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General-relativistic celestial mechanics. III. Rotational equations of motion

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The rotational laws of motion for arbitrarily shaped, weakly self-gravitating bodies, members of gravitationally interacting N-body systems, are obtained at the first post-Newtonian approximation of general relativity. The derivation uses our previously introduced framework, characterized by the combined use of N local (body-attached) reference systems with one global reference system, and by the introduction of new sets of relativistic multipole moments, and relativistic tidal moments. We show how to associate with each body (considered in its corresponding local frame) a first-post-Newtonian-accurate spin vector, whose local-time evolution is entirely determined by the coupling between the multipole moments of that body and the tidal moments it experiences. The leading relativistic effects in the spin motion are discussed: gravitational Larmor theorem (de Sitter–Fokker–Eddington precession) and post-Newtonian contributions to the torque associated with the quadrupole moment and the quadrupole tidal tensor.

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I. INTRODUCTION

In two previous papers [1, 2] (hereafter referred to as papers I and II, respectively) we presented a new formalism for treating the general-relativistic celestial mechanics of systems of N, arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies. In paper I we laid the foundations of our formalism which uses, in a complementary manner, $N+1$ coordinate charts (or "reference systems") to treat the general-relativistic N-body problem: one "global" chart [spacetime coordinates $(x^\mu) = (ct, x^i)$] is used for describing the overall dynamics of the N bodies ("ephemerides"), while N "local" charts [spacetime coordinates $(X_A^\alpha) = (cT_A, X_A^a)$, where $A = 1, \ldots, N$ labels the bodies] are used for describing the intrinsic dynamics, and the local gravitational environment of each body. We showed in paper I (Sec. II there) how to write in closed form the transformation mapping the local coordinates X_A^α onto the global ones x^μ:

$$x^\mu = f^\mu(X_A^\alpha\ ;\ \mathcal{D}_A)\ , \tag{1.1}$$

where $\mathcal{D}_A$ denotes some structures ("world line data") which determine the precise choice of the Ath local reference system. After the systematic use of local reference systems [with a specific algebraic way of freezing down the spatial coordinate freedom, see Eqs. (2.27)–(2.29) in paper I], the two other basic tools of our formalism are (i) the use of new field variables [global, $w_\mu(x^\lambda)$, and local, $W_\alpha^A(X_A^\beta)$, gravitational potentials], and new matter variables [$\sigma_\mu(x^\lambda)$, $\Sigma_\alpha^A(X_A^\beta)$] which lead to linear field equations and simple (affine) "transformation laws" for the gravitational potentials under Eq. (1.1) (see Sec. IV of paper I), and (ii) the introduction of new relativistic "multipole moments" of each body [of the "mass" $M_L^A(T_A)$ and "spin" type $S_L^A(T_A)$, where $L \equiv a_1, \ldots, a_l$ is a multispatial index of order $l = 0, 1, 2, \ldots$ in the corresponding Ath local frame; see Eq. (6.11) in paper I], and relativistic "tidal moments" experienced by each body [of the "gravitoelectric" $G_L^A(T_A)$ and "gravitomagnetic" type $H_L^A(T_A)$; see Eq. (6.13) in paper I completed by Eq. (2.25a) in paper II for a first-post-Newtonian- (1PN) accurate tidal monopole moment].

In paper II we completed the results of paper I in two directions. First, we derived, at the first-post-Newtonian (1PN) approximation, the *translational* laws of motion (including the 1PN law for the evolution of the masses) for an N-body system. This result was first obtained in the form of an integrodifferential system

$$\frac{dM^A}{dT_A} = F_0^A[\overline{W}_\alpha^A]\ , \tag{1.2a}$$

$$\frac{d^2 M_a^A}{dT_A^2} = F_a^A[\overline{W}_\alpha^A]\ , \tag{1.2b}$$

where M^A is the 1PN mass monopole of body A, M_a^A its 1PN mass dipole, and where the "force terms" F_α^A are some explicit spatial integrals over the volume of body A [Eq. (4.7) in paper II] exhibiting the localized interaction between the mass and current densities of body A, (Σ^A, Σ_a^A), and the "external gravitational potentials" $\overline{W}_\alpha^A(X_A^\beta)$ [i.e., the potentials in the local A frame remaining after subtracting the local-frame self-potentials $W_\alpha^{+A}(X_A^\beta)$]. Then, by inserting in the force terms on the right-hand sides of Eqs. (1.2) the relativistic tidal expansion of the external potentials [see Eqs. (6.23) in paper I and (4.15) and (4.16) in paper II], we found, after the occurrence of subtle cancellations, that the result was expressible as a series of bilinear couplings between the post-Newtonian multipole moments of body A (M_L^A, S_L^A) and the post-Newtonian tidal moments felt by A (G_L^A, H_L^A):

$$\frac{dM^A}{dT_A} = \overline{F}_0^A \left[M_L^{(p)A},\, G_{L'}^{(p')A}\right] + O(4) \ , \tag{1.3a}$$

$$\frac{d^2M_a^A}{dT_A^2} = \overline{F}_a^A \left[M_L^{(p)A},\, S_L^{(q)A},\, G_{L'}^{(p')A},\, H_{L'}^{(q')A}\right] + O(4) \ , \tag{1.3b}$$

where $M^{(p)} \equiv d^pM/dT^p$.

Then, using the results of the "transformation theory" of paper I, we worked out the explicit expressions of the tidal moments experienced by body A. Our results (see Sec. V of paper II) have the form

$$G_L^A = \sum_{B\neq A} G_L^{B/A}[M_K^B, S_K^B, \mathcal{D}_A, \mathcal{D}_B] + G^{A}{}''_L[\mathcal{D}_A] \,+\, O(4) \ , \tag{1.4a}$$

$$H_L^A = \sum_{B\neq A} H_L^{B/A}[M_K^B, S_K^B, \mathcal{D}_A, \mathcal{D}_B] + H^{A}{}''_L[\mathcal{D}_A] + O(2) \ , \tag{1.4b}$$

where the $N-1$ "B over A" terms $G_L^{B/A}$, $H_L^{B/A}$, represent the contributions of body B (with multipole moments M_K^B, S_K^B ; $K = b_1,\ldots,b_k,\ k = 0,1,2,\ldots$) to the tidal moments felt by body A, while the last terms $G_L^{A\prime\prime}$, $H_L^{A\prime\prime}$ represent the inertial effects of the nonlinear ("accelerated") transformation (1.1) between the global and the local frames. The latter inertial contributions were worked out in Sec. VI E of paper I, while the former B-over-A contributions have been computed in Sec. V D of paper II (see also Appendix A there). Finally, let us note that Sec. II of paper II contains a summary of the main results of paper I, and that a pedagogical overview of our formalism can be found in [3].

The purpose of the present paper is to complete the results of papers I and II by deriving some 1PN-accurate *rotational* laws of motion for an N-body system (thereby completing the proof of theorem 7 in paper I). More precisely, we are going to show that there exists a 1PN-accurate definition of the spin dipole (or spin vector), say $S_a^{A(1\mathrm{PN})}(T_A)$, of body A (member of a gravitationally interacting N-body system) with the following properties.

(i) In the Newtonian approximation $S_a^A(T_A)$ reduces to the standard Newtonian spin of an extended body (taken with respect to the origin of the local A frame), i.e.,

$$S_a^{A(1\mathrm{PN})}(T_A) = \int_A d^3X_A\, \epsilon_{abc} X_A^b \Sigma_A^c(T_A, \mathbf{X}_A) + O(2) \ . \tag{1.5}$$

(ii) When body A is an isolated system (with negligible gravitational interactions with the external universe) $S_a^{A(1\mathrm{PN})}$ reduces to the well-known 1PN-accurate conserved total spin of an isolated system (first given as a well-defined compact-support integral by Fock [4]).

(iii) $S_a^{A(1\mathrm{PN})}(T_A)$ is entirely defined in the local frame of body A as a compact-support spatial integral extending only over the volume of body A.

(iv) Similar to the translational local laws of motion (1.3), the time evolution of $S_a^{A(1\mathrm{PN})}(T_A)$ can be entirely expressed in terms of bilinear couplings between the (time derivatives of the) multipole moments of A and the (time derivatives of the) tidal moments experienced by A:

$$\frac{dS_a^{A(1\mathrm{PN})}}{dT_A} = \overline{D}_a^A \left[M_L^{(p)A},\, S_L^{(q)A},\, G_{L'}^{(p')A},\, H_{L'}^{(q')A}\right] + O(4) \ . \tag{1.6}$$

The reader should note that in all the developments of our formalism up to now we could consistently work with definitions of the spin moments (the spin dipole S_a^A, the spin quadrupole S_{ab}^A, etc.) that had only Newtonian accuracy. Namely,

$$S_L^A(T_A) = \mathrm{STF}_L \left\{ \int_A d^3X\, X^{L-1} \left(\epsilon_{a_l bc} X^b \Sigma_A^c\right) \right\} + O(2) \ , \tag{1.7}$$

where STF_L denotes the symmetric-trace-free projection with respect to the multispatial index $L = a_1,\ldots,a_l$, and where $X^{L-1} \equiv X^{a_1}X^{a_2}\cdots X^{a_{l-1}}$. [Note that, because of the properties of the STF projection, the term $X^{L-1}X^b$ in Eq. (1.7) can equivalently be replaced by $\widehat{X}^{L-1}X^b$ or even by $\widehat{X}^{L-1b}$ where $\widehat{X}^K \equiv X^{<K>} \equiv \mathrm{STF}_K(X^K)$.] The basic reason for this is that the spin moments enter the spacetime metric only at the post-Newtonian level (i.e., they always appear multiplied by a factor c^{-2}). As a consequence the right-hand sides of Eqs. (1.3) and (1.6) can all be computed with post-Newtonian accuracy [i.e., modulo $O(4) \equiv O(c^{-4})$] from a knowledge of the spin moments at Newtonian accuracy only. The only exception to this rule arises for the *left-hand side* of Eq. (1.6) in which we need a 1PN-accurate definition of the spin dipole. To find such an improved definition is precisely the purpose of the present paper.

Before tackling the technical details of our 1PN-accurate theory of spin evolution a few historical and methodological remarks seem in order (see [1, 2] for a more complete discussion and further references). The problem of the general relativistic gravitational interaction of N spinning bodies has a long and checkered history full of confusion and errors. The confusion stemmed partly from the lack of a comprehensive treatment compatible both with the formal results for "test particles" (notably by Mathisson [5] and Papapetrou [6]) and the explicit post-Newtonian results for extended objects (notably by Fock [4]). In particular, the straightforward post-Newtonian approach of Fock [4], which used only one global chart to treat the motion of N bodies, led him to derive a Lagrangian for the motion of spinning bodies which predicted a complicated coupling between the translational and the rotational motions of the bodies. (A coupling whose consequences were further analyzed by Brumberg [7].) However, most of this coupling concerned only spurious "coordinate effects." On the other hand, more formal methods (δ-function techniques [8] or quantum-spin calculations [9]) succeeded in deriving the correct general-relativistic spin-orbit coupling, at the price, however, of serious ambiguities in the physical meaning of the result. These ambiguities were two sided. On the one hand, the meanings of the spin vector and of the central world line of each spinning body, as seen in the global frame used to represent the motion of the N-body system, were unclear. On the other hand, a further heuristic step was needed to deduce from the spin-orbit interaction term in the Lagrangian an evolution equation for the spin vector of each body. These ambiguities were clarified, in the case of the motion of strongly self-gravitating spinning bodies, by methods based on the matching between expansions for the gravitational field in local frames attached to each body and a usual-type weak-field expansion for the field in the global frame [10, 11]. (Note in particular that the acceleration-dependent Lagrangian describing the spin-orbit interaction when using the Minkowski-covariant spin condition to fix the central world line was not derived until 1982 [12].) The present work shares with the matched-expansions approaches the use of several coordinate frames, but has the further advantage of being fully explicit (the structure of the gravitational field in all frames is known in complete detail, and so are its transformation properties from one frame to another). As a consequence, we feel that the results presented below (taken in conjunction with our previous results [1, 2]) are the first ones to bring to light all aspects of the motion of weakly self-gravitating spinning bodies, especially the interdependence between the rotational and translational motions. In particular, the fact that our formalism is flexible enough to allow for an arbitrary $O(v^2/c^2)$ slow spatial rotation of the local coordinate grids will enable us to describe from different, complementary, points of view the mixing of torqued and "geodetic" ("de Sitter") contributions in the rotational evolution of a spinning body (or spinning system of bodies).

The organization of the present paper is as follows. Section II starts from a preliminary definition of a 1PN-accurate spin vector for body A, and computes its derivative with respect to the local time T_A. The result is not satisfactory because it contains new moments, beyond the basic ones of our formalism: M_L^A, S_L^A, G_L^A, H_L^A. In Sec. III we show how the introduction of certain radial operators allow one to define "improved" 1PN spin vectors whose local-time evolution is entirely determined by the "good" moments, M, S, G, H. Section IV discusses the physical content of the 1PN rotational laws of motion derived in Sec. III. In particular, we discuss the "gravitational Larmor theorem" (de Sitter–Fokker–Eddington precession), the link with the notion of Fermi-Walker transport, and we give the fully explicit expression of the leading v^2/c^2 terms in the torqued precession.

II. PRELIMINARY FORM OF THE 1PN ROTATIONAL EQUATIONS OF MOTION

As mentioned in requirement (ii) of Sec. I, the looked-for definition of the spin vector of a body A, a member of a gravitationally interacting N-body system, say $S_a^{A(1\mathrm{PN})}(T_A)$, should reduce to the usual 1PN-accurate (conserved) total spin of an isolated system, say $S_i^{\text{isolated (1PN)}}$, when body A becomes infinitely separated from the other bodies $B \neq A$ of the considered N-body system. Several expressions exist for $S_i^{\text{isolated (1PN)}}$ [13, 4, 14, 15]. In particular Landau and Lifshitz [13] proposed an expression of the type

$$S_i^{\text{isolated (1PN)}} = \epsilon_{ijk} \int d^3x\, x^j \tau^{0k}/c \,, \tag{2.1}$$

where $\tau^{\mu\nu} = g(T^{\mu\nu} + t_{LL}^{\mu\nu})$ denotes an effective total stress-energy tensor, including the effect of the stress energy of the gravitational field itself. Note that Eq. (2.1) expresses the total spin as an integral over the whole space because of the contribution of the momentum of the gravitational field. Fock [4] succeeded in deriving, at the 1PN level, another expression for $S_i^{\text{isolated (1PN)}}$ containing only compact-support integrals: namely,

$$S_i^{\text{isolated (1PN)}} = \epsilon_{ijk} \int d^3x\, x^j \left[\sigma^k \left(1 + \frac{4}{c^2} U\right) - \frac{\sigma}{c^2}\left(4U^k - \frac{1}{2} Q^k\right)\right] , \tag{2.2a}$$

where U is the Newtonian potential generated by the mass density $\sigma \equiv (T^{00} + T^{ss})/c^2$, U^k the Newtonian potential generated by $\sigma^k \equiv T^{0k}/c$, and

$$Q^k(x) \equiv G \int d^3y\, \sigma^l(y)\, \frac{\delta^{kl} - n_{xy}^k n_{xy}^l}{|\mathbf{x} - \mathbf{y}|} \,, \tag{2.2b}$$

with $n_{xy}^k \equiv (x^k - y^k)/|\mathbf{x} - \mathbf{y}|$. In Ref. [15] it was shown how to construct several other compact-support expressions for $S_i^{\text{isolated (1PN)}}$. It will be important to remember in the following that all the expressions for the total spin contain, in one form or another, explicit nonlocal contributions due to the gravitational field (these nonlocalities get worse for the higher-order spin moments, see Ref. [15]). These field contributions introduce an irreducible element of nonlocalizability in the total spin which is the first indication that it is highly nontrivial to find a "good" definition of the "individual spin" to be attributed to one body, a member of a gravitationally interacting system. (By contrast, the work of Ref. [16] had succeeded in expressing the 1PN-accurate mass multipole moments of a gravitationally interacting system in an ultralocal form, free of any field contributions, thereby allowing us to introduce [1] a natural definition of the local [Blanchet-Damour (BD)] mass moments of one body, a member of an N-body system.)

The above expressions for the total spin were expressed in a global coordinate system, covering the entire gravitationally interacting system. Let us consider now the case of a system made of N, spatially disjoint, bodies. We wish to define the spin vector of one body A, a member of such an N-body system (note that our formalism allows for "the body A" to be itself made of several, gravitationally interacting, components; e.g., "body A" could be the Earth-Moon system). In keeping with the approach laid down in our previous papers, we look for a definition expressed in the local reference system X^α_A associated with body A, say

$$S_a^{A(1\mathrm{PN})}(T_A) = \mathcal{S}_a\left[\Sigma^\alpha_A(X^\beta_A),\, W^\alpha_A(X^\beta_A)\right] . \tag{2.3}$$

In view of the nonlocalizability of the spin vector, we cannot guess beforehand what is the "good" definition of $S_a^{A(1\mathrm{PN})}$ (i.e., the one which will satisfy the requirements listed in Sec. I). We shall therefore start by working with a preliminary definition that we complement later by additional terms. Inspiring ourselves from the definition (2.2), let us define a quantity S_a^{+A} ("self-part" of the spin) in the local A system by

$$S_a^{+A}(T_A) \equiv \epsilon_{abc}\int_A d^3X\, X^b\left[\Sigma^c\left(1+\frac{4}{c^2}W\right) - \frac{\Sigma}{c^2}\left(4W^{+c}+\frac{1}{2}\partial_c\partial_T Z^+\right)\right] , \tag{2.4}$$

where

$$W^{+c}(T,\mathbf{X}) \equiv G\int_A \Sigma^c(T,\mathbf{X}')|\mathbf{X}-\mathbf{X}'|^{-1}d^3X' , \tag{2.5a}$$

$$Z^+(T,\mathbf{X}) \equiv +G\int_A \Sigma(T,\mathbf{X}')|\mathbf{X}-\mathbf{X}'|d^3X' . \tag{2.5b}$$

In keeping with the notation of our previous papers the superscript plus refers to the self, or locally generated, part of a nonlocal quantity. Note, however, that the scalar potential W appearing in Eq. (2.4) denotes the complete potential in the local frame, $W = W_A^+ + \overline{W}_A$ in the notation of papers I and II. At this stage, this seems to be an arbitrary choice. However, its usefulness appears as soon as we take the time derivative of Eq. (2.4). Indeed, using the local 1PN evolution equations of the matter, Eqs. (5.6) of paper I (in which it is the full W which enters), one finds

$$\frac{dS_a^{+A}}{dT_A} = D_a^A , \tag{2.6a}$$

with a 1PN local torque D_a^A given by

$$D_a^A = \epsilon_{abc}\int_A d^3X\, X^b \times\left[\mathcal{F}^c - \frac{\partial}{\partial T}\left\{\frac{\Sigma}{c^2}\left(4W^{+c}+\frac{1}{2}\partial_c\partial_T Z^+\right)\right\}\right] . \tag{2.6b}$$

The "force density" $\mathcal{F}^c$ appearing on the right-hand side of Eq. (2.6b) is the one introduced in Eqs. (5.5) of paper I or (4.3) of paper II, namely, $\mathcal{F}^a = \Sigma E^a + B_{ab}\Sigma^b/c^2 = (\Sigma\mathbf{E} + \mathbf{\Sigma}\times\mathbf{B}/c^2)^a$. Both the local gravitoelectric field E^a, and the local gravitomagnetic one, $B^a = \frac{1}{2}\epsilon^{abc}B_{bc}$, admit a natural decomposition into self + external parts, $\mathbf{E} = \mathbf{E}^+ + \overline{\mathbf{E}}$, $\mathbf{B} = \mathbf{B}^+ + \overline{\mathbf{B}}$. As the force density is linear in $\mathbf{E}$ and $\mathbf{B}$ we can correspondingly decompose it into self + external parts, $\mathcal{F}^a = \mathcal{F}^{+a} + \overline{\mathcal{F}}^a$. Inserting the latter decomposition in Eq. (2.6b) yields a natural split of the torque:

$$D_a^A = D_a^{+A} + \overline{D}_a^A , \tag{2.7a}$$

$$D_a^{+A} \equiv \epsilon_{abc}\int_A d^3X\, X^b\left[\mathcal{F}^{+c} - \frac{\partial}{\partial T}\left\{\frac{\Sigma}{c^2}\left(4W^{+c}+\frac{1}{2}\partial_c\partial_T Z^+\right)\right\}\right] , \tag{2.7b}$$

$$\overline{D}_a^A \equiv \epsilon_{abc}\int_A d^3X\, X^b\overline{\mathcal{F}}^c . \tag{2.7c}$$

A straightforward calculation, using the explicit expression for W^+ given in Eq. (4.10) of paper II, allows one to prove the following lemma.

Lemma 1. The self-part of the first post-Newtonian local torque, defined by Eq. (2.7b), vanishes:

$$D_a^{+A} = O(4) . \tag{2.8}$$

We are then left with the problem of evaluating Eq. (2.7c), i.e., the usual torque exerted by the external force $\overline{\mathcal{F}}^c$. Using the tidal expansions (6.23) of paper I of the external gravitoelectric and gravitomagnetic fields, the external force density can be written in terms of the two sets of relativistic tidal moments, (G_L, H_L), introduced in Eq. (6.13) of paper I:

$$\overline{\mathcal{F}}^b = \sum_{l\geq 0}\frac{1}{l!}\left(\widehat{X}^L\Sigma\, G_{bL} + \frac{1}{2(2l+3)c^2}\mathbf{X}^2\widehat{X}^L\Sigma\, G^{(2)}_{bL} - \frac{7l-4}{(2l+1)c^2}\widehat{X}^{bL-1}\Sigma\, G^{(2)}_{L-1} + \frac{l}{(l+1)c^2}\epsilon_{bcd}\widehat{X}^{cL-1}\Sigma\, H^{(1)}_{dL-1} + \frac{1}{c^2}\epsilon_{bcd}\Sigma^c\widehat{X}^L H_{dL} - \frac{4l}{c^2(l+1)}\epsilon_{bcd}\epsilon_{def}\widehat{X}^{eL-1}\Sigma^c G^{(1)}_{fL-1}\right) , \tag{2.9}$$

with $H_L^{(1)} \equiv (d/dT)H_L$, etc.

Let us now introduce, in addition to

$$\widehat{N}_L \equiv \int_A d^3X\, \mathbf{X}^2 \widehat{X}^L \Sigma \ , \tag{2.10a}$$

$$\widehat{P}_L \equiv \int_A d^3X\, \widehat{X}^{aL} \Sigma^a \ , \tag{2.10b}$$

already introduced in paper II, the auxiliary quantity

$$\widehat{Q}_L \equiv \int_A d^3X\, \mathbf{X}^2 \widehat{X}^{\langle L-1}\Sigma^{a_l\rangle} \ . \tag{2.11}$$

Using some of the formulas involving symmetric and trace-free (STF) tensors presented in papers I and II, one can check that the quantities (2.10) and (2.11) are related as follows:

$$\widehat{N}_L^{(1)} = 2\widehat{P}_L + \frac{l(2l+3)}{(2l+1)}\widehat{Q}_L \ . \tag{2.12}$$

Then, using Eq. (2.12) and Eqs. (4.23)–(4.27) of paper II, the external torque $\overline{D}_c^A$ can be written as

$$\overline{D}_a^A = \frac{dS_a^{+A}}{dT} = \overline{D}_a^{(MSGH)} + \overline{D}'_a \ , \tag{2.13}$$

with

$$\overline{D}_a^{(MSGH)} = \epsilon_{abc}\sum_{l\geq 0}\frac{1}{l!}\left(M_{bL}G_{cL} + \frac{1}{c^2}\frac{l+1}{l+2}S_{bL}H_{cL}\right) + \frac{1}{c^2}\sum_{l\geq 0}\frac{1}{l!}\frac{l}{l+1}\frac{d}{dT}(M_{aL}H_L) \tag{2.14}$$

and

$$\overline{D}'_a = -\frac{d}{dT}C_a\left[MSGH;P,N\right] \ , \tag{2.15}$$

with

$$\begin{aligned} C_a = {} & \frac{1}{c^2}\sum_l \frac{1}{l!}\frac{1}{2l+3}H_{aL}N_L \\ & -\frac{1}{c^2}\epsilon_{abc}\sum_l \frac{1}{l!}\frac{4(2l+3)}{(l+2)(2l+5)}P_{bL}G_{cL} \\ & -\frac{1}{c^2}\epsilon_{abc}\sum_l \frac{1}{l!}\frac{(l+6)}{2(l+2)(2l+5)}(N_{bL}G_{cL}^{(1)} - N_{bL}^{(1)}G_{cL}) \\ & -\frac{1}{c^2}\epsilon_{abc}\sum_l \frac{1}{l!}\frac{2}{(l+2)(2l+5)}\frac{d}{dT}(N_{bL}G_{cL}) \ . \end{aligned} \tag{2.16}$$

Note that the quantity $\widehat{Q}_L$ has been eliminated from the results (2.13)–(2.16).

The result (2.13) says that the external torque $\overline{D}_a^A$ can be decomposed into two parts: $\overline{D}_a^{(MSGH)}$ and $\overline{D}'_a$. The first part $\overline{D}_a^{(MSGH)}$ is entirely expressed as a bilinear form in the relativistic multipole (M_L, S_L) and tidal (G_L, H_L) moments introduced in our papers, and their local-time derivatives. By contrast, the remaining part $\overline{D}'_a$ also depends upon the auxiliary quantities P_L and N_L, although only through a total time derivative.

Equations (2.13)–(2.16) constitute the preliminary form of the 1PN rotational equations of motion, corresponding to the preliminary definition (2.4) of the individual, local spin vector of body A. As already announced in theorem 7 of paper I there is, however, a way to define an improved local spin vector at the 1PN approximation such that the corresponding torque is entirely given as a bilinear form in our 1PN moments $(M_L, S_L; G_L, H_L)$. As we shall see below, there are even different possible definitions satisfying this requirement.

III. FINAL FORM OF THE 1PN ROTATIONAL EQUATIONS OF MOTION

We are going to prove that each of the four contributions $C_a^{[j]}$, $j = 1-4$, to C_a in Eq. (2.16) can be written in the form

$$C_a^{[j]} = \overline{S}_a^{[j]} + F_a^{[j]}[MSGH] \ , \tag{3.1}$$

where $F_a^{[j]}$ is a bilinear form in the "good" moments $(M_L, S_L; G_L, H_L)$ and their time derivatives, while $\overline{S}_a^{[j]}$ is a compact-support integral of the form

$$\overline{S}_a^{A[j]} = \int_A d^3X\, X^L \Sigma^\alpha(X)\Phi_{aL\alpha}^{[j]}[\overline{E},\overline{B}] \ , \tag{3.2}$$

i.e., a (non-STF) moment over the matter distribution of body A, $(\Sigma_A^\alpha) = (\Sigma_A, \Sigma_A^a)$, of some linear functional $\Phi^{[j]}$ of the external gravitoelectric and gravitomagnetic field vectors $\overline{E}_a[\overline{W}]$ and $\overline{B}_a[\overline{W}]$. The new feature of the linear functionals $\Phi[\overline{E},\overline{B}]$ is, as we are going to see, that they are "radially" nonlocal in $\overline{E}$ and $\overline{B}$, in the sense that the value of Φ at the spatial point $\mathbf{X}$ depends on the values of $\overline{E}$ and $\overline{B}$ at all the points of the segment connecting the origin $\mathbf{X} = 0$ to the field point $\mathbf{X}$.

To derive the form indicated in Eq. (3.1) we will introduce certain radial integral operators. Let $\phi(T,\mathbf{X})$ be some field in the local frame. We then define the operator R_α ("radial operator of order α") acting on ϕ by

$$R_\alpha[\phi](T,\mathbf{X}) \equiv \int_0^1 d\lambda\, \lambda^\alpha \phi(T,\lambda\mathbf{X}) \ . \tag{3.3}$$

We will also use the shorthand notation $R_{\alpha,\beta}$ for the composition $R_\alpha \circ R_\beta$, i.e.,

$$R_{\alpha,\beta}[\phi](T,\mathbf{X}) \equiv R_\alpha[R_\beta[\phi]](T,\mathbf{X}) \ , \tag{3.4}$$

where the order among (α,β) turns out not to be important because $R_\alpha \circ R_\beta = R_\beta \circ R_\alpha$.

The integral operators R_α have the properties[1]

$$\text{(i)}\quad \frac{\partial}{\partial T}R_\alpha[\phi] = R_\alpha\left[\frac{\partial\phi}{\partial T}\right] \ , \tag{3.5a}$$

$$\text{(ii)}\quad \frac{\partial}{\partial X^a}R_\alpha[\phi] = R_{\alpha+1}\left[\frac{\partial\phi}{\partial X^a}\right] \ , \tag{3.5b}$$

[1] One could have more generally introduced the class of radial operators defined by replacing the power λ^α by an arbitrary function $f(\lambda)$ (general radial average) (and the origin $\mathbf{X} = \mathbf{0}$ by any fixed point $\mathbf{X}_0$). In that language, $R_{\alpha,\beta}$ corresponds to $f(\lambda) = (\lambda^\alpha - \lambda^\beta)/(\beta - \alpha)$.

(iii) $$R_\alpha[R_\beta[\phi]] = R_\beta[R_\alpha[\phi]] = \frac{1}{\beta-\alpha}\left(R_\alpha[\phi] - R_\beta[\phi]\right) , \tag{3.5c}$$

(iv) $$R_\alpha\left[\sum_{l\geq 0} X^L S_L(T)\right] = \sum_{l\geq 0}\frac{X^L S_L}{(\alpha+l+1)} , \tag{3.5d}$$

(v) $$R_{\alpha,\beta}\left[\sum_{l\geq 0} X^L S_L(T)\right] = \sum_{l\geq 0}\frac{X^L S_L}{(\alpha+l+1)(\beta+l+1)} . \tag{3.5e}$$

Let us give a simple example of the use of the R_α operators in transforming the various terms in Eq. (2.16). Let us consider

$$\overline{E}_c = \sum_{l\geq 0}\frac{1}{l!}\hat{X}^L G_{cL}(T) + O(2)$$

[see Eq. (4.16a) of paper II]. Then from Eq. (3.5e) above we get

$$R_{1,3/2}[\overline{E}_c] = \sum_{l\geq 0}\frac{1}{l!}\frac{2}{(l+2)(2l+5)}\hat{X}^L G_{cL}(T) + O(2) ,$$

and therefore

$$\epsilon_{abc}\int_A d^3X\,\mathbf{X}^2 X^b \Sigma R_{1,3/2}[\overline{E}_c] = \epsilon_{abc}\sum_{l\geq 0}\frac{1}{l!}\frac{2}{(l+2)(2l+5)}G_{cL}(T)\int_A d^3X\,\mathbf{X}^2 X^b\Sigma X^L + O(2)$$

$$= \epsilon_{abc}\sum_{l\geq 0}\frac{1}{l!}\frac{2}{(l+2)(2l+5)}N_{bL}G_{cL} + O(2) .$$

We recognize on the last right-hand side a series that appears in the last term of Eq. (2.16). This allows us to write $C_a^{[4]}$ in the form (3.1) with a $\overline{S}_a^{[4]}$ piece of the form (3.2) with $\Phi^{[4]}[\overline{E}] = R_{1,3/2}[\overline{E}]$, and $\overline{F}_a^{[4]} = 0$.

Proceeding along the lines we just exemplified, we derive the following reduction formulas for the various terms in Eq. (2.16):

$$C_a^{[1]} \equiv \frac{1}{c^2}\sum_l \frac{1}{l!}\frac{1}{2l+3}H_{aL}N_L = \overline{S}_a^{[1]} + F_a^{[1]} , \tag{3.6a}$$

$$C_a^{[2]} \equiv -\frac{1}{c^2}\epsilon_{abc}\sum_l\frac{1}{l!}\frac{4(2l+3)}{(l+2)(2l+5)}P_{bL}G_{cL} = \overline{S}_a^{[2]} , \tag{3.6b}$$

$$C_a^{[3]} \equiv -\frac{1}{c^2}\epsilon_{abc}\sum_l\frac{1}{l!}\frac{l+6}{2(l+2)(2l+5)}\left(N_{bL}G_{cL}^{(1)} - N_{bL}^{(1)}G_{cL}\right) = \overline{S}_a^{[3]} , \tag{3.6c}$$

$$C_a^{[4]} \equiv -\frac{1}{c^2}\epsilon_{abc}\sum_l\frac{1}{l!}\frac{2}{(l+2)(2l+5)}\frac{d}{dT}(N_{bL}G_{cL}) = \overline{S}_a^{[4]} . \tag{3.6d}$$

In Eqs. (3.6) we have denoted

$$F_a^{[1]}[MSGH] = -\frac{1}{c^2}\sum_{l\geq 1}\frac{1}{l!}M_{aL}H_L \tag{3.7}$$

(and $F_a^{[j]} \equiv 0$ for $j = 2,3,4$), and

$$\overline{S}_a^{[1]} = \frac{1}{c^2}\int_A d^3X\,\Sigma X^a X^b R_0[\overline{B}_b] , \tag{3.8a}$$

$$\overline{S}_a^{[2]} = +\frac{2}{c^2}\epsilon_{abc}\int_A d^3X\,\mathbf{X}^2[X^b\partial_T\Sigma(R_1[\overline{E}_c] - 2R_{3/2}[\overline{E}_c]) + \Sigma^b R_1[\overline{E}_c] + X^b\Sigma^d\partial_d R_1[\overline{E}_c]] , \tag{3.8b}$$

$$\overline{S}_a^{[3]} = -\frac{1}{c^2}\epsilon_{abc}\int_A d^3X\,\mathbf{X}^2 X^b\left[\Sigma\left(2R_1[\partial_T\overline{E}_c] - \frac{7}{4}R_{3/2}[\partial_T\overline{E}_c]\right) - \partial_T\Sigma\left(2R_1[\overline{E}_c] - \frac{7}{4}R_{3/2}[\overline{E}_c]\right)\right] , \tag{3.8c}$$

$$\overline{S}_a^{[4]} = -\frac{1}{c^2}\epsilon_{abc}\frac{d}{dT}\int_A d^3X\,\mathbf{X}^2 X^b\Sigma R_{1,3/2}[\overline{E}_c] . \tag{3.8d}$$

We are now in position to complete the proof of theorem 7 of paper I. Let us associate to each body A, a member of an N-body system, the following 1PN-accurate spin vector, defined in the local reference system of A, (X_A^α):

$$S_a^{A(1\mathrm{PN})} \equiv S_a^{+A} + \overline{S}_a^A \ , \tag{3.9a}$$

where the "self-part" S_a^{+A} is defined by Eq. (2.4) above, while the "external part" is defined by summing the right-hand sides of Eqs. (3.8a)–(3.8d):

$$\overline{S}_a^A \equiv \overline{S}_a^{A[1]} + \overline{S}_a^{A[2]} + \overline{S}_a^{A[3]} + \overline{S}_a^{A[4]} \ . \tag{3.9b}$$

It is easy to check that $S_a^{A(1\mathrm{PN})}$ satisfy all the requirements we listed in the Introduction.

From Eqs. (2.13)–(2.15) and (3.1) above, we see that

$$\frac{dS_a^{A(1\mathrm{PN})}}{dT_A} = \overline{D}_a^{(MSGH)} - \frac{d}{dT_A}\sum_{j=1}^{4} F_a^{[j]}[MSGH] \ . \tag{3.10}$$

Replacing the explicit expressions (2.14) and (3.7), we get our final form of the rotational equations of motion as the following theorem.

Theorem 1. The local-time evolution of the individual, 1PN-accurate spin vector of body A [defined in its local reference system by Eqs. (3.9)] is given by the following bilinear form in the 1PN-accurate multipole (M_L^A, S_L^A) and tidal (G_L^A, H_L^A) moments introduced in paper I:

$$\frac{dS_a^{A(1\mathrm{PN})}}{dT_A} = \sum_l \frac{1}{l!}\left[\epsilon_{abc}M_{bL}^A G_{cL}^A + \frac{1}{c^2}\frac{l+1}{l+2}\epsilon_{abc}S_{bL}^A H_{cL}^A + \frac{1}{c^2}\alpha_l \frac{d}{dT_A}(M_{aL}^A H_L^A)\right] + O(4) \ , \tag{3.11}$$

where

$$\alpha_l = \frac{2l+1}{l+1} \qquad (\text{for } l \geq 1). \tag{3.12}$$

(According to our conventions there is no need to define α_0 because H_L is defined only for $l \geq 1$.) The rationale for introducing special notation for the coefficient of the last term in Eq. (3.11) will be discussed in the next section.

We shall not attempt to relate in detail our results (3.11) and (3.12) with previous investigations on the rotational equations of motion in general relativity. This would be a nontrivial task because of differences in the definitions of the various objects (spin vector, multipole moments, tidal moments) entering them, as well as in the realm of applicability of such equations of motion. Let us only remark that our result (3.11) is structurally similar to the laws of precession derived in Refs. [17–19]. However, it should be noted that our definition of tidal moments differs from the one proposed in Refs. [17, 20], and that our approach is fully constructive in that all quantities and concepts that we use have a clear technical definition. (See Sec. VI C of paper II for a more detailed discussion of the connection with previous work.)

IV. DISCUSSION OF THE 1PN ROTATIONAL EQUATIONS OF MOTION

A. Nonuniqueness of the definition of the 1PN spin

Although we were able to find a definition of a local spin vector satisfying requirements (i)–(iv) of Sec. I, nothing guarantees *a priori* that such a definition is unique. In fact it is not unique, but this lack of uniqueness is of no concern (both theoretically and practically). To show the nonuniqueness it is enough to remark that the term $C_a^{[1]}$, Eq. (3.6a), could also have been transformed to a pure $\overline{S}^{[1]}$ form. Namely,

$$C_a^{[1]} = \overline{S}_a^{[1]\prime} \tag{4.1}$$

with $\overline{F}_a^{[1]\prime} \equiv 0$ and

$$\overline{S}_a^{[1]\prime} = \frac{1}{2c^2}\int_A d^3X\, \mathbf{X}^2 \Sigma R_{1/2}[\overline{B}_a] + \frac{4}{c^2}\epsilon_{abc}\int_A d^3X\, \mathbf{X}^2 X^b \Sigma (R_1[\partial_T \overline{E}_c] - R_{3/2}[\partial_T \overline{E}_c]) \ . \tag{4.2}$$

In that reformulation all the $F_a^{[j]}$'s vanish and we would have been led to defining a different spin vector $S_a^{A\prime(1\mathrm{PN})} = S_a^{+A} + \overline{S}_a^{A\prime}$, with $\overline{S}_a^{A\prime}$ defined by replacing $\overline{S}_a^{A[1]}$ by $\overline{S}_a^{A[1]\prime}$ in Eq. (3.9b). Then, $S^{A(1\mathrm{PN})\prime}$ satisfies

$$\frac{dS_a^{A(1\mathrm{PN})\prime}}{dT} = \overline{D}_a^{(MSGH)} \ , \tag{4.3}$$

where $\overline{D}_a^{(MSGH)}$ was defined in Eq. (2.14). In other words, $S_a^{A(1\mathrm{PN})\prime}$ obeys an evolution equation of the form (3.11) with different values for the α coefficients: namely,

$$\alpha_l' = \frac{l}{l+1} \ . \tag{4.4}$$

More generally, it is clear that, by using some suitable combination of radial operators acting on terms similar to the ones appearing in Eqs. (3.8a) and (4.2), we could define still another spin vector, say $S_a^{A(1\mathrm{PN})\prime\prime}$, obeying the evolution equation (3.11) with

$$\alpha_l'' = 0 \ . \tag{4.5}$$

Working in the other direction [i.e., complicating rather than simplifying Eq. (3.11)], one could also define more complicated spin vectors, satisfying the criteria (i)–(iv) of Sec. I and evolving via an equation containing additional terms with respect to the form (3.11). However, the form of such additional terms is strongly constrained by the criteria (i)–(iv) of Sec. I. Basically they should be of order $1/c^2$, they should add only a total time derivative to Eq. (3.11), and they should be bilinear in the "good" moments $\{M, S; G, H\}$. Dimensional analysis and parity considerations show that such additional terms can only be of the form

$$\sum_l \frac{1}{c^2 l!}\beta_l \frac{d}{dT}(S_{aL}G_L) \ . \tag{4.6}$$

The physical root of the α-β ambiguities in the definition of an "individual" 1PN spin vector for a body member of an N-body system is clearly related to the fact (recalled above) that, at the 1PN level, the total spin of an isolated system contains irreducible contributions coming from the gravitational field binding the system. These field contributions can be written in various forms [see Eqs. (2.1) and (2.2) above, and the various other forms derived in [15]] but nobody succeeded in eliminating them. We see *a posteriori* that it was a lucky accident that, by contrast, the 1PN-accurate mass multipole moments (whose usual presentations also contain field contributions) could be rewritten entirely in terms of the matter variables Σ^α. This simplification very probably holds only at the 1PN level, and gets lost at higher post-Newtonian orders.

It should be noted that the nonuniqueness of the definition of the 1PN spin does not mean that there is any theoretical ambiguity in the framework we have been developing. The important (and nontrivial) result was to show the *existence* of such definitions (as well-defined compact-support integrals in the local frame of the considered body). To each precise definition corresponds a unique, unambiguous evolution equation [e.g., (3.11) with (3.12) for the definition (3.9)]. From the practical point of view, the possibility to define several different individual spin vectors is also of no concern. Indeed, the main usefulness of theorem 1 above is to give one a handle on relativistic effects in the long-term evolution of the spinning motion of bodies members of gravitationally bound systems. As the α terms (and the β terms, if one wishes to consider them) enter only as total time derivatives they do not contribute to the secular evolution of the spin. Therefore, for practical applications it suffices to work with the definition $S_a^{A(1\mathrm{PN})\prime\prime}$ alluded to above satisfying $\alpha_l'' = 0$. [Anyway, one can note that from the numerical point of view the difference between the two spin vectors $\delta S_a^A \equiv S_a^{A(1\mathrm{PN})} - S_a^{A(1\mathrm{PN})\prime\prime}$ will be extremely small in solar-system applications. In the case of the Earth one finds $\delta S^{(1\mathrm{PN})}/S \sim 10^{-20}$ if one uses a Coriolis-effacing geocentric frame (see below).]

B. Effect of the external gravitomagnetic field on the local spin motion ("gravitational Larmor theorem")

Our final result (3.11) for the local-time evolution of the spin of one body member of an N-body system exhibits various types of contributions. If we follow the recommendation of our previous papers of always using mass-centered local frames (vanishing of the BD mass dipole, $M_a^A = 0$) the first type of terms on the right-hand side (RHS) of Eq. (3.11), $\epsilon_{abc}M_{bL}^A G_{cL}^A/l!$, start contributing only for $l = 1$, i.e., for the coupling between the (BD) mass quadrupole of body A and the 1PN-quadrupolar tidal tensor. This term (as well as the corresponding higher-order terms) has a well-known Newtonian counterpart (torque responsible for the precession of equinoxia). However, one should carefully note that our result $\epsilon_{abc}M_{bL}G_{cL}$ differs from any purely Newtonian result in that we have here used some precise relativistic definitions for the moments M_L^A and G_L^A (more about this below). By contrast, the second type of terms on the RHS of Eq. (3.11) have no formal Newtonian counterparts. Let us concentrate on the lowest-order term in the multipole series, i.e., $l = 0$, which couples the spin vector S_a^A to H_a^A. In fact, if we were to truncate all the multipole expansions by keeping only the mass monopoles (M^A, $A = 1 - N$) and the spin dipoles (S_a^A, $A = 1 - N$) ("monopole-dipole model," see paper II) only the latter term would survive:

$$\left(\frac{dS_a^{A(1\mathrm{PN})}}{dT_A}\right)^{\text{monopole-dipole}} = \frac{1}{2c^2}\,\epsilon_{abc}S_b^A H_c^A \ . \tag{4.7}$$

(Note that in the monopole-dipole limit all the α and β ambiguities disappear.) Let us recall that the quantity H_a^A represents just the value, at the origin of the local A frame, of the external gravitomagnetic field $\overline{B}_a^A(X_A^\alpha)$: $H_a^A(T_A) \equiv \overline{B}_a^A(T_A, \mathbf{0})$. Writing Eq. (4.7) in vectorial form $d\mathbf{S}/dT = \mathbf{S}\times\mathbf{H}/2c^2$, we see that the dominant effect of the presence of an external gravitomagnetic field in the local frame of body A is to cause a precession of the spin vector of A, $d\mathbf{S}/dT = \mathbf{\Omega}\times\mathbf{S}$, with angular velocity

$$\mathbf{\Omega}^A_{\text{precession}} = -\frac{1}{2c^2}\,\mathbf{H}^A \ . \tag{4.8}$$

[One should, however, keep in mind that for spinning bodies endowed with quadrupole, or higher, mass moments the result (4.8) is just a small additional contribution to Newtonian order torques.]

The result (4.8) is the gravitational analogue of the "Larmor theorem," i.e., the fact that a system of electric charges with constant e/m ratio, embedded in a uniform external magnetic field $\mathbf{H}$, undergoes a global precession with angular velocity [21]:

$$\mathbf{\Omega}_{\text{Larmor}} = -\frac{e}{2mc^2}\,\mathbf{H} \ . \tag{4.9}$$

The gravitational result (4.8) is simply obtained by taking $e/m = 1$.

In the case where the considered spinning body is the Earth-Moon system and where (in lowest approximation) the gravitomagnetic field is that induced by the motion of the Earth-Moon system in the gravitoelectric field of the Sun,[2] the precession (4.8) reduces to the well-known de Sitter precession [22] (see also Fokker [23] and Eddington [24]).

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[2]The transformation law (4.20b) of paper I shows that in a frame moving with velocity $\mathbf{V}$ with respect to a purely gravitoelectric external field $\mathbf{E}$ there appears an induced gravitomagnetic field $-4\mathbf{V}\times\mathbf{E}$. As we shall see below there is an additional term linked to the global-frame acceleration of the Earth-Moon system.

It is important to keep in mind that the numerical value of the external, central gravitomagnetic field H_a^A depends on the precise choice made for the rotational state of the local A frame. As was discussed in Sec. V D of paper I, there are two natural choices that can be made to fix the latter rotational state. A technically simple choice, which simplifies many of the transformation formulas between the various reference frames entering our approach, is to tie the rotational state of the local A system X_A^α, to that of the global system x^μ by choosing a trivial rotation matrix R_{ia}^A:

$$R_{ia}^A(T_A) = \delta_{ia} \ . \tag{4.10}$$

Another choice would be to efface all relativistic Coriolis effects by choosing a slowly rotating local A frame with respect to which the central, external gravitomagnetic field vanishes:

$$H_a^A(T_A) = 0 \ . \tag{4.11}$$

By definition, in choice (4.11) the gravitational Larmor precession (4.8) will vanish, while it will generically not vanish in choice (4.10).

To see explicitly how the use of a nontrivial rotation matrix $R_{ia}^A(T)$ can be used to annul H_a^A we can use the results of paper I. There, it was shown that H_a^A can be decomposed according to

$$H_a^A = H_a^{A\prime} + H_a^{A\prime\prime} \ , \tag{4.12a}$$

where $H_a^{A\prime}$ is linear in the external gravitoelectric and gravitomagnetic fields recorded in the *global* frame, $\bar{e}_i(x^\mu)$ and $\bar{b}_i(x^\mu)$, respectively, while $H_a^{A\prime\prime}$ comes from the inertial effects in the time-dependent transformation between x^μ and X_A^α. More precisely, Eq. (4.20b) of paper I (taken at $X^a = 0$, and for the external fields) yields

$$H_a^{A\prime} = R_{ia}^A \left[\bar{b}_i^A - 4\epsilon_{ijk} v_j^A \bar{e}_k^A\right]_{\mathbf{X}_A=0} + O(2) \ , \tag{4.12b}$$

while Eq. (6.30c) of paper I reads

$$H_a^{A\prime\prime} = \epsilon_{abc}\left[V_b^A A_c^A + c^2 \frac{dR_{ib}^A}{dT} R_{ic}^A\right] + O(2) \ . \tag{4.12c}$$

In Eqs. (4.12) $dz_i^A/dt \equiv v_i^A \equiv R_{ia}^A V_a^A$ are the (global or local) components of the three-velocity of the origin of the local A frame [which needs to be defined only modulo $O(2)$, $dz_i/dt = dz_i/dT + O(2)$], $d^2 z_i^A/dt^2 \equiv a_i^A \equiv R_{ia}^A A_a^A$ are the components of the acceleration of the local A frame, and

$$\bar{e}_i^A \equiv \partial_i \overline{w}^A + \frac{4}{c^2}\,\partial_t \overline{w}_i^A \ , \tag{4.13a}$$

$$\bar{b}_i^A \equiv -4\,\epsilon_{ijk}\partial_j \overline{w}_k^A \ , \tag{4.13b}$$

are the A-external global-frame gravitoelectric and gravitomagnetic fields, generated by the A-external gravitational potentials in the global frame:

$$\overline{w}_\alpha^A \equiv \sum_{B\neq A} w_\alpha^B \ . \tag{4.14}$$

The orthogonal matrix $R_{ia}^A(T)$ represents a time-dependent rotation connecting the local coordinates to the global ones $[x_i = z_i(T) + R_{ia}(T)X^a + O(2)]$. We can as usual define the global components of the angular velocity of the local frame with respect to the global one by

$$\omega_{ij} \equiv -\frac{dR_{ia}}{dT} R_{ja} \equiv -\omega_{ji} \equiv \epsilon_{ijk}\omega_k \ . \tag{4.15a}$$

The local components of the rotation vector $\boldsymbol{\omega}(\omega_i \equiv R_{ia}\Omega_a$, or $\omega_{ij} = R_{ia}R_{jb}\Omega_{ab})$ can also be written as

$$\Omega_{ab} \equiv +\frac{dR_{ia}}{dT} R_{ib} \equiv -\Omega_{ba} \equiv \epsilon_{abc}\Omega_c \ . \tag{4.15b}$$

Then one can rewrite Eqs. (4.12) in vectorial notation (dropping an overall superscript A),

$$\mathbf{H} = \mathbf{H}_0 + 2c^2\boldsymbol{\omega} \ , \tag{4.16}$$

where $\mathbf{H} = \mathbf{H}' + \mathbf{H}''$ is the total, central gravitomagnetic field in the local frame rotating with the angular velocity $\boldsymbol{\omega}$, Eqs. (4.15), while

$$\mathbf{H}_0 = \mathbf{v}\times\mathbf{a} + \bar{\mathbf{b}} - 4\mathbf{v}\times\bar{\mathbf{e}} \tag{4.17}$$

denotes the value of $\mathbf{H}$ in a local frame which does not rotate with respect to the global one (i.e., such that $dR_{ia}/dT = 0$). Note that the vectorial notation implicitly assumes that vectors belonging to different frames are identified modulo the appropriate transformation of indices using the (instantaneous) value of R_{ia} (in that sense $\boldsymbol{\omega} = \boldsymbol{\Omega}$). Equation (4.16) makes very clear the fact that there is one and only one rotational state of the local frame which can annul $\mathbf{H}$ (Coriolis-effacing frame), namely, that of angular velocity:

$$\boldsymbol{\omega}^{\text{Coriolis effacing}} = -\frac{1}{2c^2}\mathbf{H}_0 \ . \tag{4.18}$$

More explicitly one finds

$$\boldsymbol{\omega}^{\text{Coriolis effacing}} = -\frac{1}{2c^2}\mathbf{v}\times\mathbf{a} + \frac{2}{c^2}\mathbf{v}\times\bar{\mathbf{e}} - \frac{1}{2c^2}\bar{\mathbf{b}} \ , \tag{4.19a}$$

which, after using Eqs. (4.13), with the required accuracy, gives the following evolution equation for R_{ia}:

$$\left[R_{ia}\frac{dR_{ja}}{dT}\right]^{\text{Coriolis effacing}} = -\frac{1}{2c^2}(v_i a_j - v_j a_i) + \frac{2}{c^2}(v_i\partial_j\overline{w} - v_j\partial_i\overline{w}) + \frac{2}{c^2}(\partial_i\overline{w}_j - \partial_j\overline{w}_i) + O(4) \ . \tag{4.19b}$$

In Eqs. (4.17)–(4.19) the external, global-frame gravitoelectric and gravitomagnetic fields must all be evaluated at the center of the local A frame ($X^a = 0$), i.e., on the central world line $x^\mu = z^\mu(T)$.

Equation (4.18) is nicely consistent with Eq. (4.8) above because it shows that, starting for example from a local frame tied to the global one ($\dot{R}_{ia} = 0$), in which there are Coriolis effects and where all (monopole-dipole) gyroscopes precess according to Eq. (4.8) with the universal velocity $\mathbf{\Omega}_{\rm precession} = -\mathbf{H}_0/2c^2$, one can efface all the Coriolis effects by letting the local frame rotate at the same velocity as the gyroscopes ("dragging of inertial frames").

The various contributions to the RHS of Eq. (4.19a) have already appeared in studies of the motion of non-self-gravitating gyroscopes, notably in the work of Schiff [25]. The first term $-\mathbf{v}\times\mathbf{a}/2c^2$ is a special relativistic effect (Thomas precession [26]), the second term (combined with the effect of the first one when the global-frame acceleration $\mathbf{a}$ is of gravitational origin[3]) is essentially the effect found by de Sitter [22], the third one is conventionally named after Lense and Thirring [27] because the latter authors were the first to study some (other) effects linked to nonzero, global-frame gravitomagnetic fields. One should note however that our derivation of (4.19) is the first one valid for self-gravitating extended spinning bodies, members of N-body systems described at the first post-Newtonian approximation. Previous works were generically restricted to the consideration of test spinning bodies, and had to assume that their results could be applied to self-gravitating bodies. In fact, working backwards, and remembering that most treatments start from the assumption that test gyroscopes are "Fermi-Walker transported" in an external gravitational field we can express our result (4.18) in the form of the following theorem.

Theorem 2. Condition (4.11) of effacement of relativistic Coriolis effects is equivalent to the condition that the vectorial basis $\mathbf{e}^A_\alpha(T_A) = [\partial/\partial X^\alpha_A]_{\mathcal{L}_A}$ [Eq. (2.2) of paper I], associated with the local coordinate grid constructed around the central world line $\mathcal{L}_A(X^a_A = 0)$, be Fermi-Walker transported along $\mathcal{L}_A$ with respect to the external metric $d\bar{s}^2_A = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \overline{G}_{\alpha\beta}dX^\alpha dX^\beta$, defined in Eqs. (7.7) of paper I or (6.9) of paper II. [We assume for simplicity that the weak effacement conditions $\overline{W}^A_\alpha(T,\mathbf{0}) = 0$ are enforced so that $\mathbf{e}^A_\alpha$ is orthonormalized with respect to the external metric, see Sec. V E of paper I.]

We recall that the condition of Fermi-Walker (FW) transport applied to an orthonormalized vectorial basis (or "tetrad") $\mathbf{e}_\alpha$ defined along a world line $\mathcal{L}$, with $\mathbf{e}_0$ being tangent to $\mathcal{L}$, is that the three spatial vectors $\mathbf{e}_a$ be as nearly as possible parallel transported along $\mathcal{L}$, i.e.,

$$\overline{\nabla}_{\mathbf{e}_0}\mathbf{e}_a = \lambda_a \mathbf{e}_0 \quad \text{(FW condition)} \tag{4.20}$$

for some coefficient λ_a. [The latter coefficient can be computed from the orthogonality constraints, namely, $\lambda_a = \bar{g}(\mathbf{e}_a, \overline{\nabla}_{\mathbf{e}_0}\mathbf{e}_0)$; λ_a vanishes only when $\mathcal{L}$ is a geodesic with respect to $\bar{g}$.]

On the other hand, the definition of the connection coefficients of the external metric with respect to the coordinate basis $\mathbf{e}_\alpha = \partial/\partial X^\alpha$ reads

$$\overline{\nabla}_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \overline{\Gamma}^\gamma_{\alpha\beta}\,\mathbf{e}_\gamma\ . \tag{4.21}$$

(Here, we have extended $\mathbf{e}_\alpha$ around $\mathcal{L}$. That extension was denoted $\boldsymbol{\epsilon}_\alpha$ in paper I.) Therefore the Fermi-Walker condition (4.20) is equivalent to requiring

$$\left[\overline{\Gamma}^b_{a0}\right]_{\mathcal{L}} = 0 \qquad \text{(FW condition)}. \tag{4.22}$$

(And $\lambda_a = \left[\overline{\Gamma}^0_{a0}\right]_{\mathcal{L}}$.) As $\mathbf{e}_\alpha$ is a coordinate basis, the connection coefficients $\overline{\Gamma}^\gamma_{\alpha\beta}$ are simply the Christoffel symbols of $\overline{G}_{\alpha\beta}(X^\gamma)$, the coefficients of the external metric in the local frame. Therefore

$$\begin{aligned}\left[\overline{\Gamma}^b_{a0}\right]_{\mathcal{L}} &= \frac{1}{2}\left[\partial_0\overline{G}_{ab} + \partial_a\overline{G}_{0b} - \partial_b\overline{G}_{0a}\right]_{\mathcal{L}} \\ &= \frac{1}{2}\left[\overline{B}_{ab}\right]_{\mathcal{L}} \equiv \frac{1}{2}\epsilon_{abc}H_c\ ,\end{aligned} \tag{4.23}$$

where we have used the normalization condition $\left[\overline{G}_{ab}\right]_{\mathcal{L}} = \delta_{ab}$, and the definition of the gravitomagnetic field. The comparison between Eqs. (4.22) and (4.23) concludes the proof of theorem 2. We leave as an exercise for the reader to give an alternative proof of theorem 2 by using the result (3.37b) of paper I.

C. Relativistic contributions to the quasi-Newtonian torque

Let us for simplicity work with the S'' definition of the spin vector, as defined by Eq. (4.5), i.e.,

$$S^{A(1\text{PN})''}_a = S^{A(1\text{PN})}_a - \sum_{l\geq 1}\frac{1}{c^2 l!}\frac{2l+1}{l+1}M^A_{aL}H^A_L\ , \tag{4.24}$$

where some more work would be needed to write the l series in closed form. $S^{A(1\text{PN})''}_a$ obeys the evolution equation (3.11) with α_l replaced by zero. The successive terms in the second l–series on the RHS of (3.11) fall off (as the terms in the first l series) proportionally to the lth power of the (generally small) ratio ϵ_A between the radius of body A and the characteristic scale of variation of the external gravitational field experienced by A. As the first term in the series (Larmor effect; see above) is already very small in most solar-system applications, we expect the following truncated evolution equation to represent a useful approximation in practice:

$$\frac{dS^{A(1\text{PN})''}_a}{dT_A} \simeq \epsilon_{abc}\sum_l \frac{1}{l!}M^A_{bL}G^A_{cL} + \frac{1}{2c^2}\epsilon_{abc}S^A_b H^A_c\ . \tag{4.25}$$

At first sight, it would seem that the only "relativistic" contribution in Eq. (4.25) is the previously discussed "Larmor–de Sitter" term (which can be eliminated by using a suitably rotating local frame). However, one must remember that the Newtonian-looking torque terms $\sim \epsilon_{abc}M_{bL}G_{cL}$ involve many hidden relativistic contributions. Indeed, the mass multipole moments M_L appearing in Eq. (4.25) are the relativistic (BD) moments

[3]In de Sitter's approximate (and heuristic) treatment one assumes $\mathbf{a} \simeq \bar{\mathbf{e}}$. By contrast, note that in our treatment $\mathbf{a}$ is the exact (Newtonian level) global–frame acceleration of the extended body A. It differs from $\bar{\mathbf{e}}$ by all the couplings between the multipole moments of A and the external tidal moments (see paper II).

[Eqs. (6.11) of paper I], while the gravitoelectric tidal moments G_L are those defined with post-Newtonian accuracy by Eq. (6.13) of paper I. As is clear from the results of paper I (and as will be discussed in more detail in a future publication concerning the motion of satellites) the BD moments are directly measurable from satellite and other external geodetic data. Therefore one should in no case try to split M_L in a "Newtonian" plus a "relativistic" contribution, but work only with the total M_L. However, the situation is different for the relativistic tidal moments, and notably for the dominant tidal quadrupole G_{ab} (the $l=0$ term, $\epsilon_{abc}M_bG_c$, vanishes when using, as we recommend, a mass-centered local frame). Indeed, it was shown in paper II that the tidal moments experienced by body A could be expressed in terms of the (BD) multipole moments of the other bodies $B\neq A$. The resulting expressions, Eqs. (5.37) or (A14)–(A16) of paper II, contain many contributions of order $1/c^2$. For applications we shall give here the explicit form of the 1PN-accurate tidal quadrupole moment G^A_{ab} in the approximation where the other bodies $B\neq A$ making up the N-body system are described by pure mass monopoles M^B in their own local frames (if needed, the results of paper II allow one to compute the effect of the higher-order multipole moments M^B_L). Taking the results from papers I and II we have

$$G^A_{ab} = G^{A\prime\prime}_{ab} + \sum_{B\neq A} G^{B/A}_{ab} \ . \tag{4.26}$$

Here

$$G^{A\prime\prime}_{ab} = \frac{3}{c^2} A^A_{\langle a} A^A_{b\rangle} \ , \tag{4.27a}$$

and [see Eq. (A14) of paper II]

$$\begin{aligned} G^{B/A}_{ab} = \mathrm{STF}_{ab}\Bigg\{ & d_{ab}W^{B/A} + \frac{2}{c^2}V^A_b d^A_a(\partial_t W^{B/A}) \\ & -\frac{2}{c^2}A^A_b d^A_a W^{B/A} \\ & + \frac{4}{c^2} d^A_a\left(\frac{\partial W^{B/A}_b}{\partial t} + v^i_A\frac{\partial W^{B/A}_b}{\partial x^i}\right)\Bigg\}_{\mathcal{L}_A} \\ & + O(4) \ , \end{aligned} \tag{4.27b}$$

with

$$\begin{aligned} d^A_a &= e^{Ai}_a\partial_i = \left(1-\frac{\overline{w}_A(\mathbf{z}_A)}{c^2}\right)\left(\delta^{ij}+\frac{1}{2c^2}v^i_A v^j_A\right)R^{Aj}_a\partial_i, \\ d^A_{ab} &= e^{Ai}_a e^{Aj}_b\partial_{ij} \ . \end{aligned} \tag{4.28}$$

Using expressions (A7)–(A10) from paper II for $W^{B/A}$ and $W^{B/A}_a$, a tedious but straightforward calculation leads to

$$\begin{aligned} G^{B/A}_{ab} = R^A_{ia}R^A_{jb}\left(\frac{3GM_B}{r^3_{AB}}\right)\Bigg[& n^{\langle ij\rangle}_{AB} + \frac{1}{c^2}\Bigg\{ n^{\langle ij\rangle}_{AB}\left(2\mathbf{v}^2_{AB} - 2\overline{w}_A(\mathbf{z}_A) - \overline{w}_B(\mathbf{z}_B) - \frac{3}{2}(\mathbf{n}_{AB}\cdot\mathbf{v}_B)^2 - \frac{1}{2}\mathbf{a}_B\cdot\mathbf{r}_{AB}\right) \\ & + a^{\langle i}_A r^{j\rangle}_{AB} + a^{\langle i}_{AB} r^{j\rangle}_{AB} + v^{\langle i}_{AB} v^{j\rangle}_{AB} + (\mathbf{n}_{AB}\cdot\mathbf{v}_B)\left[v^{\langle i}_B n^{j\rangle}_{AB} - (\mathbf{n}_{AB}\cdot\mathbf{v}_B) n^{\langle i}_{AB} n^{j\rangle}_{AB}\right] \\ & - 3(\mathbf{n}_{AB}\cdot\mathbf{v}_{BA}) n^{\langle i}_{AB} v^{j\rangle}_{BA} + \left[(\mathbf{n}_{AB}\cdot\mathbf{v}_A) v^{\langle i}_B n^{j\rangle}_{AB} - (\mathbf{n}_{AB}\cdot\mathbf{v}_B) v^{\langle i}_A n^{j\rangle}_{AB}\right]\Bigg\}\Bigg] \ , \end{aligned} \tag{4.29}$$

in which $\overline{w}_A(\mathbf{z}_A)$ is given with sufficient accuracy by

$$\overline{w}_A(\mathbf{z}_A) = \sum_{C\neq A}\frac{GM_C}{r_{AC}} + O(2) \ .$$

In Eq. (4.29) one has denoted, as usual,

$$\begin{aligned} r_{AB} &= |\mathbf{z}_A - \mathbf{z}_B|, \ \mathbf{n}_{AB} = (\mathbf{z}_A-\mathbf{z}_B)/r_{AB}, \ \mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B \ , \\ \mathbf{r}_{AB} &= \mathbf{z}_A - \mathbf{z}_B, \ \mathbf{a}_{AB} = \mathbf{a}_A - \mathbf{a}_B \ , \end{aligned} \tag{4.30}$$

$\mathbf{z}_A$, $\mathbf{v}_A$, $\mathbf{a}_A$ denoting, respectively, the position, velocity, and acceleration of the (BD) center of mass of body A.

The total quadrupole tidal tensor experienced by body A is obtained by summing Eq. (4.29) over all the $N-1$ other bodies B and by adding the inertial contribution $G^{A\prime\prime}_{ab}$, Eq. (4.27a). If we can approximate the acceleration of body A by neglecting the tidal couplings in its translational equation of motion (see paper II) we can write the inertial contribution as

$$G^{A\prime\prime}_{ab} = \frac{3}{c^2}A^A_{\langle a}A^A_{b\rangle} \simeq R^A_{ia}R^B_{jb}\sum_{B\neq A}\frac{3GM_B}{r^3_{AB}}\left(-a^{\langle i}_A r^{j\rangle}_{AB}\right) \ , \tag{4.31}$$

which precisely cancels the corresponding term in (4.29).

As a check on the result (4.29) we have also derived the PN expression for the tidal quadrupole moment by using relation (3.40) of paper I:

$$\begin{aligned} G_{ab} &= \partial_{\langle a}\overline{E}_{b\rangle}\big|_{X^a=0} \\ &= -K_{\langle ab\rangle} + \frac{3}{c^2}\,\overline{E}_{\langle a}\overline{E}_{b\rangle}\big|_{\mathcal{L}_A} + O(4) \ , \end{aligned} \tag{4.32}$$

where K_{ab} are the following tetrad components of the curvature tensor of the external metric:

$$\begin{aligned} K_{ab} &= \overline{R}_{\alpha\beta\gamma\delta}\, e^\alpha_0 e^\beta_a e^\gamma_0 e^\delta_b\big|_{\mathcal{L}_A} \\ &= -R^i_a R^j_a[\overline{w}_{,ij} + \frac{1}{c^2}\{-2\overline{w}\overline{w}_{,ij} + \delta_{ij}\overline{w}_{,m}\overline{w}_{,m} - 3\overline{w}_{,i}\overline{w}_{,j} + 2\mathbf{v}^2\overline{w}_{,ij} + \delta_{ij}\overline{w}_{,tt} + 4\overline{w}_{(i,j)t} + 2\delta_{ij}\overline{w}_{,tk}v^k + \delta_{ij}\overline{w}_{,k\ell}v^k v^l \\ &\quad -4\overline{w}_{k,ij}v^k + 4\overline{w}_{(i,j)k}v^k - 3v_{(i}\overline{w}_{,j)k}v^k - 2v_{(i}\overline{w}_{,j)t}\}] \ , \end{aligned} \tag{4.33}$$

where the comma denotes a partial derivative and $T_{(ij)} \equiv \frac{1}{2}(T_{ij} + T_{ji})$. An expression equivalent to (4.33) was derived in Ref. [28].

V. CONCLUDING REMARKS

This third paper completes the theoretical foundations of our new formulation of relativistic celestial mechanics. In the first-post-Newtonian approximation of Einstein's theory of gravity our new scheme yields a complete description of the global dynamics of systems of N, arbitrarily composed and shaped, rotating and deformable, bodies ("the external problem" of celestial mechanics), the local gravitational structure of each body ("internal problem"), and the way they fit together ("relativistic theory of reference systems"). Our formalism is built up in a purely constructive way by proving a number of theorems; each concept introduced is well defined and in acordance with Einstein's theory at the 1PN level.

We think that this formalism is well suited for dealing with the various practical aspects of relativistic celestial mechanics and astrometry in the solar system. It should also be very useful in discussing the conceptual problems that arise when trying to define a hierarchy of reference frames in the solar system that can be realized operationally. It is in a position to yield new and improved descriptions of relevant measuring techniques, such as satellite laser ranging (SLR), lunar laser ranging (LLR), GPS, very long baseline interferometry (VLBI) or astrometric techniques, as well as more accurate equations of motion for celestial bodies in the solar system like the planets, Sun, Moon, Earth, or artificial satellites. Finally, it leads to improved theories of gravimetry and gradiometry.

It is clear that for certain applications one might want to extend the scheme in a natural manner. The introduction of a whole hierarchy of frames instead of one global and several independent local ones, including frames attached to the center of mass of composite subsystems (like the Earth-Moon one), topocentric frames, etc., is quite natural. For other applications one might introduce "models" for specifying the mass- and spin-multipole moments of the various bodies as a function of time in their local rest frames. Special applications of our framework will be published in further articles.

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[1] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991), referred to as paper I.
[2] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992), referred to as paper II.
[3] T. Damour, M. Soffel, and C. Xu, in *Relativistic Gravity Research*, Proceedings of the 81st WE-Heraeus Seminar, Bad Honnef, Germany, 1991, edited by J. Ehlers and G. Schäfer (Springer-Verlag, Berlin, 1992), pp. 46–69.
[4] V.A. Fock, *Teoriya Prostranstva Vremeni i Tyagoteniya* (Fizmatgiz, Moscow, 1955); translated into English as *The Theory of Space, Time and Gravitation* (Pergamon, Oxford, 1959).
[5] M. Mathisson, Z. Phys. **67**, 270 (1931); **67**, 826 (1931); Acta Phys. Pol. **6**, 163 (1937).
[6] A. Papapetrou, Proc. R. Soc. London **A209**, 248 (1951).
[7] V.A. Brumberg, *Relativistic Celestial Mechanics* (Nauka, Moscow, 1972).
[8] W. Tulczyjew, Acta Phys. Pol. **18**, 37 (1959).
[9] B.M. Barker and R.F. O'Connell, Phys. Rev. D **12**, 329 (1975).
[10] P.D. D'Eath, Phys. Rev. D **11**, 2183 (1975).
[11] T. Damour, in *Gravitational Radiation*, Proceedings of the Les Houches Summer School, Les Houches, France, 1982, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59–144.
[12] T. Damour, C. R. Acad. Sci. Paris **294**, 1355 (1982).
[13] L.D. Landau and E.M. Lifshitz, *Teoriya Polya* (Nauka, Moscow, 1941).
[14] C.M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1981).
[15] T. Damour and B.R. Iyer, Ann. Inst. Henri Poincaré **54**, 115 (1991).
[16] L. Blanchet and T. Damour, Ann. Inst. Henri Poincaré **50**, 377 (1989).
[17] K.S. Thorne and J.B. Hartle, Phys. Rev. D **31**, 1815 (1985).
[18] X.-H. Zhang, Phys. Rev. D **31**, 3130 (1985).
[19] W.M. Suen, Phys. Rev. D **34**, 3617 (1986).
[20] X.-H. Zhang, Phys. Rev. D **34**, 991 (1986).
[21] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980), pp. 234–235.
[22] W. de Sitter, Mon. Not. R. Astron. Soc. **77**, 155 (1916).
[23] A.D. Fokker, Proc. R. Acad. Amsterdam **23**, 729 (1921).
[24] A.S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1923).
[25] L.I. Schiff, Phys. Rev. Lett. **4**, 215 (1960).
[26] L.H. Thomas, Philos. Mag. **3**, 1 (1927).
[27] H. Thirring, Phys. Z. **19**, 33 (1918); H. Thirring and J. Lense, *ibid.* **19**, 156 (1918).
[28] A.V. Voinov, Celest. Mech. **42**, 293 (1988).