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Analyticity, crossing symmetry, and the limits of chiral perturbation theory

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The chiral Lagrangian for Goldstone-boson scattering is a power-series expansion in numbers of derivatives. Each successive term is suppressed by powers of a scale, Λ_χ, which must be less than of order $4\pi f/\sqrt{N}$ where f is the Goldstone-boson decay constant and N is the number of flavors. The chiral expansion therefore breaks down at or below $4\pi f/\sqrt{N}$. Because of crossing symmetry, some "isospin" channels will deviate from their low-energy behavior well before they approach the scale at which their low-energy amplitudes would violate unitarity. The breakdown of the chiral expansion is associated with the appearance of physical states other than Goldstone bosons. We speculate that, since the bound on Λ_χ falls as N increases, the masses of resonances will decrease relative to f_π at least as fast as $1/\sqrt{N}$ and argue that the estimates of "oblique" corrections from technicolor obtained by scaling from QCD are untrustworthy.

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I. INTRODUCTION

The chiral Lagrangian [1] and [2] is a compact way of calculating the amplitudes for low-energy processes involving Goldstone bosons. Though it was originally developed to describe pions, it has more recently been applied to calculate amplitudes for processes involving the longitudinal components of the W and Z gauge bosons [3–5]. The equivalence theorem [4] ensures that, at energies large compared to M_W, the scattering amplitudes of longitudinally polarized gauge bosons are approximately the same as those of the Goldstone bosons, which would be present in the ungauged theory. In technicolor models [6], frequently there are also additional, approximate Goldstone bosons [7]. For example, the one-family model [8] has 60 pseudo Goldstone bosons, in addition to the three "swallowed" degrees of freedom. In this case, the chiral Lagrangian may also be used to describe low-energy processes involving pseudo Goldstone bosons as well as longitudinal W or Z bosons [9].

Consider a model in which the symmetry-breaking pattern is $SU(N)_L \times SU(N)_R \to SU(N)_V$ ($N=2$ in the simplest technicolor model and $N=8$ in the one-family model). The most general chirally invariant Lagrangian may be written in terms of the field

$$\Sigma = \exp(2i\pi^a T^a/f) , \tag{1.1}$$

where the π^a (which we refer to as "pions") are the Goldstone-boson fields, the T^a are the generators of SU(N), normalized to $\mathrm{tr}\,T^aT^b=\delta^{ab}/2$, and f is the analogue of the pion decay constant. Under a chiral transformation, the field Σ transforms as $\Sigma \to L\Sigma R^\dagger$, with $L \in SU(N)_L$ and $R \in SU(N)_R$. The most general chirally invariant Lagrangian can be written as an expansion in powers of derivatives. There are no nontrivial chirally invariant terms involving no derivatives and only

$$\mathcal{L}_2^{(0)} = \frac{f^2}{4}\,\mathrm{tr}\,\partial^\mu\Sigma^\dagger\partial_\mu\Sigma \tag{1.2}$$

with two derivatives. Additional terms with more derivatives are suppressed by powers of some momentum scale, denoted Λ_χ.

Chiral perturbation theory is an expansion in p^2/Λ_χ^2. The utility of the chiral Lagrangian arises from the fact that at energies less than Λ_χ, the interactions of exact Goldstone bosons are determined by $\mathcal{L}_2^{(0)}$; i.e., they are entirely determined by the symmetry structure of the theory. For this reason, these lowest-order predictions are universal [5].

At energies near or above Λ_χ, however, all terms in the expansion contribute and the chiral Lagrangian becomes effectively useless. Here the amplitudes become model dependent. In general, the expansion will fail at an energy scale associated with the appearance of additional particles or resonances. Therefore, we associate the scale Λ_χ with some kind of new physics. For example, in the case of the one-Higgs-doublet standard model the chiral ex-

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pansion fails at an energy scale of order the Higgs-boson mass. In a weakly coupled theory, these particles will generally be light and the chiral expansion will fail at low energies.

In a strongly interacting theory, such as QCD or technicolor, the resonances will generally be heavy, and the predictions of chiral perturbation theory may be used to fairly high energies. It is important, therefore, to understand how large Λ_χ may be. This question was first addressed by Weinberg [10], who argued that, since the higher-order terms are required as counterterms to loops involving the lowest-order interactions, it was inconsistent to assume that the size of these terms was smaller than that typical of the corresponding loop correction. This leads to the estimate of naive dimensional analysis (NDA) that Λ_χ must be less than or about $4\pi f$.

In this paper, we discuss how the bound on Λ_χ varies with N, i.e., how it depends on the number of Goldstone bosons. By examining the size of corrections to the one-loop effective action, Soldate and Sundrum [11] have argued that Λ_χ is bounded by $4\pi f/\sqrt{N}$. We address this question by examining pion scattering in chiral perturbation theory, and explicitly confirm their results to all orders in chiral perturbation theory. This result does not depend on any expansion in $1/N$. In particular, if N is significantly larger than 2 (e.g., 8 in the one-family model), the chiral Lagrangian ceases to be valid at energies much lower than suggested by naive dimensional analysis.

In realistic cases $\sqrt{N}$ is a number of order one, and the reader may be concerned that, since there may be other factors of order one that cannot be taken into account, one should not take this factor seriously. However, our results show that this factor appears *consistently* to all orders in perturbation theory. Moreover, since we have argued that Λ_χ corresponds to the scale of new physics, this change is extremely important in the context of electroweak symmetry-breaking phenomenology at the Superconducting Super Collider (SSC) and CERN Large Hadron Collider (LHC) [12,9,13]. Naively scaling from QCD, one would conclude that, to within a factor of a few one way or the other, pseudo Goldstone-boson interactions become strong in the one-family model technicolor model at approximately 1 TeV and may barely be accessible at proposed accelerators. These results imply that, to within a factor of a few, pion interactions become strong at an energy of order 500 GeV and are much more likely to be experimentally accessible.

We wish to state how our work is related to that of other authors. Previous papers [1] and [14] have included resonances into the chiral Lagrangian in various ways and computed the effects on low-energy parameters.[1] Our approach is the converse. We do not know what the form of the new physics is, and we make no attempt to modify the chiral Lagrangian in order to accommodate it. Instead, we use the chiral Lagrangian to infer the existence and scale of the new physics. What is new to our work is the exploration of how the scale of new physics depends on N, the number of light flavors.

The plan of this paper is as follows. In Sec. II we consider Goldstone boson scattering. Using crossing invariance we show that all amplitudes may be expressed in terms of two invariant functions. We decompose them into flavor and angular-momentum channels.

In Sec. III, we consider pion scattering from $\mathcal{L}_2$. We find that the $SU(N)_V$-singlet, spin-0 channel would violate unitarity at an energy of order $4\pi f/\sqrt{N}$. As physical amplitudes cannot violate unitarity, we can conclude that the terms of order p^4 and higher must become important and that Λ_χ cannot be larger than of order $4\pi f/\sqrt{N}$. This argument is a generalization of that applied by Lee, Quigg, and Thacker [16] to longitudinal gauge-boson scattering in the standard one-Higgs-doublet model.

In Sec. IV, we compute the corrections to Goldstone-boson scattering at one loop. Confirming Ref. [11], we find that the corrections are larger than naively expected by a factor of N.

In Sec. V, we give a simple argument that this pattern of enhancement by powers of N persists to all orders in chiral perturbation theory. In particular, we show that the contributions to Goldstone-boson scattering of order p^{2k+2} are greater than or of order

$$\frac{p^2}{f^2}\left[\frac{\sqrt{N}p}{4\pi f}\right]^{2k}, \tag{1.3}$$

implying again that Λ_χ is of order $4\pi f/\sqrt{N}$. This implies that *all* terms in the chiral expansion are relevant at energies of order $4\pi f/\sqrt{N}$ and chiral perturbation theory breaks down.

While the singlet channel saturates unitarity at energies of order $4\pi f/\sqrt{N}$, the other channels have amplitudes that are still much less than one. Is it possible that the low-energy predictions from $\mathcal{L}_2$ continue to be valid in these other channels, as in a generalization of the "conservative" model of Chanowitz and Gaillard in [4]? In Sec. VI, we argue that, in general, the chiral Lagrangian will be valid at all energies below the energy scale associated with new physical states; i.e., Λ_χ may always be interpreted as the scale of new physics and is not just a formal artifact of chiral perturbation theory. This new physics will enter in all channels, and hence the low-energy predictions for *all* channels fail. The behavior of the theory above Λ_χ depends on what new physics is present, and one cannot trust arbitrary unitarizations of the low-energy amplitudes (such as summing a subset of the chiral loop diagrams, the K matrix, or the Padé approximants) [17,18].

In QCD, the chiral Lagrangian ceases to be valid at an energy scale of order the mass of the ρ meson. The arguments we make here imply that in a theory with more than two light flavors, at least some of the resonances are lighter than would be expected by scaling from QCD, and that any results for such a theory based on scaling from QCD are suspect. In particular, the estimates of oblique

[1]Inclusion of resonances in nonlinear $O(N)\rightarrow O(N-1)$ models appears in [15].

corrections in the one-family technicolor model [19] are not trustworthy [20].

II. GENERAL PROPERTIES OF PION SCATTERING

Consider the scattering process $\pi^a\pi^b \to \pi^c\pi^d$. The amplitude for such a process may be decomposed into irreducible representations of the unbroken $SU(N)_V$. We may understand the representations contained in adjoint⊗adjoint of SU(N) as follows. Consider the object $(T^a)^i_j(T^b)^k_l$, where the indices a,b are in the adjoint of SU(N), while i,j,k,l are in the fundamental. First, we may trace i with l, and j with k, yielding a singlet. Next, we may form an adjoint: Remove the singlet already constructed, then symmetrize (i,k), antisymmetrize (j,l), and contract i with l. Another adjoint is obtained by symmetrizing both (i,k) and (j,l) before contracting. The next possibility is to antisymmetrize (i,k) and (j,l) and remove all traces. This representation has dimension $(N+1)(N-3)N^2/4$. We may symmetrize (i,k) and (j,l) and remove traces, yielding a representation of dimension $(N-1)(N+3)N^2/4$. Lastly there are two complex representations, conjugates of each other, in which the traces are removed and either (i,k) is symmetric and (j,l) is antisymmetric or vice versa. These complex representations have dimension $(N+2)(N+1)(N-1)(N-2)/4$. We refer to the seven representations as Δ, F, D, Y, X, T, and $\overline{T}$, respectively. The representations Δ, D, Y, and X are symmetric under the exchange of a and b, and therefore only the angular momentum J=even partial waves can contribute. The others are antisymmetric under $a \leftrightarrow b$, and only the odd angular-momentum partial waves contribute. For $N=2$ only Δ, F, and X exist, corresponding to the isospin 0, 1, and 2 channels. If $N=3$, Y does not exist, while the others are, respectively, the **1**, $\mathbf{8}_a$, $\mathbf{8}_s$, **27**, **10**, and $\overline{\mathbf{10}}$.

Next, we construct the most general amplitude for $\pi\pi$ scattering consistent with Bose symmetry, crossing invariance, and $SU(N)_V$ conservation. There are nine invariant tensors with four adjoint indices, corresponding to the nine singlets in (adjoint)^4. A set of nine[2] linearly independent invariants is $\delta^{ab}\delta^{cd}$, $\delta^{ac}\delta^{bd}$, $\delta^{ad}\delta^{bc}$, $d^{abe}d^{cde}$, $d^{ace}d^{bde}$, $d^{ade}d^{bce}$, $d^{abe}f^{cde}$, $d^{ace}f^{bde}$, and $d^{ade}f^{bce}$, where d^{abc} and f^{abc} are defined by

$$f^{abc} = -2i\,\mathrm{tr}[T^a,T^b]T^c \quad \text{and} \quad d^{abc} = 2\,\mathrm{tr}\{T^a,T^b\}T^c\ . \tag{2.1}$$

There can be no part of the amplitude proportional to $d^{abe}f^{cde}$: Because of Bose symmetry and the symmetry of d^{abc} under $a \leftrightarrow b$, the initial state would have to be in an even angular-momentum state, while by the antisymmetry of f^{cde}, the final state would have to be in an odd angular-momentum state. Therefore, the most general amplitude is

$$\begin{aligned} a(s,t,u)^{a,b;c,d} = {} & \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) \\ & + \delta^{ad}\delta^{bc}A(u,t,s) \\ & + d^{abe}d^{cde}B(s,t,u) \\ & + d^{ace}d^{bde}B(t,s,u) \\ & + d^{ade}d^{bce}B(u,t,s)\ , \end{aligned} \tag{2.2}$$

where s, t, and u are the Mandelstam variables and A and B are unknown functions. Bose symmetry also implies that the functions A and B must be symmetric under the exchange of their second and third arguments.

Applying the projection operators defined in Appendix A to the amplitude (2.2), the amplitudes for pion scattering in the various $SU(N)_V$ channels can all be written in terms of A and B:

$$\begin{aligned} a_\Delta(s,t,u) &= (N^2-1)A(s,t,u) + A(t,s,u) + A(u,t,s) + \frac{N^2-4}{N}[B(t,s,u)+B(u,t,s)]\ , \\ a_F(s,t,u) &= A(t,s,u) - A(u,t,s) + \frac{N^2-4}{2N}[B(t,s,u)-B(u,s,t)]\ , \\ a_D(s,t,u) &= A(t,s,u) + A(u,t,s) + \frac{N^2-4}{N}B(s,t,u) + \frac{N^2-12}{2N}[B(t,s,u)+B(u,t,s)]\ , \\ a_Y(s,t,u) &= A(t,s,u) + A(u,t,s) - \frac{N+2}{N}[B(t,s,u)+B(u,t,s)]\ , \\ a_X(s,t,u) &= A(t,s,u) + A(u,t,s) + \frac{N-2}{N}[B(t,s,u)+B(u,t,s)]\ , \\ a_T(s,t,u) &= a_{\overline{T}}(s,t,u) = A(t,s,u) - A(u,t,s) - \frac{2}{N}[B(t,s,u)-B(u,t,s)]\ . \end{aligned} \tag{2.3}$$

[2]There are nine when $N>3$. Using the relations in Appendix A, all the other invariants can be shown to be dependent on these nine. For $N=2$ the d symbols do not exist, and there are only three invariants. For $N=3$ the relation (Burgoyne's identity [21]) $3(d^{abe}d^{cde}+d^{ace}d^{bde}+d^{ade}d^{bce}) = \delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}$ permits us to eliminate one of the invariants.

The partial wave amplitudes are defined by

$$a_{Il}(s)=\frac{1}{64\pi}\int_{-1}^{1} a_I(s,\cos\theta)P_l(\cos\theta)d\cos\theta \ , \qquad (2.4)$$

where P_l is the Legendre polynomial of order l, and I runs over $\Delta,F,D,Y,X,T,\overline{T}$. The functions A and B will be such that all these partial wave amplitudes obey the usual unitarity relations.

III. PION SCATTERING FROM $\mathcal{L}^2$

The chiral symmetries of technicolor are generally only approximate symmetries. In addition to the three absorbed, exact Goldstone bosons, there are often additional pseudo Goldstone bosons. If the mass of a typical pseudo Goldstone boson is m, chiral perturbation theory is also an expansion in m^2/Λ_χ^2; i.e., Λ_χ also sets the scale for the size of corrections due to chiral-symmetry breaking. Chiral-symmetry-breaking interactions induce nonderivative terms in the chiral Lagrangian. For simplicity, and so that the general considerations of the preceding section continue to apply, we consider a chiral-symmetry-breaking interaction, which gives the same mass to all Goldstone bosons. At lowest order in the symmetry breaking the Lagrangian is

$$\mathcal{L}_2=\mathcal{L}_2^{(0)}+\frac{m^2f^2}{4}\mathrm{tr}(\Sigma+\Sigma^\dagger) \ . \qquad (3.1)$$

We will show that in a theory with a large number of pseudo Goldstone bosons, since Λ_χ is smaller than expected, the effects of chiral-symmetry breaking are correspondingly larger than expected. While this particular kind of symmetry breaking cannot occur in a technicolor theory—such a symmetry-breaking term would break the weak gauge symmetry—the calculation will illustrate the enhancement of symmetry-breaking effects in a theory with $N>2$.

Using (3.1) to compute the invariant functions defined in the preceding section, we find $A(s,t,u)=(2/N)(s-m^2)/f^2$, and $B(s,t,u)=(s-m^2)/f^2$. The scattering amplitudes in the various channels from $\mathcal{L}_2$ [17] are then

$$a_{\Delta 0}=\frac{Ns}{32\pi f^2}-\frac{m^2}{16\pi N f^2}, \quad a_{F1}=\frac{Ns}{192\pi f^2}-\frac{Nm^2}{48\pi f^2} \ ,$$
$$a_{D0}=\frac{Ns}{64\pi f^2}-\frac{m^2}{8\pi N}, \quad a_{Y0}=\frac{s}{32\pi f^2}-\frac{m^2}{16\pi f^2} \ , \qquad (3.2)$$
$$a_{X0}=-\frac{s}{32\pi f^2}+\frac{m^2}{16\pi f^2} \ .$$

All other partial wave amplitudes are zero (including a_{T1} and $a_{\overline{T}1}$). Note that $a_{\Delta 0}$, a_{F1}, and a_{D0} are enhanced by a factor of N.

The amplitude $a_{\Delta 0}$ calculated at tree level is real, and (for small m^2) would exceed 1 when $\sqrt{s}>4\pi f/\sqrt{N}$. A physical scattering amplitude must lie on or inside the Argand circle. The point $a=1$ is far outside the Argand circle. At these energies, therefore, loop corrections and higher-order terms in the chiral Lagrangian must make as large a contribution as the two-derivative term, and the calculation using $\mathcal{L}_2$ ceases to be useful. This suggests that Λ_χ is less than or of order $4\pi f/\sqrt{N}$, as was emphasized in [11]. Note that this result holds independent of the largeness of N. No expansion in powers of $1/N$ need be made.

IV. PION SCATTERING AT ORDER p^4

An alternative approach to put a limit on Λ_χ is based on an estimate of the size of loop corrections [10]. Since the theory is not renormalizable, the terms of order p^4 are required as counterterms to loops involving the lowest-order interactions. In calculating the scattering amplitude to order p^4, one must consider tree-level diagrams with interactions coming from operators of fourth order in momenta, and one-loop diagrams using $\mathcal{L}_2$. It is unnatural to assume that the contribution from the former is much larger than the latter, since such a statement could only be true for a particular choice of renormalization scale. Similarly, the two-loop calculation using $\mathcal{L}_2$ will require counterterms of order p^6, etc. In this section we compute the one-loop corrections to Goldstone boson scattering.

The next-to-leading-order Lagrangian is made up of terms containing four derivatives, two derivatives and one power of the symmetry-breaking parameter, or two powers of the symmetry breaking:

$$\begin{aligned}\mathcal{L}_4=&\,l_1\{\mathrm{tr}(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma)\}^2+l_2\{\mathrm{tr}(\partial^\mu\Sigma^\dagger\partial^\nu\Sigma)\}^2+l_3\mathrm{tr}(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma\partial^\nu\Sigma^\dagger\partial_\nu\Sigma)+l_3'\mathrm{tr}(\partial^\mu\Sigma^\dagger\partial^\nu\Sigma\partial_\mu\Sigma^\dagger\partial_\nu\Sigma)\\&+l_4m^2\mathrm{tr}(\Sigma+\Sigma^\dagger)\mathrm{tr}(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma)+l_5m^2\mathrm{tr}\{(\Sigma+\Sigma^\dagger)(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma)\}\\&+l_6m^4\{\mathrm{tr}(\Sigma+\Sigma^\dagger)\}^2+l_7m^4\{\mathrm{tr}(\Sigma-\Sigma^\dagger)\}^2+l_8m^4\mathrm{tr}(\Sigma^2+\Sigma^{\dagger 2}) \ . \end{aligned} \qquad (4.1)$$

All other possible terms vanish by the equations of motion. This notation agrees with that of Gasser and Leutwyler [22]. In their case the term proportional to l_3' is not linearly independent, because they were considering an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ chiral symmetry.

The calculations of the $\pi\pi\rightarrow\pi\pi$ amplitudes at one loop is straightforward, if tedious. The result is

$$A(s,t,u)=\frac{2}{N}\frac{s-m^2}{f^2}$$
$$+\frac{J(s)}{f^4}\left[-\frac{1}{2}s^2+\frac{2}{N^2}m^4\right]+\frac{J(t)}{f^4}\left[-\frac{1}{3}t^2+\frac{5}{3}m^2t-\frac{7}{3}m^4-\frac{1}{6}st+\frac{2}{3}m^2s\right]$$
$$+\frac{J(u)}{f^4}\left[-\frac{1}{3}u^2+\frac{5}{3}m^2u-\frac{7}{3}m^4-\frac{1}{6}su+\frac{2}{3}m^2s\right]$$
$$+\frac{s^2}{f^4}\left[-\frac{2}{3}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}+8l_1(\mu)+4l_2(\mu)+\frac{8}{N}l_3(\mu)+\frac{23}{18}\frac{1}{16\pi^2}\right]$$
$$+\frac{tu}{f^4}\left[\frac{2}{3}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}-8l_2(\mu)-\frac{16}{N}l_3'(\mu)-\frac{13}{9}\frac{1}{16\pi^2}\right]$$
$$+\frac{m^2s}{f^4}\left[\frac{2}{3}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}-32l_1(\mu)-16l_2(\mu)-\frac{32}{N}l_3(\mu)+16l_4(\mu)+\frac{16}{N}l_5(\mu)-\frac{28}{9}\frac{1}{16\pi^2}\right]$$
$$+\frac{m^4}{f^4}\left[\left[-2+\frac{2}{N^2}\right]\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}+32l_1(\mu)+32l_2(\mu)+\frac{32}{N}l_3(\mu)+\frac{32}{N}l_3'(\mu)\right.$$
$$\left.-32l_4(\mu)-\frac{32}{N}l_5(\mu)+32l_6(\mu)+\frac{32}{N}l_8(\mu)+\left[\frac{20}{3}-\frac{4}{N^2}\right]\frac{1}{16\pi^2}\right], \tag{4.2}$$

$$B(s,t,u)=\frac{s-m^2}{f^2}$$
$$+\frac{NJ(s)}{f^4}\left[-\frac{1}{8}s^2+\frac{2}{N^2}m^4\right]+\frac{NJ(t)}{f^4}\left[-\frac{1}{24}t^2+\frac{1}{3}m^2t-\frac{2}{3}m^4-\frac{1}{12}st+\frac{1}{3}m^2s\right]$$
$$+\frac{NJ(u)}{f^4}\left[-\frac{1}{24}u^2+\frac{1}{3}m^2u-\frac{2}{3}m^4-\frac{1}{12}su+\frac{1}{3}m^2s\right]$$
$$+\frac{Ns^2}{f^4}\left[-\frac{1}{12}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}+\frac{4}{N}l_3(\mu)+\frac{5}{36}\frac{1}{16\pi^2}\right]$$
$$+\frac{Ntu}{f^4}\left[+\frac{1}{12}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}-\frac{8}{N}l_3'(\mu)-\frac{2}{9}\frac{1}{16\pi^2}\right]$$
$$+\frac{Nm^2s}{f^4}\left[-\frac{1}{6}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}-\frac{16}{N}l_3(\mu)+\frac{8}{N}l_5(\mu)-\frac{5}{9}\frac{1}{16\pi^2}\right]$$
$$+\frac{Nm^4}{f^4}\left[\frac{2}{N^2}\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}+\frac{16}{N}l_3(\mu)+\frac{16}{N}l_3'(\mu)-\frac{16}{N}l_5(\mu)+\frac{16}{N}l_8(\mu)+\left[\frac{4}{3}-\frac{4}{N^2}\right]\frac{1}{16\pi^2}\right]. \tag{4.3}$$

Here we have used dimensional regularization in $4-\epsilon$ dimensions with a scale μ and, as detailed in Appendix B, the parameters $l_i(\mu)$ are renormalized at this scale.[3] The function J is defined as

$$J(x)=\frac{1}{16\pi^2}\sqrt{1-4m^2/x}\,\ln\frac{\sqrt{1-4m^2/x}+1}{\sqrt{1-4m^2/x}-1}, \tag{4.4}$$

where $\ln(z)$ and $\sqrt{z}$ are both understood to have a cut under the negative real axis. The quantities m and f, which appear in (4.2) and (4.3), are the physical mass and decay constant. Their renormalizations are also given in Appendix B.

From (4.2) and (4.3) we see that the typical corrections to the tree-level results are of order $Ns/(16\pi^2f^2)$ or $Nm^2/(16\pi^2f^2)$. In agreement with the preceding section and Ref. [11], then, it is inconsistent to assume that the corrections to pion scattering of order p^4 are less than or of order $(\sqrt{N}p/4\pi f)^2$, where p is a typical momentum in the process.

Moreover, following Ref. [10], we see that a change in the renormalization scale μ by order one will result in a change in the renormalized parameters $l_i(\mu)$ by order of $N/16\pi^2f^2$. It is therefore inconsistent to assume that the

[3]Gasser and Leutwyler have computed the function $A(s,t,u)$ for the case of $N=2$. Our formula (4.2) agrees with their result. They do not use the modified minimal subtraction ($\overline{\text{MS}}$) prescription. Their subtractions are proportional to $D+1/16\pi^2$ instead of D.

l_i are smaller than this order of magnitude. This implies that the parameters l_i are larger by a factor of N than expected by NDA and that, insofar as $\mathcal{L}_4$ is concerned, it is inconsistent to assume that Λ_χ is much greater than $4\pi f/\sqrt{N}$.

V. PION SCATTERING TO ALL ORDERS

The result that loop corrections are larger by powers of N than expected by naive dimensional analysis persists to all orders in chiral perturbation theory, as we now show. At order p^{2k+2}, there are several contributions to be considered: tree level from interactions of order p^{2k+2} in the chiral Lagrangian, k loops using interactions from $\mathcal{L}_2$, $k-1$ loops using one interaction from $\mathcal{L}_4$ and the rest from $\mathcal{L}_2$, etc.

In order to keep track of powers of N, we may use a double line notation for each Goldstone boson, analogous to the double line notation used by 't Hooft [23] in large-N_C QCD. Each Goldstone boson is a member of the adjoint representation of $\mathrm{SU}(N)_V$, and is analogous to a gluon in the adjoint of the $\mathrm{SU}(N_C)$ color. Analogous to large-N_C QCD, then, the contributions to a process at L loops that have the highest power of N arising from traces of the flavor indices are proportional to $N^L/(16\pi^2)^L$.

First, consider pion scattering to order p^6. There are three types of contributions: two-loop diagrams with vertices from $\mathcal{L}_2$, one-loop diagrams with one vertex from $\mathcal{L}_4$ and any others from $\mathcal{L}_2$, and tree-level contributions from terms in the sixth-order chiral Lagrangian $\mathcal{L}_6$. The argument given above shows that the leading contributions from two-loop diagrams with interactions from $\mathcal{L}_2$ are larger than the estimate of NDA by a factor of N^2. Similarly, the leading contributions from a one-loop diagram with one vertex from $\mathcal{L}_4$ is also enhanced by at least N^2 relative to the expectation of naive dimensional analysis: one power of N from the loop and one power of l_i, which, as shown in the preceding section, is larger than assumed in NDA by a factor at least as large as N. The tree-level contributions are required as counterterms to the loop corrections and must therefore also be at least as large as the other two contributions: the unknown coefficients in $\mathcal{L}_6$, therefore, are also larger by a factor of at least N^2 than expected by NDA.

Similarly, we can show that the corrections of order p^8 are larger than expected by at least a factor of N^3, the corrections of order p^{10} by at least N^4, etc. In general, the contributions of order p^{2k+2} to pion scattering are at least of order

$$\frac{p^2}{f^2}\left[\frac{\sqrt{N}p}{4\pi f}\right]^{2k}, \tag{5.1}$$

where p is a typical momentum in the process. Since *all* higher-momentum corrections are enhanced, chiral perturbation theory as a whole breaks down by energies of order $4\pi f/\sqrt{N}$, since by this scale all terms are equally important.

This is exactly the same argument as (and is exactly as rigorous as) the one given by Weinberg in Ref. [10] to show that all counterterms are suppressed by a factor of $(4\pi)^{2k}$. There is always uncertainty in this answer—there could be a small numerical factor that goes one way or the other. For any fixed N, it is *possible* that the subleading-in-N terms, which arise from nonplanar contractions of the $\mathrm{SU}(N)_V$ group indices, are numerically as important as those that are leading. However, making Λ_χ much bigger than $4\pi f/\sqrt{N}$ would require, at *every* order in chiral perturbation theory, either a cancellation among the leading terms or both an enhancement of the subleading terms by a numerical factor as large as N and a cancellation of these with the leading terms.

VI. IMPLICATIONS OF ANALYTICITY AND CROSSING

Some interesting questions arise at this point. We have argued that chiral perturbation theory breaks down at or before Λ_χ, but what actually happens to the amplitudes as s increases beyond this value? What is the significance of Λ_χ? The amplitudes for the partial waves other than $a_{\Delta 0}$ are all below their unitarity limits when $\sqrt{s}=4\pi f/\sqrt{N}$. Is it possible that these other channels continue to behave like the prediction of the lowest-order chiral Lagrangian, as in the $\mathrm{SU}(N)_V$ generalization of the "conservative model" of Ref. [4]?

The pion-scattering amplitudes are determined by the two functions A and B. We will examine the analytic structure of these functions in the complex s,t hypersurface, where u is determined by the mass shell condition. Let us consider the case of a small $m^2>0$, so as to avoid the subtleties associated with infrared problems. Amplitudes in chiral perturbation theory are expansions in s and t. The S matrix is analytic on this hypersurface except for cuts on the physical (s,t) plane and poles or cuts on the unphysical hypersheets. The cuts in the physical (s,t) plane are due to multipion states, and the appearance of poles or other structure on the unphysical hypersheets correspond to resonances or physical states other than pions. The masses of the new physical states are not protected by a chiral symmetry and the corresponding structures in the S matrix are therefore away from the origin.

Consider the region R with s, t, and u all less than $4m^2$. Here, assuming there are no particles lighter than twice the pion mass, the S matrix is analytic. The functions A and B computed from the chiral Lagrangian to arbitrary order have an infinite set of adjustable coefficients multiplying the terms $s^i t^j$ for all i and j. We can regard this as a convergent expansion about any point in the region R. Therefore, the amplitude computed in chiral perturbation theory can be adjusted to match the S matrix exactly in R. Going outside R, the singularities of the S matrix on the physical plane correspond to cuts from multipion states and are determined by unitarity. They are correctly included in chiral-loop calculations. Moving away from this region, therefore, the chiral Lagrangian calculation should reproduce the S matrix so long as there are not any singularities closer to the origin associated with physical states *other* than multipion states. That is, chiral perturbation theory is a good approximation to the S matrix *all the way* out to an energy scale associated with the appearance of new physics.

In chiral perturbation theory, ignoring pion masses, the arbitrary polynomial in the dispersive part of the scattering amplitude is a function of the form

$$\sum_k a_k \frac{p^2}{f^2}\left[\frac{p^2}{\Lambda_\chi^2}\right]^{k-1} \tag{6.1}$$

where all the a_i are the numbers of order 1. In the preceding section, we gave an all-orders argument that the contributions to pion scattering of order p^{2k+2} are enhanced by powers of N^k relative to the expectation of naive dimensional analysis, and therefore that Λ_χ was bounded by $4\pi f/\sqrt{N}$.

When does such a series fail to converge? Since the a_k are of order one, the radius of convergence is Λ_χ. Because the series diverges at energies higher than Λ_χ, chiral perturbation theory (to any, arbitrarily high, finite order) cannot give good approximation to the scattering amplitude at energies beyond Λ_χ. We have previously argued the chiral Lagrangian can accurately match the scattering amplitude out to the first nonanalytic structure representing new physics. It follows that the mass of the lightest nonanalytic structure in the S matrix corresponding to new physics is lighter than a scale of order $4\pi f/\sqrt{N}$.

Admittedly, the argument given above for the range of validity of the chiral expansion is not truly rigorous. This is because it works only if one computes without expanding in m^2. As emphasized by Pagels and Li [24], the S matrix is not analytic in m^2. However, as motivated by the theory of critical phenomena [25], we assume that the coefficients of the chiral Lagrangian are analytic in m^2 and that the nonanalyticity of the S matrix is reproduced by chiral-loop calculations.

If this standard assumption is correct, then the arguments given above are correct so long as m^2 is small enough, and the chiral expansion is valid for all energies below the first nonanalytic structure in the S matrix corresponding to some real, new physics. The breakdown of the chiral Lagrangian is not a calculation artifact; the scale Λ_χ has direct physical significance, unlike the scale $\Lambda_{\rm QCD}$ in perturbation theory. Moreover, the arguments above show that this physical scale cannot be larger than about $4\pi f/\sqrt{N}$.

In general, there are many possibilities for the new physics at Λ_χ. In QCD with $N=2$, the ρ pole gives a singularity in A when t or u is $m_\rho^2 - i m_\rho \Gamma_\rho$. In the one-doublet-Higgs model, there is a pole in A whenever s is $M_H^2 - i M_H \Gamma_H$. The first singularity to appear could be a branch cut instead of a pole. Consider a world in which the π's are massless, but the K's weigh 200 MeV. (The pion decay constant is still 93 MeV.) Imagine constructing an SU(2)-invariant chiral Lagrangian to describe the massless pions. The first new singular structure one encounters in the S matrix is the two-K branch cut, at 400 MeV, and at this energy the chiral expansion breaks down.

Since the scale Λ_χ is associated with nonuniversal structure, such as the ρ, the Higgs boson, or the KK states in the examples above, the behavior of the scattering amplitudes at energies at or above Λ_χ *cannot* be universal. Therefore, there is no reason to trust predictions based on an arbitrary, e.g., K matrix, Padé approximant [18], or bubble sum [17], unitarization of the universal, lowest-order, chiral amplitudes.

Finally, we address the issue of whether the channels other than $a_{\Delta 0}$ can follow their low-energy predictions beyond energies of order Λ_χ. Consider again the functions A and B. Because of the crossing relations (2.3), all channels will be affected by whatever new physics enters at the scale Λ_χ. In particular, we expect that at Λ_χ the amplitudes in all channels will deviate strongly from the low-energy predictions, even though the low-energy predictions may be well below 1.

Consider pion scattering in QCD. The data[4] and lowest-order predictions for the low isospin channels are plotted in Figs. 1, 2, and 3. The amplitude in the isospin-0 spin-0 channel ($a_{\Delta 0}$) starts to deviate from the lowest-order prediction at an energy of about 600 MeV. This is not surprising because the amplitude is at that point a substantial fraction of its unitarity limit. Examining the other partial waves, we find that they deviate from the lowest-order chiral Lagrangian formulas above approximately *the same energy scale.* For example, in the isospin-1-spin-1 channel (a_{F1}), the ρ resonance appears at 770 MeV. While we cannot predict the appearance of the ρ, the fact that the isospin-1 and -2 amplitudes deviate strongly from their low-energy predictions, while they would still be relatively weakly coupled, is not a surprise.

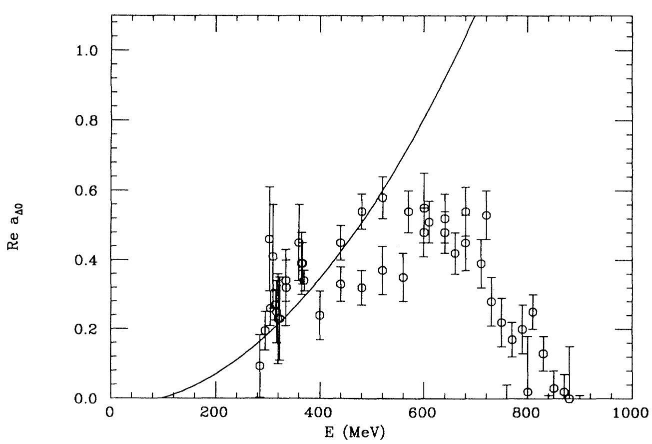

FIG. 1. Data and lowest-order prediction for Re$a_{\Delta 0}$. The data are from a compilation in Ref. [26] of the experimental results in Ref. [28].

[4]We thank G. Valencia for providing us with the data compiled for Ref. [26].

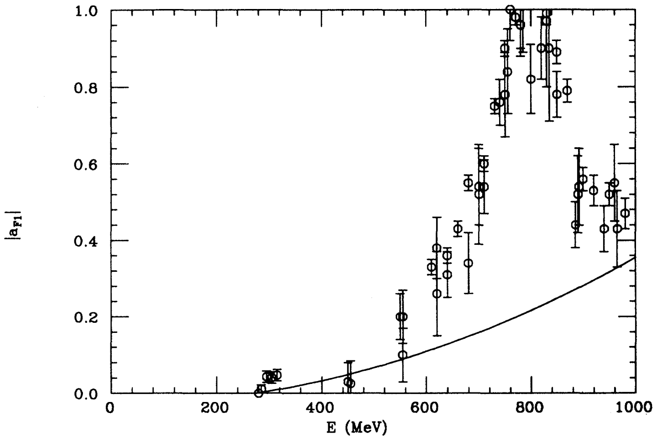

FIG. 2. Data and lowest-order prediction as in Fig. 1, for $|a_{F1}|$.

VII. CONCLUSIONS AND SPECULATIONS

In this paper we have argued that, in general, the chiral expansion will be valid at all energies below the energy scale associated with new physical states. That is, Λ_χ may always be interpreted as the scale of new physics and is not just a formal artifact of chiral perturbation theory. In general, there is no procedure to infer the high-energy structure of the theory from the universal lowest-order chiral Lagrangian. In the three examples given in the preceding section, the high-energy physics is different, while the low-energy behavior is precisely the same. In particular, there is no reason to trust the amplitudes constructed by unitarizing the lowest-order predictions.

In a theory with a spontaneously broken $SU(N)_L \times SU(N)_R$ chiral symmetry, the scale Λ_χ is bounded by $4\pi f/\sqrt{N}$. This result depends only on the low-energy effective theory and is independent of the precise form of the fundamental theory. If the theory is QCD-like, Λ_χ is presumably associated with the appearance of resonances such as the ρ. When $N=2$ and $f_\pi=93$ MeV, the limit on Λ_χ is about 825 MeV, and the ρ is close to saturating this bound.

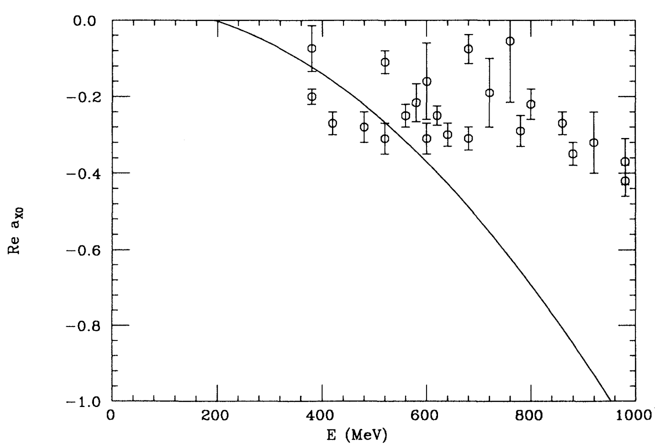

FIG. 3. Data and lowest-order prediction as in Fig. 1, Re$a_{\chi 0}$.

In QCD with two light flavors the new physics associated with Λ_χ is the formation of vector mesons. If this pattern persists for larger N, then we may speculate that the masses of the vector mesons will decrease relative to f_π at least as fast as $1/\sqrt{N}$ as the number of flavors increases. For example, we know that QCD, with three colors, is well defined for the six quarks that actually exist. If they were all light and f_π were fixed at 93 MeV, then if the vector mesons appearing in the channel a_{F1} remained the lightest, the above argument would say that the masses of the analogues of the ρ would have to be less than 500 MeV.

In the one-family technicolor model, $N=8$ and $f \approx 125$ GeV. The low-energy prediction for the singlet spin zero channel would saturate unitarity at an energy of order 550 GeV. Because of the arguments we have given we expect that the new physics in this model has a mass of this order of magnitude. In technicolor phenomenology, however, one usually scales from QCD [27] by multiplying all mass scales by f_{TC}/f_π and applying large N_{TC} arguments [23]. When $N_{\mathrm{TC}}=3$ in the one-family model, this gives a technirho mass of about 1000 GeV. While we cannot say conclusively that there are *resonances* as light as 550 GeV, it is clear that the one-family model is quite different from QCD and that predictions based on scaling from QCD, e.g., [12], cannot be trusted in detail. In particular, if the masses of the resonances are much lighter than naively expected, then the estimates of oblique corrections from technicolor obtained by scaling from QCD [19] are untrustworthy [20].

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APPENDIX A: PROJECTION OPERATORS AND IDENTITIES FOR SU(N)

The following relations may be derived from the SU(N) Fierz identities:

$$
\begin{aligned}
&f^{abe}f^{cde}=d^{ace}d^{bde}-d^{ade}d^{bce}+\frac{2}{N}(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc}) ,\\
&f^{abe}d^{cde}=d^{ade}f^{bce}+d^{ace}f^{bde} ,\\
&d^{aef}d^{bef}=\frac{N^2-4}{N}\delta^{ab} ,\\
&d^{aef}d^{bfg}d^{cge}=\frac{N^2-12}{2N}d^{abc} ,\\
&d^{aef}d^{bfg}d^{cgh}d^{dhe}=\frac{N^2-4}{N^2}(\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc})\\
&\qquad+\frac{N^2-16}{4N}(d^{abe}d^{cde}+d^{ade}d^{bce})\\
&\qquad-\frac{N}{4}d^{ace}d^{bde} .
\end{aligned}
\tag{A1}
$$

The operators that project a state $|\pi^a\pi^b\rangle$ onto the various irreducible representations of $SU(N)_V$ are

$$
\begin{aligned}
&P_\Delta=\frac{1}{N^2-1}\delta^{ab}\delta^{cd} ,\\
&P_F=\frac{1}{N}(d^{ace}d^{bde}-d^{ade}d^{bce})+\frac{2}{N^2}(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc}) ,\\
&P_D=\frac{N}{N^2-4}d^{abe}d^{cde} ,\\
&P_Y=\frac{N-2}{4N}(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})+\frac{N-2}{2N(N-1)}\delta^{ab}\delta^{cd}\\
&\qquad-\frac{1}{4}(d^{ace}d^{bde}+d^{ade}d^{bce})+\frac{N-4}{4(N-2)}d^{abe}d^{cde} ,\\
&P_X=\frac{N+2}{4N}(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})-\frac{N+2}{2N(N+1)}\delta^{ab}\delta^{cd}\\
&\qquad+\frac{1}{4}(d^{ace}d^{bde}+d^{ade}d^{bce})-\frac{N+4}{4(N+2)}d^{abe}d^{cde} ,\\
&P_T=P_{\bar{T}}^{*}=\frac{N^2-4}{4N^2}(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc})\\
&\qquad-\frac{1}{2N}(d^{ace}d^{bde}-d^{ade}d^{bce})\\
&\qquad+\frac{i}{4}(d^{ace}f^{bde}-d^{ade}f^{bce}) .
\end{aligned}
\tag{A2}
$$

Using the identities (A1), one can verify that these are projection operators satisfying

$$P_I^{a,b;c,d}P_K^{c,d;g,h}=\delta_{IK}P_I^{a,b;g,h} , \tag{A3}$$

and

$$\sum_I P_I^{a,b;c,d}=\delta^{ac}\delta^{bd} . \tag{A4}$$

We may write the amplitude in terms of the seven $SU(N)_V$ channels:

$$a(s,t,u)^{a,b;c,d}=\sum_I a_I(s,t,u)P_I^{a,b;c,d} , \tag{A5}$$

where I runs over $\Delta,F,D,Y,X,T,\bar{T}$, yielding Eq. (2.3).

APPENDIX B: RENORMALIZATION OF $\mathcal{L}_4$

Calculating at one loop, there are renormalizations to the pion mass, the decay constant, and the pion wave function. We define

$$
\begin{aligned}
&\pi^a=Z_\pi^{-1/2}\pi_0^a ,\\
&m=Z_m m_0 ,\\
&f=Z_f f_0 ,
\end{aligned}
\tag{B1}
$$

where the subscript 0 denotes the bare quantities. The calculation of the mass and wave-function renormalizations is straightforward. We demand that the pion propagator have a pole of residue 1 at m^2. There is only one one-loop diagram, which contributes. In addition, some terms in $\mathcal{L}_4$ make a contribution, and the total is

$$
\begin{aligned}
Z_\pi&=1-\frac{m^2}{f^2}\left[8Nl_4+8l_5+\frac{N}{3}\left[D+\frac{1}{16\pi^2}-\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}\right]\right] ,\\
Z_m&=1-\frac{m^2}{f^2}\left[4Nl_4+4l_5-8Nl_6-8l_8+\frac{1}{2N}\left[D+\frac{1}{16\pi^2}-\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}\right]\right] .
\end{aligned}
\tag{B2}
$$

Here μ is scale of dimensional regularization and D is the quantity $(1/16\pi^2)(2/\epsilon-\gamma_E+\ln 4\pi)$.

To renormalize f, we must define the axial-vector current. To do this we replace $\partial^\mu\Sigma$ by $(\partial^\mu-ia^\mu)\Sigma-i\Sigma a^\mu$ in $\mathcal{L}$. The axial-vector current A^μ is what multiplies a^μ in $\mathcal{L}$; at lowest order it is $A_\mu^a=(if^2/2)\mathrm{tr}[T^a(\Sigma^\dagger\partial_\mu\Sigma-\Sigma\partial_\mu\Sigma^\dagger)]$. The decay constant f is then defined by

$$\langle 0|A^{a\mu}|\pi^b\rangle=ifp^\mu\delta^{ab} . \tag{B3}$$

The operators multiplying $l_1,\ldots,l_3'$ make a contribution to A^μ, but not to the matrix element in (B3). The only contributions in $\mathcal{L}_4$ are from the l_4 and l_5 terms. In addition there is a one-loop diagram from $\mathcal{L}_2$. The result is

$$Z_f=1+\frac{m^2}{f^2}\left[4Nl_4+4l_5+\frac{N}{2}\left[D+\frac{1}{16\pi^2}-\frac{1}{16\pi^2}\ln\frac{m^2}{\mu^2}\right]\right] . \tag{B4}$$

Finally, the infinities in the computation of the scattering amplitudes can all be absorbed by defining renormalized quantities $l(\mu)$ by

$$
\begin{aligned}
&l_1(\mu)=l_1+\frac{1}{32}D,\quad l_2(\mu)=l_2+\frac{1}{16}D,\quad l_3(\mu)=l_3+\frac{N}{48}D ,\\
&l_3'(\mu)=l_3'+\frac{N}{96}D,\quad l_4(\mu)=l_4+\frac{1}{16}D,\quad l_5(\mu)=l_5+\frac{N}{16}D ,\\
&l_6(\mu)=l_6+\left[\frac{1}{32}+\frac{1}{16N^2}\right]D,\\
&l_8(\mu)=l_8+\left[\frac{N}{32}-\frac{1}{8N}\right]D .
\end{aligned}
\tag{B5}
$$

This computation does not yield the renormalization of l_7 because that term makes no contribution to $\pi\pi\to\pi\pi$ scattering. Equation (B5) agrees with the results of Gasser and Leutwyler [22].

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