frac{1}{c}\frac{d}{dt}\left((\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\sum_{i=1}^{3}u^i_{(1),i}\right) + (5H + 2\kappa c\eta)(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\sum_{i=1}^{3}u^i_{(1),i} = 0. \tag{98}$$

Using Eq. (95) to eliminate the time derivative of $\sum_{k=1}^{3}u^k_{(1),k}$ from this equation we find

$$\left\{\frac{1}{c}\frac{d}{dt}(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H) + (3H + 2\kappa c\eta)(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\right\}\sum_{k=1}^{3}u^k_{(1),k} = 0. \tag{99}$$

If we eliminate the time derivative of the unperturbed energy density $\dot{\varepsilon}_{(0)}$ with the help of Eq. (33) we finally arrive at

$$\left\{\frac{1}{c}\frac{d}{dt}(p_{(0)} - 3c\eta_v H) + 2\kappa c\eta(\varepsilon_{(0)} + p_{(0)} - 3c\eta_v H)\right\} \times\sum_{k=1}^{3}u^k_{(1),k} = 0. \tag{100}$$

Since the expression between curly brackets need not vanish necessarily, Eq. (100) implies

$$\sum_{k=1}^{3}u^k_{(1),k} = 0. \tag{101}$$

Hence, with (93), we find

$$\sum_{k=1}^{3}\dot{h}^k{}_k = 0. \tag{102}$$

Equation (92) implies

$$\sum_{k=1}^{3}\hat{h}^k{}_k = \sum_{k=1}^{3}h^k{}_k - 2\sum_{k=1}^{3}\chi^k{}_{,k}. \tag{103}$$

Hence, by a proper choice of the sum of the first-order derivatives, $\sum_{k=1}^{3}\chi^k{}_{,k}(x^i)$, we can achieve that, in the initial hypersurface $t = t_0$,

$$\sum_{k=1}^{3}\hat{h}^k{}_k(t_0, x^i) = 0. \tag{104}$$

From Eq. (102), which holds for all times t, it then follows that

$$\sum_{k=1}^{3}\hat{h}^k{}_k = 0 \quad \text{for all } t. \tag{105}$$

We have found that it is always possible to choose the functions $\chi^k(x^i)$ such that (105) holds. We now do actually choose a coordinate system such that for all t and x^i we have

$$\sum_{k=1}^{3} h^k{}_k = 0. \tag{106}$$

Upon substituting Eqs. (101), (106), $\varepsilon_{(1)} = 0$, and $p_{(1)} = 0$ into the Einstein equation (68) for the trace of the perturbation tensor, we arrive at

$$\sum_{k=1}^{3}\sum_{l=1}^{3} h^{kl}{}_{,k,l} = 0. \tag{107}$$

With (101), (106), and (107) the initial value condition (97) is identically satisfied. We thus have proved that Eqs. (90) are *necessary* conditions for perturbations for which $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$. It may be verified by substitution that the conditions (90) are also *sufficient* conditions for perturbations for which $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$. We thus have proved the lemma.

Note, that in case the fluid is perfect ($\eta = 0$ and $\eta_v = 0$) and pressureless ($p_{(0)} = 0$), Eq. (100) does not imply Eq. (101) since the expression between curly brackets vanishes identically. In that case we simply require Eq. (101) to hold, for reasons of continuity.

In Lemma 1 we have shown that, for flat FRW universes, Eqs. (90) are *necessary* and *sufficient* conditions for perturbations which have $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$. Since the perturbation theories for a flat FRW universe and a Bianchi type-I universe have the Einstein equation (68) for the trace of the perturbation tensor $h^i{}_j$ in common, Eqs. (90) are also *necessary* conditions for perturbations which have $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$ in the *anisotropic* case. This is the key difference between our treatment and other treatments [11–14] of the subject.

The conditions (90) need not be *sufficient* in an anisotropic universe, since the latter has a greater number of degrees of freedom in its expansion. These considerations lead us to the definition of what we shall call *nonmaterial perturbations* in Bianchi type-I universes.

Definition 1. *Non-material perturbations in Bianchi type-I universes are solutions of the perturbation equations (67) and (71)–(74) with the properties*

$$\varepsilon_{(1)} = 0, \quad p_{(1)} = 0, \quad \sum_{k=1}^{3} u^k_{(1),k} = 0, \quad \sum_{k=1}^{3} h^k{}_k = 0,$$
$$\sum_{k=1}^{3}\sum_{l=1}^{3} h^{kl}{}_{,k,l} = 0. \tag{108}$$

The physical interpretation of this definition is as follows. The first two relations express the fact that there are no local condensations or rarefactions of matter. Hence, there can be no local sources or sinks. This is expressed by the third relation, which says that the divergence of the velocity vanishes. The fourth relation expresses the fact that the perturbation on the determinant of the metric tensor vanishes [see Eq. (43)]. The last two equations of (108) together imply [see Eq. (57)], in an isotropic as well as in an anisotropic Bianchi type-I universe,

$$^3R_{(1)} = 0. \tag{109}$$

Hence, nonmaterial perturbations have the physical interpretation that they do not locally curve the three-dimensional hyperspaces of constant time.

The metric perturbation tensor h_{ij} associated with nonmaterial perturbations in a flat FRW universe has precisely the properties given by Landau and Lifshitz [1]. Lemma 1 shows that these properties follow directly from the integrability conditions for the perturbed Einstein equations.

Solutions of the perturbation equations for which $\varepsilon_{(1)}(t, x^i) \neq 0$ are called *material perturbations.*

B. Bianchi type-I universes

In this section we study nonmaterial perturbations, defined by Eqs. (108), in an anisotropic Bianchi type-I universe. We will prove a theorem which will help us to classify the general solution. They will turn out to belong to three different types of solutions. In order to prove the theorem we need the following lemma.

Lemma 2. *Let $H_i(t)$ ($i = 1, 2, 3$) be Hubble parameters satisfying the Einstein equations for a Bianchi type-I universe, Eq. (32). Let $X_i(t, x^j)$ ($i = 1, 2, 3$) be arbitrary functions of time and space coordinates satisfying the linear equations*

$$\sum_{i=1}^{3} X_i(t, x^j) = 0, \quad \sum_{i=1}^{3} H_i(t) X_i(t, x^j) = 0. \tag{110}$$

Then the system of Eqs. (110) is equivalent to

$$\sum_{i=1}^{3} X_i(t, x^j) = 0, \quad \sum_{i=1}^{3} C_i X_i(t, x^j) = 0, \tag{111}$$

where the $C_i = H_i(t_a)$ ($i = 1, 2, 3$) are the Hubble parameters at an arbitrary moment t_a in the evolution of a Bianchi type-I universe.

Proof. In case the universe is isotropic ($H_1 = H_2 = H_3$) the proof is trivial.

If a Bianchi type-I universe is initially axially symmetric [i.e., $H_1(t_0) \neq H_2(t_0) = H_3(t_0)$] then it is axially symmetric for all times [see Eqs. (32)]. In this case the proof is also trivial.

In case the universe is completely anisotropic ($H_1 \neq H_2 \neq H_3$) the solution of Eqs. (110) is

$$X_1 = \lambda(t, x^i)\big[K(t) - 1\big], \quad X_2 = -\lambda(t, x^i) K(t),$$
$$X_3 = \lambda(t, x^i), \tag{112}$$

where $\lambda(t, x^i)$ is an arbitrary function of time and space coordinates and where the function $K(t)$ is given by

$$K(t) := \frac{H_1(t) - H_3(t)}{H_1(t) - H_2(t)}. \tag{113}$$

With the help of the Einstein equations (32) it follows that $dK(t)/dt = 0$, implying that $K(t) = K(t_a)$; i.e., only the values of the Hubble parameters at an *arbitrary* moment t_a, $H_i(t_a)$ $(i = 1, 2, 3)$, enter into the solution (112). Hence, $C_i = H_i(t_a)$ $(i = 1, 2, 3)$. We thus have proved the lemma.

We now come to the main result of this section. The following theorem lists the conditions on the perturbations on the metric $\mathbf{h}$ and the material velocity $\mathbf{u}_{(1)}$ for nonmaterial perturbations defined by Eqs. (108) for a Bianchi type-I universe.

Theorem 1. *Consider a Bianchi type-I universe filled with a viscous or nonviscous fluid. The conditions on the perturbations on the fluid velocity* $u^i_{(1)}$ *and the perturbations on the metric* $h^i{}_j$ *given by*

$$\sum_{i=1}^{3} C_i h^i{}_i(t, x^k) = 0, \tag{114}$$

$$\sum_{i=1}^{3} C_i u^i_{(1),i}(t, x^k) = 0, \quad \sum_{i=1}^{3} C_i^2 u^i_{(1),i}(t, x^k) = 0, \tag{115}$$

$$\begin{gathered}\sum_{i=1}^{3}\sum_{j=1}^{3} C_j h^{ij}{}_{,i,j}(t, x^k) = 0, \\ \sum_{i=1}^{3}\sum_{j=1}^{3} C_i C_j h^{ij}{}_{,i,j}(t, x^k) = 0,\end{gathered} \tag{116}$$

where the C_i *are the constants defined by Lemma 2, are necessary and sufficient conditions for the perturbed Einstein equations (67) to be integrable for nonmaterial perturbations defined by Eqs. (108).*

In the relations (114)–(116) the constants C_i *may be replaced by the Hubble parameters* $H_i(t)$.

To prove the theorem we make use of the perturbed conservation laws (71) and (72), the constraint equations (73) and (74), and the gauge transformations (87)–(89). The gauge transformation (87) now reads

$$\hat{h}^i{}_j = h^i{}_j - \chi^i{}_{,j} - \frac{a_j^2}{a_i^2}\chi^j{}_{,i}. \tag{117}$$

Proof. We first prove the necessity. Upon substituting Eqs. (108) into the constraint equation (73) we get

$$\sum_{k=1}^{3} H_k \dot{h}^k{}_k = 0. \tag{118}$$

Since $\sum_{k=1}^{3} \dot{h}^k{}_k = 0$ for nonmaterial perturbations we have, using Lemma 2,

$$\sum_{k=1}^{3} C_k \dot{h}^k{}_k = 0. \tag{119}$$

From Eq. (117) it follows that

$$\sum_{k=1}^{3} C_k \hat{h}^k{}_k = \sum_{k=1}^{3} C_k h^k{}_k - 2\sum_{k=1}^{3} C_k \chi^k{}_{,k}. \tag{120}$$

By a proper choice of the derivatives $\chi^k{}_{,k}(x^i)$ we can achieve that in the initial hypersurface $t = t_0$,

$$\sum_{k=1}^{3} C_k \hat{h}^k{}_k(t_0, x^i) = 0. \tag{121}$$

From Eqs. (119) and (121) it follows that

$$\sum_{k=1}^{3} C_k \hat{h}^k{}_k = 0 \quad \text{for all } t. \tag{122}$$

We have found that it is always possible to choose the functions $\chi^k{}_{,k}(x^i)$ such that Eq. (121) holds. We now choose a coordinate system such that Eq. (114) holds.

To prove the first relation of Eq. (115) we substitute Eqs. (108) into the conservation law (71). We then find the solubility condition

$$\sum_{k=1}^{3} H_k (u^k_{(1),k} - \tfrac{1}{2}\dot{h}^k{}_k) = 0. \tag{123}$$

Using Eq. (118) we get

$$\sum_{k=1}^{3} H_k u^k_{(1),k} = 0. \tag{124}$$

For nonmaterial perturbations $\mathbf{u}_{(1)}$ is divergenceless [see Eqs. (108)]. Using Lemma 2 we find the first relation of Eqs. (115).

To prove the first relation of Eqs. (116) we make use of the constraint equation (74). Upon substituting Eqs. (108) and (114), raising the index i in the left-hand side of this equation, and differentiating with respect to x^i we get

$$\begin{aligned}\sum_{k=1}^{3}(\dot{h}^{ki}{}_{,k,i} &+ 2H_k h^{ki}{}_{,k,i}) \\ &= -2\kappa u^i_{(1),i}\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right].\end{aligned} \tag{125}$$

Summing over the index i we find, using Eqs. (108) and (124),

$$\sum_{i=1}^{3}\sum_{k=1}^{3} H_k h^{ki}{}_{,k,i} = 0. \tag{126}$$

Applying Lemma 2 to the functions $X_k := \sum_{i=1}^{3} h^{ki}{}_{,k,i}$ we arrive at the first relation of Eq. (116):

$$\sum_{i=1}^{3}\sum_{k=1}^{3} C_k h^{ki}{}_{,k,i} = 0. \tag{127}$$

The second relation of Eqs. (115) can be found by differentiating Eq. (72) with respect to x^i, summing over the repeated index i, substituting the definition (108), and using Eqs. (114) and the first relation of Eqs. (115). One finds

$$4\sum_{i=1}^{3} H_i^2 u^i_{(1),i} - \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{1}{a_i^2}[2(H_k - H_i)h^k{}_{i,k,i} + \dot{h}^k{}_{i,k,i}] = 0. \quad (128)$$

Raising the index i in the second term yields

$$4\sum_{i=1}^{3} H_i^2 u^i_{(1),i} + \sum_{i=1}^{3}\sum_{k=1}^{3}(2H_k h^{ki}{}_{,k,i} + \dot{h}^{ki}{}_{,k,i}) = 0. \quad (129)$$

Upon substituting (108) and (126) one gets

$$\sum_{i=1}^{3} H_i^2 u^i_{(1),i} = 0. \quad (130)$$

Applying Lemma 2 to the functions $X_i := H_i u^i_{(1),i}$ we find, using Eq. (130),

$$\sum_{i=1}^{3} C_i H_i u^i_{(1),i} = 0. \quad (131)$$

Finally, applying Lemma 2 to the functions $X_i := C_i u^i_{(1),i}$ and using Eq. (131) yields the second relation of Eqs. (115).

The last relation of Eqs. (116) can be proved as follows. Multiplying Eq. (125) by H_i and summing over the index i we find, using Eqs. (115), (124), and the time derivative of Eq. (127),

$$\sum_{i=1}^{3}\sum_{k=1}^{3} H_i H_k h^{ki}{}_{,k,i} = 0. \quad (132)$$

Applying Lemma 2 to the functions $X_i := \sum_{k=1}^{3} H_k h^{ki}{}_{,k,i}$ we find, using Eqs. (126) and (132),

$$\sum_{i=1}^{3}\sum_{k=1}^{3} C_i H_k h^{ki}{}_{,k,i} = 0. \quad (133)$$

Applying Lemma 2 to the functions $X_k := \sum_{i=1}^{3} C_i h^{ki}{}_{,k,i}$ we finally find, using Eqs. (127) and (133), the second relation of Eqs. (116).

We thus have proved that Eqs. (114)–(116) are *necessary* conditions for the nonmaterial perturbations defined by Eqs. (108). It may be verified by substitution that the conditions (114)–(116) are also *sufficient* conditions for nonmaterial perturbations characterized by Eqs. (108). We thus have completed the proof of Theorem 1.

Notice that if the viscosity coefficients η and η_v vanish Eq. (123) does not follow from Eq. (71). In that case, however, Eq. (124) follows directly from the conservation law (72) by differentiating the latter with respect to x^i, summing over the repeated index, and using Eqs. (108). Moreover, if $\eta = 0$ and $\eta_v = 0$ then Eq. (130) can be proved by differentiation of Eq. (72) with respect to x^i, multiplying by H_i, summing over the repeated index i and using Eqs. (108), (124), and Lemma 2.

Theorem 1 is a central point in our discussion. It has as a consequence that the general solution of the perturbation equations can be split up in material and nonmaterial solutions, which are independent (see Table I). The nonmaterial solutions can again be split up in gravitational waves and vortices (see Table II). We come to these points in detail in the next section, where we will use the theorem to classify the various solutions of the perturbation equations.

Restrictions on the gauge transformations. If we require that, in a synchronous system of reference, nonmaterial perturbations are transformed again into nonmaterial perturbations, then the functions $\psi(x^j)$ and $\chi^i(x^j)$, occurring in Eqs. (87)–(89), are restricted to

$$\psi(x^j) = 0, \quad \sum_{i=1}^{3}\chi^i{}_{,i}(x^j) = 0, \quad \sum_{i=1}^{3} C_i\chi^i{}_{,i}(x^j) = 0, \quad (134)$$

where the C_i ($i = 1,2,3$) are the constants defined by Lemma 2. The first identity follows from $\varepsilon_{(1)} = 0$ and Eq. (89), the second equation follows from $\psi = 0$, Eqs. (87), and the fact that $h^i{}_j$ is traceless. Finally, the third condition follows from the additional condition (114) in an anisotropic universe. The consequence of Eqs. (134) is that the decomposition given in Definition 1 and Theorem 1 is *gauge invariant.*

TABLE I. The independent material and nonmaterial perturbations found in an anisotropic Bianchi type-I universe. The general solution of the perturbation equations is a linear combination of these two types of solutions. The tensor $\tilde{\mathbf{h}}$ and the vector $\mathbf{u}_{(1)\perp}$ have the properties specified in Definition 1 and Theorem 1.

General solution	Nonmaterial perturbations	Material perturbations	Mathematical properties
$\mathbf{h}$	$\tilde{\mathbf{h}}$	$\mathbf{h}_{\parallel}$	$\mathrm{Tr}(\mathbf{h}) = \mathrm{Tr}(\mathbf{h}_{\parallel})$
$\mathbf{u}_{(1)}$	$\mathbf{u}_{(1)\perp}$	$\mathbf{u}_{(1)\parallel}$	$\mathrm{div}(\mathbf{u}_{(1)}) = \mathrm{div}(\mathbf{u}_{(1)\parallel})$
$\varepsilon_{(1)}$	0	$\varepsilon_{(1)}$	$\mathrm{rot}(\mathbf{u}_{(1)}) = \mathrm{rot}(\mathbf{u}_{(1)\perp})$

TABLE II. Three independent types of perturbations found in a flat FRW universe and in an anisotropic Bianchi type-I universe. The general solution of the perturbation equations is a linear combination of these three types of solutions. The tensors $\mathbf{h}_*$, $\mathbf{h}_\perp$ and the vector $\mathbf{u}_{(1)\perp}$ have the properties specified in Definition 1 and Theorem 1.

General solution	Gravitational wave ("tensor wave")	Vortex perturbation ("vector wave")	Density perturbation ("scalar wave")
$\mathbf{h}$	$\mathbf{h}_*$	$\mathbf{h}_\perp$	$\mathbf{h}_\parallel$
$\mathbf{u}_{(1)}$	0	$\mathbf{u}_{(1)\perp}$	$\mathbf{u}_{(1)\parallel}$
$\varepsilon_{(1)}$	0	0	$\varepsilon_{(1)}$

VI. CLASSIFICATION OF THE SOLUTIONS

We will now show that we can split up the general solution of the perturbation equations (67) and (71)–(74) uniquely into three distinct classes.

Perturbation equations for the nonmaterial case. The integrability conditions (114)–(116) of Theorem 1 for the nonmaterial perturbations of Definition 1 can now be used to derive the equations describing nonmaterial perturbations, defined by Eqs. (108), from the full set of perturbation equations (67), the conservation laws (71) and (72), and the constraint equations (73) and (74).

Applying Definition 1 and Theorem 1 to the perturbed Einstein equations (67) we find the evolution equations for nonmaterial perturbations:

$$\tfrac{1}{2}\ddot{h}^i{}_j + \dot{h}^i{}_j\left(H_i - H_j + \tfrac{3}{2}H + \kappa c\eta\right) + {}^3R^i_{(1)j} = \kappa c\eta\left(u^i_{(1),j} + \frac{a_j^2}{a_i^2}u^j_{(1),i}\right), \quad (135)$$

where

$${}^3R^i_{(1)j} = \frac{1}{2}\sum_{k=1}^{3}\left(\frac{1}{a_i^2}\left(h^k{}_{i,j,k} + h^k{}_{j,i,k}\right) - \frac{1}{a_k^2}h^i{}_{j,k,k}\right), \quad (136)$$

is the curvature tensor of the three-dimensional hypersurfaces of constant time t. Recall that for nonmaterial perturbations the curvature scalar ${}^3R_{(1)}$ vanishes [see Eqs. (57) and (108)].

Applying Definition 1 and Theorem 1 to the conservation laws (71) and (72) we find that Eq. (71) is identically satisfied, whereas Eq. (72) reduces to

$$\frac{1}{c}\frac{d}{dt}\{u^i_{(1)}\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right]\} + u^i_{(1)}(3H + 2H_i)\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right] + c\eta\sum_{k=1}^{3}\frac{1}{a_i^2}\left[2(H_k - H_i)h^k{}_{i,k} + \dot{h}^k{}_{i,k}\right] - c\eta\sum_{k=1}^{3}\frac{1}{a_k^2}u^i_{(1),k,k} = 0. \quad (137)$$

Finally, applying Definition 1 and Theorem 1 to the constraint equations (73) and (74) we find that Eq. (73) is identically satisfied, and that Eq. (74) reduces to

$$-\sum_{k=1}^{3}\frac{1}{a_i^2}\left[\dot{h}^k{}_{i,k} + 2(H_k - H_i)h^k{}_{i,k}\right] = -2\kappa u^i_{(1)}\left[\varepsilon_{(0)} + p_{(0)} + 2c\eta(H - H_i) - 3c\eta_v H\right]. \quad (138)$$

The set of equations (135)–(138) completely replaces the former set of perturbation equations, constraint equations, and conservation laws in the case $\varepsilon_{(1)}(t,x^i) = 0$ and $p_{(1)}(t,x^i) = 0$. We will use these equations to classify the various types of solutions of the perturbation equations. Note that the solution $\mathbf{u}_{(1)}(t,x^i)$ is, by virtue of (134), gauge independent, as can be seen from Eq. (88). The solution $\mathbf{h}(t,x^i)$, however, is determined up to time-independent functions $\chi^i(x^k)$ $(i = 1,2,3)$.

Independency of material and nonmaterial perturbations. With the help of Definition 1 and Theorem 1 we are able to distinguish two different types of solutions which are independent of one another: namely, the material and the nonmaterial perturbations. This can be done as follows. Let us denote the solutions of Eqs. (135)–(138), for which $\varepsilon_{(1)} = 0$ and $p_{(1)} = 0$, by $(0, \mathbf{u}_{(1)\perp}, \tilde{\mathbf{h}})$, where $\mathbf{u}_{(1)\perp}$ and $\tilde{\mathbf{h}}$ have the properties given in Definition 1 and Theorem 1. The general solution $(\varepsilon_{(1)}, \mathbf{u}_{(1)}, \mathbf{h})$, for which $\varepsilon_{(1)} \neq 0$, follows from Eqs. (67), (71), (72), (73), and (74). We now make use of the linearity of the perturbation equations by subtracting Eqs. (135), (137), and (138) from their counterparts (67), (72), and (74), respectively. Now, we replace in the resulting equations $\mathbf{u}_{(1)} - \mathbf{u}_{(1)\perp}$ by $\mathbf{u}_{(1)\parallel}$ and $\mathbf{h} - \tilde{\mathbf{h}}$ by $\mathbf{h}_\parallel$. (This *defines* in fact $\mathbf{u}_{(1)\parallel}$ and $\mathbf{h}_\parallel$.) We thus end up with a system of equations which contain only the functions $\mathbf{h}_\parallel$, $\mathbf{u}_{(1)\parallel}$, and $\varepsilon_{(1)}$. We thus arrive at Table I. The solutions with $\varepsilon_{(1)} \neq 0$ are called *material*, those with $\varepsilon_{(1)} = 0$ are called *nonmaterial.* From their construction, it follows that the material and nonmaterial solutions are independent. Note, that Eqs. (71) and (73) are satisfied identically for the nonmaterial part of the solution. The further consequences of this construction are that, since $\mathrm{div}(\mathbf{u}_{(1)\perp}) = 0$ according to Definition 1, the rotation of $\mathbf{u}_{(1)\|}$ vanishes, i.e., $\mathrm{rot}(\mathbf{u}_{(1)\|}) = 0$, and that $\mathbf{h}_\|$ contains the trace of the perturbation tensor $\mathbf{h}$. Hence, the nonmaterial perturbations $(0, \mathbf{u}_{(1)\perp}, \tilde{\mathbf{h}})$ contain the rotational perturbations, since $\mathbf{u}_{(1)\perp}$ is the rotational part of the vector $\mathbf{u}_{(1)}$. In other words, vortices are not coupled to density perturbations.

In Sec. VI B we will show that the nonmaterial perturbations $(0, \mathbf{u}_{(1)\perp}, \tilde{\mathbf{h}})$ can be split up further into two groups of independent solutions: namely, gravitational waves $(0, 0, \mathbf{h}_*)$ and vortices $(0, \mathbf{u}_{(1)\perp}, \mathbf{h}_\perp)$.

A. Gravitational waves

We define gravitational waves as solutions of wave equations that have no material consequences, i.e., $\varepsilon_{(1)}(t, x^i)$, $p_{(1)}(t, x^i)$, and $\mathbf{u}_{(1)}(t, x^i)$ vanish for all times t and for all points x^i. In other words, gravitational waves are perturbations of the gravitational field in which matter remains at rest and uniformly distributed throughout space. We start from Eqs. (135)–(138). Upon substituting $\mathbf{u}_{(1)} = 0$ into the dynamical equations (135) and the conservation law (137) we find, respectively,

$$\tfrac{1}{2}\ddot{h}^i{}_j + \dot{h}^i{}_j \left(H_i - H_j + \tfrac{3}{2}H + \kappa c\eta\right) + {}^3R^i_{(1)j} = 0 \qquad (139)$$

and

$$-\sum_{k=1}^{3} \frac{1}{a_i^2}[2(H_k - H_i)h^k{}_{i,k} + \dot{h}^k{}_{i,k}] = 0, \qquad (140)$$

where ${}^3R^i_{(1)j}$ is given by (136). Notice that the conservation law (137) and the constraint equation (138) are identical for nonmaterial perturbations for which $\mathbf{u}_{(1)} = 0$. Hence, Eq. (140) is satisfy for all times t, once the initial conditions $h^i{}_j(t_0, x^k)$ satisfy Eq. (140). We will take Eq. (139) with initial conditions satisfying Eq. (140) as the defining property of gravitational waves in an anisotropic Bianchi type-I universe. Solutions satisfying Eqs. (139) and (140) will be distinguished by an asterisk: $\mathbf{h}_*$.

Nontransversality. We will now show that the gravitational waves defined in this way are in general *nontransversal* in a Bianchi type-I universe.

The covariant transversality condition is given by

$$h^\mu{}_{\nu;\mu} = 0. \qquad (141)$$

Using the connection coefficients (22)–(24) for a Bianchi type-I metric, the left-hand side of Eq. (141) can be written as

$$h^\mu{}_{\nu;\mu} = h^\mu{}_{\nu,\mu} + 3Hh^0{}_\nu - \delta^0{}_\nu \sum_{k=1}^{3} H_k h^k{}_k. \qquad (142)$$

Since we use a synchronous system of reference, the second term on the right-hand side vanishes [cf. Eq. (38)]. Moreover, for gravitational waves the third term on the right-hand side vanishes also, as follows directly from Theorem 1. Hence, in a synchronous coordinate system the transversality condition reads

$$\sum_{k=1}^{3} h^k{}_{i,k} = 0; \qquad (143)$$

i.e., the covariant derivative in Eqs. (141) may be replaced by the partial derivative in case of a Bianchi type-I universe.

For nonmaterial perturbations the gauge transformation (87) reduces to

$$\hat{h}^i{}_j = h^i{}_j - \chi^i{}_{,j} - \frac{a_j^2}{a_i^2}\chi^j{}_{,i} \qquad (144)$$

since $\psi(x^i) = 0$ for nonmaterial perturbations. Hence,

$$\sum_{k=1}^{3} \hat{h}^k{}_{i,k} = \sum_{k=1}^{3} h^k{}_{i,k} - \sum_{k=1}^{3} \chi^k{}_{,i,k} - \sum_{k=1}^{3} \frac{a_i^2}{a_k^2}\chi^i{}_{,k,k}. \qquad (145)$$

The second term on the right-hand side of Eq. (145) vanishes by virtue of Eq. (134), so that

$$\sum_{k=1}^{3} \hat{h}^k{}_{i,k} = \sum_{k=1}^{3} h^k{}_{i,k} - \sum_{k=1}^{3} \frac{a_i^2}{a_k^2}\chi^i{}_{,k,k}. \qquad (146)$$

Differentiation of (146) with respect to ct gives

$$\frac{1}{c}\sum_{k=1}^{3} \frac{d\hat{h}^k{}_{i,k}}{dt} = \frac{1}{c}\sum_{k=1}^{3} \frac{dh^k{}_{i,k}}{dt} + 2\sum_{k=1}^{3}(H_k - H_i)\frac{a_i^2}{a_k^2}\chi^i{}_{,k,k}, \qquad (147)$$

where we have used Eq. (20). From the fact that integration of (147) with respect to ct yields (146) it follows that we can choose a coordinate system in the initial hypersurface $t = t_0$ such that $\sum_{k=1}^{3} h^k{}_{i,k}$ and its time derivative vanish simultaneously. In other words, it is possible to choose the functions $\chi^i(x^j)$ in the initial hypersurface $t = t_0$ such that the gravitational waves are transversal initially:

$$\sum_{k=1}^{3} h^k{}_{i,k}(t_0, x^j) = 0, \qquad \sum_{k=1}^{3} \dot{h}^k{}_{i,k}(t_0, x^j) = 0 \quad (i = 1, 2, 3). \qquad (148)$$

From the constraint equation (140) it now follows that the transversality condition in the initial hypersurface must satisfy the additional condition

$$\sum_{k=1}^{3} H_k(t_0) h^k{}_{i,k}(t_0, x^j) = 0 \quad (i = 1, 2, 3). \qquad (149)$$

Upon substituting Eqs. (148) and (149) into the dynamical equations (139) we find that the second-order time derivative in the initial hypersurface $t = t_0$ is given by

$$\sum_{k=1}^{3} \ddot{h}^k{}_{i,k}(t_0, x^j) = -2\sum_{k=1}^{3} H_k(t_0)\dot{h}^k{}_{i,k}(t_0, x^j) \quad (i = 1, 2, 3). \qquad (150)$$

In the case of a flat FRW universe where the three Hubble parameters H_i are equal, the right-hand side vanishes, in view of (148), implying that in a flat FRW universe gravitational waves are transversal everywhere and for all times. In general, however, in an anisotropic Bianchi type-I universe the right-hand side of Eq. (150) need not vanish necessarily, implying that in general gravitational waves are *nontransversal.*

Note that gravitational waves can be made transversal by a gauge transformation *everywhere* in an arbitrary hypersurface $t = t_a$. Since this gauge transformation is, in contrast to the isotropic case, time dependent [see Eqs. (146) and (147)], transversal gravitational waves evolve into nontransversal gravitational waves. However, the longitudinal components of the gravitational waves have no physical meaning, since they can be transformed away, by the gauge transformation (144), in the *entire* space sections of constant time.

B. Vortex perturbations

We now use again the linearity of the perturbation equations and subtract (139) from (135) and (140) from (138). Then we replace in the resulting equations $\tilde{\mathbf{h}} - \mathbf{h}_*$ by $\mathbf{h}_\perp$. (This *defines* in fact $\mathbf{h}_\perp$.) We thus end up with equations for $\mathbf{h}_\perp$ and $\mathbf{u}_{(1)\perp}$. Note that the equations (137) are satisfied identically for the gravitational wave part of the solutions. Hence, what remains of these three equations are equations for $\mathbf{h}_\perp$ and $\mathbf{u}_{(1)\perp}$.

The solutions $(0, \mathbf{u}_{(1)\perp}, \mathbf{h}_\perp)$ are the so-called *rotational perturbations* or *vortices* since $\mathrm{div}(\mathbf{u}_{(1)\perp}) = 0$ and $\mathrm{rot}(\mathbf{u}_{(1)\|}) = 0$ by virtue of Definition 1, implying that $\mathrm{rot}(\mathbf{u}_{(1)}) = \mathrm{rot}(\mathbf{u}_{(1)\perp})$.

In conclusion, the equations for vortex perturbations $(0, \mathbf{u}_{(1)\perp}, \mathbf{h}_\perp)$ are obtained by replacing $\mathbf{h}$ and $\mathbf{u}_{(1)}$ by $\mathbf{h}_\perp$ and $\mathbf{u}_{(1)\perp}$ in the dynamical equations (135) and the conservation law (137). Furthermore, the initial conditions $\mathbf{h}_\perp(t_0, x^k)$ and $\mathbf{u}_{(1)\perp}(t_0, x^k)$ must obey the constraint equations (138).

The set of equations for vortices $(0, \mathbf{u}_{(1)\perp}, \mathbf{h}_\perp)$ is, by construction, independent of the sets for gravitational waves $(0, 0, \mathbf{h}_*)$ and density perturbations $(\varepsilon_{(1)}, \mathbf{u}_{(1)\|}, \mathbf{h}_\|)$. Combining the results of Table I with the results of Secs. VI A and VI B, we arrive at Table II.

VII. SUMMARY

We considered a universe filled with an imperfect fluid characterized by the viscosity coefficients η and η_v. We derived the evolution equations for small perturbations in the metric, the energy density, and the velocity of matter in a homogeneous, anisotropic Bianchi type-I universe.

We defined nonmaterial perturbations as solutions of the perturbation equations, (67) and (71)–(74), which obey Eqs. (108). As a consequence, we find that the metric perturbations must obey Eqs. (114) and (116), and the material velocity must obey Eq. (115). Theorem 1 tells us that we may not make $h^i{}_j$ traceless by simply subtracting its trace, as can be done in a flat FRW universe. Instead, we are forced to make a decomposition appropriate to the anisotropic case. Only if we make *this* decomposition we find that, in a linear perturbation theory, the nonmaterial and material perturbations are independent. This leads us to consider nontransversal gravitational waves.

The nonmaterial perturbations can be split up into gravitational waves which have $\mathbf{u}_{(1)} = 0$, and vortex perturbations for which $\mathbf{u}_{(1)} \neq 0$. These two types of perturbations are again independent, if one chooses the decomposition dictated by the integrability conditions. In contrast to what one finds in a flat FRW universe, the gravitational waves need not be transversal in an anisotropic Bianchi type-I universe.

We recall that the decomposition given by Definition 1 and Theorem 1 is independent of the properties of the cosmic fluid, i.e., the results of the theorem are independent of the equation of state and the viscosity of the fluid.

In a forthcoming paper we will analyze in detail the consequences of our approach for the evolution of density fluctuations.

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