Measurable distributions of unpolarized neutron decay

F. Glück
Central Research Institute for Physics, H-1525 Budapest 114, P.O.B. 49, Hungary
(Received 26 May 1992)

Several two- and one-dimensional distributions of unpolarized free-neutron decay are calculated. The results of the order-α model-independent radiative correction calculations are tabulated numerically. With these corrections the theoretical distributions become precise enough to make possible the determination of the ratio of the axial-vector to the vector weak coupling constants to a precision of ~ 0.001.

PACS number(s): 13.40.Ks, 13.30.Ce, 23.40.−s

I. INTRODUCTION

The measurement of free neutron β decay provides precise values for weak vector and axial-vector coupling constants, which allow precise tests of basic symmetries such as conservation of the weak vector current, the unitarity of the weak quark-mixing matrix, SU(3)-flavor symmetry, and right-handed weak currents. In addition, neutron β-decay data are needed to calculate weak cross sections, for applications in big bang cosmology, astrophysics, solar physics, and the solar neutrino problem.

Recently, several high-quality experiments have been carried out aimed at determining the neutron-decay parameters with improved accuracy: neutron lifetime measurements using in-beam neutrons [1–5] and ultracold stored neutrons [6–9], and electron asymmetry measurements with polarized in-beam neutrons [10,11] (see Ref. [12] for a recent review of free-neutron lifetime measurements and the third paper of Ref. [13] for a review of particle physics with cold neutrons). These precise measurements allowed one for the first time to derive the G_V and G_A weak-coupling constants and the V_{ud} mixing matrix element from neutron-decay data alone [13,14]. They provide also important constraints on the free parameters of the $SU(2)_L \times SU(2)_R \times U(1)$ left-right-symmetric model of electroweak interactions [13,15–17] and on the scalar and tensor coupling constants [18].

The neutron lifetime and electron asymmetry measurements provide independent values for the $\lambda = G_V/G_A$ ratio of the axial-vector to vector weak couplings (λ_τ and λ_A). Within the framework of the standard model, λ_τ and λ_A should be equal. The latest polarization measurement [11] indicates, however, a significant discrepancy:

$$|\lambda_\tau - \lambda_A| = 0.010 \pm 0.003 \ .$$

This difference in the two measured λ values could be explained by assuming the existence of right-handed currents [16] (although other particle-physics constraints seem to contradict this explanation [9]). It is obvious that further, more precise measurements are needed to confirm or refute this discrepancy.

There is a third method of determining the $\lambda = G_V/G_A$ ratio: the electron-neutrino correlation measurement in unpolarized neutron decay, providing a $\lambda_{e\nu}$ value which is experimentally independent of λ_τ and λ_A. A precise $\lambda_{e\nu}$ result of this experiment would test both the right-handed currents and the CVC (conserved vector current) hypothesis (and the reliability of the various corrections applied to the ft values of the superallowed Fermi decays).

Our paper is devoted to give the order-α model-independent radiative corrections, in the framework of the standard model, for several two- and one-dimensional distributions of unpolarized neutron decay. With these corrections the theoretical distributions become precise enough so as to make possible the determination of $\lambda_{e\nu}$ within an error of ~ 0.001.

The plan of this paper is the following. In Sec. II we summarize the most important results of earlier publications on the electron energy spectrum and the total decay rate, and we make some comments on the order-α^2 radiative corrections. Section III contains the description of the (E_2, E_f) Dalitz distribution and the proton energy spectrum (E_2 and E_f denote the electron and proton energies, respectively). In Secs. IV and V the $(E_2, \cos\theta_{e\nu})$ and $(E_2, \cos\theta_{ep})$ two-dimensional distributions are presented. Finally, the Appendix is devoted to describing a simple method of the order-α radiative correction calculation to the (E_2, E_f) distribution.

II. ELECTRON ENERGY SPECTRUM AND TOTAL DECAY RATE

The general theoretical framework of our calculations can be found in Refs. [19,20]. We use the conventions and notation of Ref. [20] (unless otherwise stated). Indices 1, 2, i, and f refer to antineutrino, electron, initial (decaying) baryon (neutron), and final baryon (proton), respectively. p, $\mathbf{p}$, E, and m denote four-momentum, three-momentum, energy, and mass, respectively.

The electron energy spectrum, up to order α, can be written as

$$w_{0C\alpha}(E_2) = w_0(E_2) F_C(E_2) [1 + 0.01 r_e(x)] \ . \quad (2.1)$$

The first factor here is the zeroth-order spectrum

$$w_0(E_2)=\frac{G_V^2(1+3\lambda^2)}{2\pi^3}|\mathbf{p}_2|E_2(E_{2m}-E_2)^2 \times[1+0.01R_0(E_2)]\ , \tag{2.2}$$

$$R_0(E_2)=\frac{100}{1+3\lambda^2}\left\{2\frac{E_2}{m_i}+\lambda^2\left[10\frac{E_2}{m_i}-2\frac{m_2^2}{m_iE_2}-2\frac{E_{2m}}{m_i}\right] + \lambda(1+2\kappa)\times\left[2\frac{E_{2m}}{m_i}-4\frac{E_2}{m_i}+2\frac{m_2^2}{m_iE_2}\right]\right\},$$

$$E_{2m}=\Delta-\frac{\Delta^2-m_2^2}{2m_i},\quad \Delta=m_i-m_f\ , \tag{2.3}$$

$$G_V=G_\mu V_{ud}f_1',\quad \lambda=\frac{g_1'}{f_1'},\quad \kappa=\frac{f_2}{f_1'}\approx\frac{\mu_p-\mu_n}{2}$$

(see Ref. [21]).

The following notation has been used: E_{2m} is the electron end-point energy; G_μ and V_{ud} denote the muon-decay coupling constant and the up-down Cabibbo-Kobayashi-Maskawa matrix element, respectively; and μ_p and μ_n are the anomalous magnetic moments of the proton and neutron. The f_1' and g_1' form factors contain the model-dependent parts of the order-α radiative corrections [22]. Time-reversal invariance is assumed. We have neglected the q^2 dependence of the form factors and other very small terms in the complete zeroth-order expressions (see Appendix A of Ref. [20]). Their effect on the spectrum is less than 0.002%.

The second factor in Eq. (2.1) is the Coulomb correction

$$F_C(E_2)=F(Z=1,E_2)Q(Z=1,E_2)\ , \tag{2.4}$$

where $F(Z=1,E_2)$ denotes the Fermi function and the effect of the recoil on this function is taken into account with the aid of $Q(Z=1,E_2)$ (see Ref. [21]). For $E_2-m_2>5$ keV electron energies, the following expression is a very good approximation of the Coulomb correction (with a relative error less than 0.01%):

$$F_C(E_2)\approx 1+\frac{\alpha\pi}{\beta}+\alpha^2\left[\frac{11}{4}-\gamma_E-\ln(2\beta E_2R)+\frac{\pi^2}{3\beta^2}\right], \tag{2.5}$$

where

$$\beta=\frac{|\mathbf{p}_2|}{E_2},\quad R\approx 1\text{ fm}\approx\frac{0.01}{4m_2},\quad \gamma_E\approx 0.5772\ .$$

The third factor in Eq. (2.1) contains the model-independent part of the order-α radiative correction. We introduce the dimensionless variable x as

$$x:=\frac{E_2-m_2}{E_{2m}-m_2}\ . \tag{2.6}$$

The $r_e(x)$ model-independent correction can be written as [22]

$$r_e(x)=100\frac{\alpha}{2\pi}g(E_2)\ ,$$

$$g(E_2)=3\ln\left[\frac{m_f}{m_2}\right]-\frac{3}{4}+4\left[\frac{N}{\beta}-1\right]\left\{\frac{E_{2m}-E_2}{3E_2}-\frac{3}{2}+\ln\left[\frac{2(E_{2m}-E_2)}{m_2}\right]\right\}+\frac{4}{\beta}L\left[\frac{2\beta}{1+\beta}\right] +\frac{N}{\beta}\left[2(1+\beta^2)+\frac{(E_{2m}-E_2)^2}{6E_2^2}-4N\right], \tag{2.7}$$

$$N=\frac{1}{2}\ln\left[\frac{1+\beta}{1-\beta}\right],\quad L(z)=\int_0^z dt\frac{\ln|1-t|}{t}\ .$$

The $r_e(x)$ correction is tabulated in Table I. We can see that the model-independent order-α correction has a rather large effect ($\sim 1\%$) on the shape of the electron energy spectrum. The measurement of the electron spectrum with 0.1% experimental error could reveal this correction.

On the other hand, the electron spectrum shape is rather insensitive to the λ and κ form-factor ratios. The weak magnetism part of the $R_0(E_2)$ zeroth-order correction [terms containing κ in Eq. (2.3)] varies between -0.1% and 0.2%. This seems to be too small for a meaningful check of the "strong" CVC hypothesis [23] in free-neutron-decay experiments. This hypothesis has been verified by nuclear β-decay measurements to the $\sim 10\%$ level [24,25]; therefore, we can safely use the $\kappa=(\mu_p-\mu_n)/2$ CVC value in our calculations.

The total decay rate is obtained by integrating Eq. (2.1) over E_2:

TABLE I. Radiative correction to the electron energy spectrum.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$r_e(x)$	1.82	1.74	1.65	1.53	1.40	1.25	1.07	0.84	0.50	0.21

$$\rho_{0C\alpha}=\int_{m_2}^{E_{2m}} dE_2 w_{0C\alpha}(E_2) \ . \tag{2.8}$$

We write this integral as

$$\rho_{0C\alpha}=\frac{G_V^2(1+3\lambda^2)}{2\pi^3} m_2^5 f_{0C\alpha} \ ,$$
$$f_{0C\alpha}=f_{0C}(1+0.01r_\rho) \ , \tag{2.9}$$

$$f_{0C}=\frac{1}{m_2^5}\int_{m_2}^{E_{2m}} dE_2 |\mathbf{p}_2| E_2 (E_{2m}-E_2)^2 \times[1+0.01R_0(E_2)]F_C(E_2) \ . \tag{2.10}$$

Using the mass values of Ref. [26], we get

$$f_{0C}=1.6887\pm0.0001 \ ,$$
$$r_\rho=1.505 \ , \tag{2.11}$$
$$f_{0C\alpha}=1.7141\pm0.0001$$

(the errors given here reflect the uncertainties of the measured mass values).

Expression (2.9) determines the decay rate for a point nucleon, up to order α. The model-dependent part of the order-α radiative correction might contain small electron energy-dependent terms [22] (the relative effect of these terms on the electron spectrum is probably smaller than 0.01%). The contribution of these terms is unknown for the time being; therefore, we should increase the error of $f_{0C\alpha}$ to 0.0002. According to Wilkinson [21], the finite nucleon radius correction yields a -0.0001 contribution to $f_{0C\alpha}$. On the other hand, no complete order-α^2 calculation exists for the neutron decay. The so-called order-$Z\alpha^2$ corrections, relevant for the CVC analysis of superallowed Fermi decays, have been studied extensively in the literature [27–31]. In these calculations the transverse-wave part of one virtual photon is neglected; only the instantaneous Coulomb potential is included in the integrals (this photon interacts between the outgoing electron or positron and the nucleus with charge Ze). The other photon is treated exactly. The leading-logarithmic term of these corrections is $Z\alpha^2\ln(m_f/m_2)$. For the free-neutron decay ($Z=1$), this gives a 0.04% relative correction to the decay rate (which corresponds to a $\sim$0.0007 absolute correction to $f_{0C\alpha}$). It is, however, obvious that the result of the exact order-α^2 calculation might considerably differ from this value. The leading-logarithmic term of the order-α model-independent calculation yields a 2.5% relative correction to the decay rate, while the exact result for neutron decay is 1.5% [see Eqs. (2.9)–(2.11)]. The new results for the order-$Z\alpha^2$ corrections of superallowed Fermi decays [29–31] show also large deviations from the 0.04Z% leading-logarithmic predictions. Moreover, the order-α^2 correction might contain also double-logarithmic terms [$\sim\alpha^2\ln^2(m_f/m_2)$].

We suspect that the electron-energy-dependent part of the complete order-α^2 correction can be computed approximately in a model-independent way, similarly to the order-α correction [22]. This calculation would be useful for the CVC analysis of the superallowed Fermi decays (see Ref. [32] for a new survey) and for the derivation of λ_τ from the neutron lifetime. The electron energy spectrum might be altered as a result of this correction by a few times 0.01%. On the other hand, the model-dependent (electron-energy-independent) part can be absorbed into G_V and λ. This correction might effect considerably the V_{ud} value derived from neutron and nuclear decay lifetimes. We intend to discuss in detail these problems in a later publication.

III. (E_2,E_f) DALITZ DISTRIBUTION AND PROTON ENERGY SPECTRUM

We write the electron-energy–proton-energy correlational distribution, up to order α, as a product of three factors (similarly to the electron energy spectrum):

$$W_{0C\alpha}(E_2,E_f)= W_{0C}(E_2,E_f) \times[1+0.01r_e(x)+0.01r(x,y)] \ ,$$
$$W_{0C}(E_2,E_f)=W_0(E_2,E_f)\hat{F}_C(E_2,E_f) \ . \tag{3.1}$$

The zeroth-order distribution can be written as

$$W_0(E_2,E_f)=m_i\frac{G_V^2}{4\pi^3}[D_V+\lambda^2 D_A+\lambda(1+2\kappa)D_I] \ ,$$
$$D_{V/A}=E_2(E_{2m}-E_2)+E_1(E_{1m}-E_1)\mp m_f(E_{fm}-E_f) \ , \tag{3.2}$$
$$D_I=2[E_2(E_{2m}-E_2)-E_1(E_{1m}-E_1)] \ ,$$

where

$$E_{2m/1m}=\Delta-\frac{\Delta^2\mp m_2^2}{2m_i}, \quad E_{fm}=m_f+\frac{\Delta^2-m_2^2}{2m_i} \ ,$$
$$\Delta=m_i-m_f, \quad E_1=m_i-E_2-E_f \tag{3.3}$$

(E_{2m} and E_{fm} are the maximum energies of the electron and proton, respectively) and λ,κ are the form-factor ratios defined in the previous section. E_1 is the antineutrino energy in zeroth order. This expression is a very good approximation of the complete zeroth-order distribution [the omitted terms give a few times 10^{-4}% relative correction to Eq. (3.2)]. It is in agreement with the result of Nachtmann [33].

In the $\hat{F}_C(E_2,E_f)$ Coulomb correction, we take into account the proton energy dependence of the proton recoil. This correction is obtained by replacing the $\beta=|\mathbf{p}_2|/E_2$ electron velocity in the $F(Z=1,E_2)$ Fermi function with the

$$\beta_r=|\beta-(1-\beta^2)v_f c_f| \tag{3.4}$$

relative velocity, where $v_f=|\mathbf{p}_f|/E_f$, $c_f=\mathbf{p}_2\cdot\mathbf{p}_f/(|\mathbf{p}_2||\mathbf{p}_f|)$.

In the third factor of Eq. (3.1), $r_e(x)$ is the model-independent order-α correction to the electron energy spectrum [see Eq. (2.7) and Table I], and $r(x,y)$ represents the proton energy dependence of the model-independent order-α correction. The dimensionless variable y is defined similarly to Eq. (2.6):

$$y := \frac{E_f - m_f}{E_{fm} - m_f} \; . \tag{3.5}$$

The $r(x,y)$ correction is tabulated in Table II. The boundaries of the zeroth-order (E_2, E_f) Dalitz region are

$$E_{f\min/\max}(E_2) = \frac{1}{2}\left[m_i - E_2 \mp |\mathbf{p}_2| + \frac{m_f^2}{m_i - E_2 \mp |\mathbf{p}_2|} \right] , \tag{3.6}$$

and

$$E_{2\min/\max}(E_f) = \frac{1}{2}\left[m_i - E_f \mp |\mathbf{p}_f| + \frac{m_2^2}{m_i - E_f \mp |\mathbf{p}_f|} \right] . \tag{3.7}$$

We mention that, in the absence of hard bremsstrahlung photons $r(x,y)$ would be about -0.01 [see Eq. (A2) in the Appendix].

Equation (3.1) determines the distribution of neutron-decay events in the $E_{f\min}(E_2) \le E_f \le E_{f\max}(E_2)$ region. On the other hand, there are also decay events in the $m_f < E_f < E_{f\min}(E_2)$, $m_2 < E_2 < E_{2h}$ region, where

$$E_{2h} = \frac{1}{2}\left[m_i - m_f + \frac{m_2^2}{m_i - m_f} \right] . \tag{3.8}$$

Denoting the bremsstrahlung distribution in this region by $W^{<}_{\rm hard}(E_2, E_f)$ (see the Appendix), the proton energy spectrum is obtained by integrating the (E_2, E_f) Dalitz distribution over E_2:

$$\omega_{0C\alpha}(E_f) = \int_{E_{2\min}(E_f)}^{E_{2\max}(E_f)} dE_2\, W_{0C\alpha}(E_2, E_f) + \omega^{<}(E_f) \; ,$$

$$\omega^{<}(E_f) = \int_{m_2}^{E_{2\min}(E_f)} dE_2\, W^{<}_{\rm hard}(E_2, E_f) \hat{F}_C(E_2, E_f) \quad \text{if } E_f < E_{fh} \; , \tag{3.9}$$

$$\omega^{<}(E_f) = 0 \quad \text{if } E_f > E_{fh} \; ,$$

where

$$E_{fh} = \frac{1}{2}\left[m_i - m_2 + \frac{m_f^2}{m_i - m_2} \right] . \tag{3.10}$$

We write

$$\omega_{0C\alpha}(E_f) = \tilde{\omega}_{0C}(E_f)[1 + 0.01 r_C(y)][1 + 0.01 r_p + 0.01 r_p(y)] \; , \tag{3.11}$$

where

$$\tilde{\omega}_{0C}(E_f) = \int_{E_{2\min}(E_f)}^{E_{2\max}(E_f)} dE_2\, W_0(E_2, E_f) \left[1 + \frac{\alpha\pi}{\beta} \right]$$

$$\approx m_i \frac{G_V^2}{4\pi^3} [\Omega(E_{2\max}(E_f)) - \Omega(E_{2\min}(E_f))] \; ,$$

$$\Omega(E_2) = (1 + \lambda^2) \left[E_{2m} E_2^2 (1 + \pi\alpha\beta) - \frac{2}{3} E_2^3 + \frac{1}{2}\pi\alpha E_{2m} m_2^2 \ln \left[\frac{1+\beta}{1-\beta} \right] - 2\pi\alpha\beta E_2 \left[m_2^2 + \frac{1}{3}\beta^2 E_2^2 \right] \right]$$

$$- (1 - \lambda^2) m_f (E_{fm} - E_f) E_2 (1 + \pi\alpha\beta) \; , \tag{3.12}$$

$$\beta = \frac{(E_2^2 - m_2^2)^{1/2}}{E_2}$$

TABLE II. Radiative correction to the (E_2, E_f) Dalitz distribution $(E_f = E_{f\min}(E_2) + [E_{f\max}(E_2) - E_{f\min}(E_2)]z = m_f + (E_{fm} - m_f)y)$.

z	$r(x,y)$								
0.99	−0.20	−0.30	−0.36	−0.40	−0.43	−0.45	−0.46	−0.46	−0.46
0.96	−0.14	−0.17	−0.19	−0.19	−0.18	−0.17	−0.15	−0.13	−0.11
0.93	−0.11	−0.12	−0.12	−0.11	−0.09	−0.07	−0.05	−0.03	−0.01
0.9	−0.09	−0.09	−0.08	−0.06	−0.04	−0.02	−0.00	0.02	0.04
0.8	−0.05	−0.04	−0.01	0.01	0.03	0.05	0.07	0.10	0.12
0.7	−0.03	−0.01	0.02	0.04	0.06	0.08	0.10	0.12	0.14
0.6	−0.01	0.01	0.03	0.06	0.07	0.09	0.11	0.13	0.14
0.4	0.01	0.03	0.05	0.06	0.08	0.09	0.10	0.11	0.11
0.3	0.01	0.04	0.05	0.06	0.07	0.07	0.07	0.07	0.07
0.2	0.02	0.04	0.04	0.04	0.04	0.03	0.02	0.01	−0.0
0.1	0.02	0.03	0.02	−0.00	−0.04	−0.07	−0.11	−0.14	−0.17
0.07	0.02	0.03	0.01	−0.03	−0.09	−0.13	−0.18	−0.23	−0.27
0.04	0.03	0.03	−0.02	−0.09	−0.17	−0.24	−0.31	−0.37	−0.43
0.01	0.03	0.02	−0.08	−0.27	−0.41	−0.53	−0.64	−0.75	−0.84
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

TABLE III. Higher-order Coulomb correction to the proton energy spectrum.

y	0.1	0.2	0.3	0.4	0.42	0.43	0.45	0.5	0.6	0.8	0.9
$r_C(y)$	0.07	0.08	0.09	0.11	0.12	0.14	0.11	0.09	0.08	0.06	0.06

(the relative error of this approximation is about 0.001% or less).

In $\tilde{\omega}_{0C}(E_f)$, the $\hat{F}_C(E_2,E_f)\approx 1+\pi\alpha/\beta$ approximation has been employed. The $r_C(y)$ function contains the contribution of the remaining part of the Coulomb correction (see Table III). For the precise computation of the $F(Z=1,E_2)$ Fermi function, we have used formula (ii) in Appendix 7 of Ref. [21]. The third factor in (3.11) includes the model-independent order-α correction. r_p is the model-independent order-α correction to the total decay rate [see Eqs. (2.9)–(2.11)]. The $r_p(y)$ additional correction (which alters the shape of the proton spectrum) is tabulated in Table IV. This correction seems to be rather small. However, neglecting this correction at the statistical fit of the theoretical and measured proton spectra would yield a $|\lambda|$ value larger than the true $|\lambda|$ by ~ 0.010.

The model-independent order-α correction to the proton spectrum was calculated in Ref. [34] by Christian and Kühnelt. The comparison of our calculation with their results shows satisfactory agreement in the lower half of the spectrum (for $E_f-m_f<400$ eV). In the upper half of the spectrum, however, the numerical results of Ref. [34] do not agree with our results. In order to study this discrepancy, we have computed the $r_e(x)$, $r(x,y)$, and $r_p(y)$ corrections using the method of Ref. [34] (see the Appendix for more details). We have found complete agreement with our previous calculations.

The order-α^2 correction and the uncertainties of the order-α calculation (due to neglected model-dependent terms) are expected to give contributions less than 0.01% to the $r(x,y)$ and $r_p(y)$ relative corrections. Therefore, from a theoretical point of view, the measurements of the (E_2,E_f) Dalitz distribution and the proton energy spectrum make possible the determination of the λ parameter with a ~ 0.0005–0.001 error. Unfortunately, the low sensitivity of these distributions to λ, the low kinetic energy of the recoil protons ($E_{fm}-m_f=751$ eV), and other systematic errors cause serious difficulties in these experiments. As far as we know, only two measurements of unpolarized neutron-decay distributions were carried out in the last three decades [35,36] (see also Ref. [37] for earlier references). We mention that the determination of λ with a ~ 0.001 error allows a $\sim 0.02\%$ experimental error in the (E_2,E_f) distribution and a $\sim 0.01\%$ error in the proton spectrum.

IV. $(E_2,\cos\theta_{e\nu})$ DISTRIBUTION

We use the "experimental" definition of $\cos\theta_{e\nu}$ (see Ref. [19]):

$$c:=\cos\theta_{e\nu}:=-\frac{\mathbf{p}_2\cdot(\mathbf{p}_2+\mathbf{p}_f)}{|\mathbf{p}_2||\mathbf{p}_2+\mathbf{p}_f|}\ . \tag{4.1}$$

The (E_2,c) distribution, up to order α, can be written as

$$W^{e\nu}_{0C\alpha}(E_2,c)=W_{0C}(E_2,E_{fc})\frac{\partial E_{fc}}{\partial c}\times[1+0.01r_e(x)+0.01r_{e\nu}(x,c)]\ , \tag{4.2}$$

where

$$E_{fc}=\frac{d^2+\mathbf{p}_2^2+m_f^2+2d|\mathbf{p}_2|c}{2(d+|\mathbf{p}_2|c)}\ ,\quad d=m_i-E_2\ , \tag{4.3}$$

$$\frac{\partial E_{fc}}{\partial c}\approx\frac{|\mathbf{p}_2|(E_{2m}-E_2)}{m_i}\left[1+\frac{2E_2}{m_i}(1-\beta c)\right]\ . \tag{4.4}$$

For the zeroth-order (E_2,c) distribution

$$W^{e\nu}_0(E_2,c)=W_0(E_2,E_{fc})\frac{\partial E_{fc}}{\partial c}\ , \tag{4.5}$$

a simple approximate formula exists [neglecting terms of order $(m_i-m_f)/m_i$]:

$$W^{e\nu}_{\mathrm{app}}(E_2,c)=\frac{G_V^2(1+3\lambda^2)}{4\pi^3}|\mathbf{p}_2|E_2(E_{2m}-E_2)^2\times\left[1+\frac{1-\lambda^2}{1+3\lambda^2}\beta c\right]\ . \tag{4.6}$$

The (E_2,c) distribution has the simplest Dalitz region among the two-dimensional distributions:

$$m_2\le E_2\le E_{2m},\quad -1\le c\le +1 \tag{4.7}$$

(this is valid for the bremsstrahlung events, too).

Table V contains the $r_{e\nu}(x,c)$ correction. This part of

TABLE IV. Radiative correction to the proton energy spectrum.

y	0.1	0.2	0.3	0.4	0.5	0.55	0.6	0.65	0.7	0.75
$r_p(y)$	0.12	0.11	0.10	0.08	0.05	0.04	0.01	−0.02	−0.06	−0.12
y	0.78	0.8	0.83	0.85	0.88	0.9	0.92	0.94	0.96	0.98
$r_p(y)$	−0.16	−0.20	−0.26	−0.32	−0.43	−0.52	−0.63	−0.79	−1.00	−1.34

TABLE V. Radiative correction to the $(E_2,\cos\theta_{e\nu})$ distribution.

c	$r_{e\nu}(x,c)$							
0.9	0.03	0.04	0.06	0.08	0.10	0.12	0.14	0.16
0.7	0.03	0.04	0.05	0.07	0.08	0.10	0.11	0.13
0.5	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.3	0.02	0.02	0.03	0.03	0.04	0.04	0.05	0.06
0.1	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03
−0.1	0.01	0.00	0.00	−0.00	−0.00	−0.00	−0.01	−0.00
−0.3	−0.00	−0.01	−0.01	−0.02	−0.02	−0.03	−0.03	−0.03
−0.5	−0.01	−0.02	−0.02	−0.05	−0.04	−0.05	−0.05	−0.06
−0.7	−0.01	−0.02	−0.05	−0.05	−0.06	−0.07	−0.08	−0.08
−0.9	−0.02	−0.03	−0.05	−0.06	−0.08	−0.09	−0.10	−0.11
x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9

the model-independent correction is largely due to the hard bremsstrahlung photons [similarly to $r(x,y)$]. These photons alter the kinematics of the decay and increase the number of events with positive $\cos\theta_{e\nu}$. The $r_{e\nu}(x,c)$ relative correction is finite at the $c=\pm 1$ boundaries [contrary to the $r(x,y)$ correction, which goes logarithmically to minus infinity in the $E_f \to E_{f\max}(E_2)$ and $E_2 > E_{2h}$, $E_f \to E_{f\min}(E_2)$ limits].

The order-α radiative correction to the (E_2,c) distribution was calculated in Ref. [38]. The $r_{e\nu}(x,c)$ correction derived from the analytic formulas (14)–(16) of Ref. [38] is about one order of magnitude smaller than our result. This discrepancy is probably due to the fact that the authors of Ref. [38] used the

$$c=\cos\theta_{e\nu}=\frac{\mathbf{p}_2\cdot\mathbf{p}_1}{|\mathbf{p}_2||\mathbf{p}_1|} \tag{4.8}$$

definition in their calculations. In the presence of hard bremsstrahlung photons, (4.1) and (4.8) are not equivalent (see also Ref. [19]).

The measurement of the $W^{e\nu}(E_2,c)$ distribution with a $\sim 0.02\%$ error would yield a λ parameter with a ~ 0.001 error. Unfortunately, this experiment requires the detection of outgoing electrons and protons in coincidence, in all directions, and the measurement of their three-momentum vectors.

V. $(E_2,\cos\theta_{ep})$ DISTRIBUTION

θ_{ep} is the angle between the outgoing electron and proton directions in the decaying neutron rest frame. We use the $c_f=\cos\theta_{ep}$ abbreviation. The (E_2,c_f) distribution, up to order α, is

$$W^{ep}_{0C\alpha}(E_2,c_f)=W^{ep}_{0C}(E_2,c_f)[1+0.01r_e(x)+0.01r_{ep}(x,c_f)]\ , \tag{5.1}$$

$$W^{ep}_{0C}(E_2,c_f)=\begin{cases} W_{0C}(E_2,E_f^+)\left|\dfrac{\partial E_f^+}{\partial c_f}\right| & \text{if } E_2<E_{2h}\ , \\ \sum\limits_{+,-} W_{0C}(E_2,E_f^\pm)\left|\dfrac{\partial E_f^\pm}{\partial c_f}\right| & \text{if } E_2>E_{2h} \end{cases} \tag{5.2}$$

[see Eq. (3.8)],

$$\begin{aligned} &E_f^\pm=[(p_f^\pm)^2+m_f^2]^{1/2}\ ,\\ &p_f^\pm=\frac{-b\pm S}{2a},\quad S=(b^2-4ah)^{1/2}\ ,\\ &a=4(d^2-\mathbf{p}_2^2c_f^2),\quad b=4|\mathbf{p}_2|c_f(d^2+m_f^2-\mathbf{p}_2^2)\ ,\\ &h=4(m_i^2-m_f^2)(E_2-E_{2h})(H-E_2)\ ,\\ &H=\frac{1}{2}\left[m_i+m_f+\frac{m_2^2}{m_i+m_f}\right],\quad d=m_i-E_2\ , \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\frac{\partial E_f^\pm}{\partial c_f}=\frac{p_f^\pm}{E_f^\pm}\left\{\frac{b\mp S}{2a^2}D_a\pm\frac{b\mp S}{2aS}D_b\mp\frac{h}{aS}D_a\right\},\\ &D_a=-8\mathbf{p}_2^2c_f,\quad D_b=4|\mathbf{p}_2|(d^2+m_f^2-\mathbf{p}_2^2)\ . \end{aligned} \tag{5.4}$$

The Dalitz region of the (E_2,c_f) distribution is

$$\begin{aligned} &-1\le c_f\le c_{fm}(E_2)\ ,\\ &c_{fm}(E_2)=+1\quad \text{if } E_2<E_{2h}\ ;\\ &c_{fm}(E_2)=-\frac{\sqrt{h}}{2|\mathbf{p}_2|m_f}=-[1-s_{fm}^2(E_2)]^{1/2}\ ,\\ &s_{fm}(E_2)=\frac{m_i}{m_f}\frac{E_{2m}-E_2}{|\mathbf{p}_2|}\quad \text{if } E_2\ge E_{2h} \end{aligned} \tag{5.5}$$

(see Sec. IV of Ref. [39]).

The $r_{ep}(x,c_f)$ part of the model-independent order-α correction is tabulated in Table VI. The hard bremsstrahlung photons shift the distribution here toward lower c_f values. We can easily understand this trend by writing c_f as

$$\begin{aligned} &c_f=\frac{1}{2|\mathbf{p}_2||\mathbf{p}_f|}[Q^2-\mathbf{p}_2^2-\mathbf{p}_f^2]\ ,\\ &Q=|\mathbf{p}_2+\mathbf{p}_f|=|\mathbf{p}_1+\mathbf{k}| \end{aligned} \tag{5.6}$$

($\mathbf{p}_1$ and $\mathbf{k}$ are the three-momenta of the antineutrino and the bremsstrahlung photon, respectively). In the presence of hard bremsstrahlung photons, Q becomes smaller than the zeroth-order $Q_0=m_i-E_2-E_f$ value, implying a decrease of c_f. The $r_{ep}(x,c_f)$ relative correction has a negative logarithmic singularity at the $E_2>E_{2h}$,

TABLE VI. Radiative correction to the $(E_2,\cos\theta_{ep})$ distribution ($c_f=-1+[c_{fm}(E_2+1)]z$; see Eq. (5.5)).

z	$r_{ep}(x,c_f)$										
0.99	−0.04	−0.14	−0.24	−0.64	−0.46	−0.47	−0.50	−0.55	−0.60	−0.64	−0.69
0.96	−0.04	−0.14	−0.24	−0.65	−0.27	−0.26	−0.25	−0.26	−0.27	−0.28	−0.29
0.93	−0.04	−0.14	−0.24	−0.66	−0.20	−0.17	−0.16	−0.14	−0.14	−0.13	−0.13
0.9	−0.04	−0.14	−0.24	−0.65	−0.15	−0.11	−0.09	−0.07	−0.05	−0.04	−0.03
0.8	−0.03	−0.13	−0.23	−0.65	−0.06	−0.01	0.02	0.07	0.10	0.13	0.15
0.7	−0.03	−0.11	−0.21	−0.64	−0.00	0.05	0.09	0.15	0.19	0.22	0.24
0.6	−0.02	−0.10	−0.19	−0.63	0.04	0.10	0.14	0.20	0.25	0.28	0.30
0.5	−0.01	−0.07	−0.15	−0.54	0.08	0.13	0.18	0.24	0.29	0.32	0.34
0.4	−0.00	−0.04	−0.10	−0.28	0.11	0.16	0.21	0.27	0.32	0.35	0.37
0.3	0.01	−0.01	−0.04	−0.12	0.14	0.19	0.23	0.30	0.34	0.37	0.39
0.2	0.02	0.02	0.02	−0.00	0.16	0.21	0.25	0.31	0.36	0.39	0.41
0.1	0.03	0.06	0.08	0.09	0.18	0.23	0.26	0.33	0.37	0.40	0.42
0.05	0.04	0.08	0.10	0.13	0.19	0.23	0.27	0.33	0.38	0.41	0.43
0.01	0.04	0.09	0.12	0.15	0.20	0.24	0.27	0.34	0.38	0.41	0.43
x	0.1	0.2	0.25	0.3	0.35	0.4	0.45	0.55	0.65	0.75	0.85

$c_f=c_{fm}(E_2)$ boundary [39] (the zeroth-order distribution is also singular at this boundary).

The λ form-factor ratio may be determined by measuring the shapes of the c_f distribution for fixed electron energies. The sensitivity of these shapes to λ is maximal in the $0.2<x<0.5$ interval; the determination of λ with a 0.001 error allows a $\sim 0.01-0.02\,\%$ relative error in the distribution measurement.

Taking into account the $r_{ep}(x,c_f)$ radiative correction in the statistical fit of the theoretical and measured distributions near $x\sim 0.4$ decreases the $|\lambda|$ result by ~ 0.01.

Another method for the determination of λ is the measurement of the energy spectrum of electrons emitted into a given range of angles referred to the proton direction (see Sec. 3 of Ref. [37]). The backward electron spectrum is defined as

$$w^B_{0C\alpha}(E_2;\theta_f)=\int_{-1}^{c_{f\max}(E_2,\theta_f)} dc_f\, W^{ep}_{0C\alpha}(E_2,c_f)\ , \quad (5.7)$$

where

$$c_{f\max}(E_2,\theta_f)=\min[\cos\theta_f, c_{fm}(E_2)]\ . \quad (5.8)$$

Our usual factorization gives

$$w^B_{0C\alpha}(E_2;\theta_f)=w^B_{0C}(E_2;\theta_f)[1+0.01r_e(x)+0.01r_e^B(x;\theta_f)]\ ,$$
$$w^B_{0C}(E_2;\theta_f)=\int_{-1}^{c_{f\max}(E_2,\theta_f)} dc_f\, W^{ep}_{0C}(E_2,c_f)\ . \quad (5.9)$$

Table VII contains the $r_e^B(x;\theta_f)$ relative corrections for a few θ_f values. We mention that, for $\theta_f=0$, Eq. (5.9) becomes identical with Eq. (2.1) $[w^B_{0C\alpha}(E_2;0)=w_{0C\alpha}(E_2), r_e^B(x;0)=0]$.

In order to determine λ with a 0.001 error, the $w^B(E_2;\theta_f)$ spectrum has to be measured with a $\sim 0.01\%$ relative error.

The measurement of the $W^{ep}(E_2,c_f)$ or $w^B(E_2;\theta_f)$ distributions makes possible the determination of λ in unpolarized neutron decay even if the proton energy measurement is impracticable.

Finally, we mention that we have calculated the radiative corrections presented in Tables II and IV–VII by several different methods (one of them is described in the Appendix). The precise agreement of the numerical results of the different methods (in addition to the tests mentioned in Sec. IV of Ref. [20]) increases substantially the reliability of our calculations.

TABLE VII. Radiative correction to backward electron spectra.

θ_f	$r_e^B(x;\theta_f)$									
170°	0.04	0.09	0.16	0.20	0.24	0.27	0.30	0.35	0.36	0.37
165°	0.04	0.09	0.16	0.19	0.23	0.27	0.30	0.33	0.31	0.00
160°	0.04	0.09	0.15	0.19	0.23	0.26	0.28	0.25	0.00	0.00
155°	0.04	0.09	0.15	0.18	0.22	0.24	0.25	0.00	0.00	0.00
150°	0.04	0.08	0.14	0.17	0.20	0.22	0.18	0.00	0.00	0.00
140°	0.04	0.08	0.12	0.15	0.16	0.00	0.00	0.00	0.00	0.00
x	0.1	0.2	0.3	0.35	0.4	0.45	0.5	0.6	0.65	0.7

ACKNOWLEDGMENTS

I would like to thank Dr. K. Tóth for his reading the manuscript and for useful discussions.

APPENDIX

We show here a simple method of calculation for the $r_D(x,y)=r_e(x)+r(x,y)$ model-independent radiative correction (see Ref. [34]). The following decomposition will be introduced:

$$r_D(x,y)=r_e(x)+r(x,y) \\ =r_{VS}(x,y,\omega)+r_{\rm hard}(x,y,\omega) \; . \tag{A1}$$

Here the first term represents the contribution of virtual and soft bremsstrahlung photons (the maximum energy of the soft photons is ω). It can be written as

$$r_{VS}(x,y,\omega)=100\frac{\alpha}{2\pi}\left[g(\beta,\omega)+2\frac{1-\lambda^2}{1+3\lambda^2}(1-\beta^2)Nc\right] ,$$

$$g(\beta,\omega)=3\ln\left[\frac{m_f}{m_2}\right]+\frac{4}{\beta}L\left[\frac{2\beta}{1+\beta}\right]-\frac{3}{4} \\ +2\frac{N}{\beta}(1+\beta^2-2N)+4\left[\frac{N}{\beta}-1\right]\ln\left[\frac{2\omega}{m_2}\right] , \tag{A2}$$

$$c\approx\frac{m_i}{|\mathbf{p}_2|(E_{2m}-E_2)}(E_f-E_{f0}) ,$$

$$E_{f0}=\frac{(m_i-E_2)^2+\mathbf{p}_2^2+m_f^2}{2(m_i-E_2)}, \quad \beta=\frac{|\mathbf{p}_2|}{E_2}$$

[see Eq. (2.7) for the definition of N and L]. The small λ-dependent term of $r_{VS}(x,y,\omega)$ was neglected in Ref. [34].

The second term in (A1) is the contribution of the hard bremsstrahlung photons with ω minimum energy:

$$r_{\rm hard}(x,y,\omega)=100\frac{W_{\rm hard}(E_2,E_f,Q_1,Q_0,\omega)}{W_0(E_2,E_f)} , \tag{A3}$$

$$Q_1=||\mathbf{p}_2|-|\mathbf{p}_f||, \quad Q_0=m_i-E_2-E_f \; . \tag{A4}$$

$W_0(E_2,E_f)$ can be found in Eq. (3.2) and

$$W_{\rm hard}(E_2,E_f,Q_A,Q_B,\omega) \\ :=\frac{m_i\alpha}{16\pi^4}\int_{Q_A}^{Q_B}dQ\int_{K_{\rm min}}^{K_{\rm max}}dK\, S(Q,K) , \tag{A5}$$

$$K_{\rm max}=(Q_0+Q)/2 , \\ K_{\rm min}=\begin{cases}(Q_0-Q)/2 & \text{if } Q<Q_0-2\omega , \\ \omega & \text{if } Q>Q_0-2\omega ,\end{cases} \tag{A6}$$

$$S(Q,K)=-(1+3\lambda^2)H_1-(1-\lambda^2)H_2 , \tag{A7}$$

$$H_1=2E_2(Q_0-K)\phi_0+2m_2^2K(Q_0-K)(\phi_2-\phi_1/m_2^2) \\ -4E_2(Q_0-K)\phi_1+2(Q_0-K)/K , \tag{A8}$$

$$H_2=H_1+2[m_2^2\phi_1+A/K^2-2E_2/K] \\ -2(N_{12}-N_{1k})(\phi_0-\phi_1) \\ -2m_2^2N_{1k}(\phi_2-\phi_1/m_2^2)+2N_{1k}\phi_1E_2/K-2Q_0/K , \tag{A9}$$

$$\phi_1=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi_k}{(p_2k)}=\frac{1}{(A^2-B^2)^{1/2}} , \\ \phi_2=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi_k}{(p_2k)^2}=\frac{A}{(A^2-B^2)^{3/2}} , \tag{A10}$$

$$\phi_0=m_2^2\phi_2+\frac{1}{K^2}-2\frac{E_2}{K}\phi_1 , \tag{A11}$$

$$(p_2k)=A-B\cos\phi_k , \\ A=E_2K-|\mathbf{p}_2|K_\parallel\cos\eta , \\ B=|\mathbf{p}_2|K_\perp\sin\eta , \\ \cos\eta=\frac{\mathbf{p}_2^2-\mathbf{p}_f^2+Q^2}{2|\mathbf{p}_2|Q} , \\ K_\parallel=\frac{Q_0^2-Q^2}{2Q}-\frac{Q_0}{Q}K , \\ K_\perp=(K^2-K_\parallel^2)^{1/2} \tag{A12}$$

(see Ref. [20], Appendix B),

$$N_{12}=m_i(E_{fm}-E_f) , \\ N_{1k}=\tfrac{1}{2}(Q_0^2-Q^2) \tag{A13}$$

[see Eq. (3.3) for E_{fm}].

The ω parameter determines the separation of the soft and hard bremsstrahlung photons. We require

$$\omega\ll Q_0, \quad \omega\le(Q_0-Q_1)/2 , \tag{A14}$$

so as to satisfy the validity of the soft photon approximation in the derivation of $r_{VS}(x,y,\omega)$. At the $E_f=E_{f\rm max}(E_2)$ and $E_2>E_{2h}$, $E_f=E_{f\rm min}(E_2)$ boundaries $Q_0=Q_1$, and in the $E_2=E_{2m}$, $E_f=E_{fm}$ point $Q_0=0$. This means that ω could not be chosen as a constant over the whole Dalitz region.

In the $m_f<E_f<E_{f\rm min}(E_2)$, $m_2<E_2<E_{2h}$ ($Q_0>Q_2=|\mathbf{p}_2|+|\mathbf{p}_f|$) points, the

$$W_{\rm hard}^{<}(E_2,E_f)=W_{\rm hard}(E_2,E_f,Q_1,Q_2,0) \tag{A15}$$

function determines the distribution of the decay events.

As mentioned in Appendix B of Ref. [20], the integrals over the K-photon energy can also be calculated analytically.

[1] J. Last *et al.*, Phys. Rev. Lett. **60**, 995 (1988).
[2] P. E. Spivak, Zh. Eksp. Teor. Fiz. **94**, 1 (1988) [Sov. Phys. JETP **67**, 1735 (1988)].
[3] R. Kossakowski *et al.*, Nucl. Phys. **A503**, 473 (1989).
[4] J. Döhner *et al.*, Nucl. Instrum. Methods A **284**, 123 (1989).
[5] J. Byrne *et al.*, Phys. Rev. Lett. **65**, 289 (1990).
[6] Yu. Yu. Kosvincev *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 444 (1986) [JETP Lett. **44**, 571 (1986)]; V. I. Morozov, Nucl. Instrum. Methods A **284**, 108 (1989); V. I. Morozov *et al.* (unpublished).
[7] W. Mampe *et al.*, Phys. Rev. Lett. **63**, 593 (1989).
[8] W. Paul *et al.*, Z. Phys. C **45**, 25 (1989).
[9] V. P. Alfimenkov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **52**, 984 (1990) [JETP Lett. **52**, 373 (1990)].
[10] P. Bopp *et al.*, Phys. Rev. Lett. **56**, 919 (1986); E. Klemt *et al.*, Z. Phys. C **37**, 179 (1988).
[11] B. G. Erozolimskii *et al.*, Yad. Fiz. **52**, 1583 (1990) [Sov. J. Nucl. Phys. **52**, 999 (1990)]; Phys. Lett. B **263**, 33 (1991).
[12] K. Schreckenbach and W. Mampe, J. Phys. G **18**, 1 (1992).
[13] D. Dubbers, W. Mampe, and J. Döhner, Europhys. Lett. **11**, 195 (1990); D. Dubbers, in *Particles and Nuclei,* Proceedings of the Twelfth International Conference, Cambridge, Massachusetts, 1990, edited by J. L. Matthews, T. W. Donnelly, E. H. Farhi, and L. S. Osborne [Nucl. Phys. **A527**, 239c (1991)]; Prog. Part. Nucl. Phys. **26**, 173 (1991).
[14] D. Thompson, J. Phys. G **16**, 1423 (1990).
[15] A. Garcia *et al.*, Phys. Lett. **157B**, 317 (1985); M. Aquino, A. Fernandez, and A. Garcia, Phys. Lett. B **261**, 280 (1991); A.-S. Carnoy, J. Deutsch, and B. R. Holstein, Phys. Rev. D **38**, 1636 (1988).
[16] Yu. V. Gaponov, N. B. Shul'gina, and P. E. Spivak, Phys. Lett. B **253**, 283 (1991); Yad. Fiz. **52**, 1653 (1990) [Sov. J. Nucl. Phys. **52**, 1042 (1990)].
[17] B. G. Erozolimskii, Nucl. Instrum. Methods A **284**, 89 (1989).
[18] B. G. Erozolimskii and Yu. A. Mostovoi, Yad. Fiz. **53**, 418 (1991) [Sov. J. Nucl. Phys. **53**, 260 (1991)].
[19] F. Glück and K. Tóth, Phys. Rev. D **41**, 2160 (1990); K. Tóth and F. Glück, *ibid.* **40**, 119 (1989).
[20] F. Glück and K. Tóth, Phys. Rev. D **46**, 2090 (1992).
[21] D. H. Wilkinson, Nucl. Phys. **A377**, 474 (1982).
[22] A. Sirlin, Phys. Rev. **164**, 1767 (1967).
[23] D. H. Wilkinson, in *Nuclear Physics with Heavy Ions and Mesons,* edited by R. Balian, M. Rho, and G. Ripka (North-Holland, Amsterdam, 1978), p. 877.
[24] L. Grenacs, Annu. Rev. Nucl. Part. Sci. **35**, 455 (1985).
[25] D. W. Hetherington, A. Alousi, and R. B. Moore, Nucl. Phys. **A494**, 1 (1989); J. B. Camp, Phys. Rev. C **41**, 1719 (1990).
[26] Particle Data Group, J. J. Hernández *et al.*, Phys. Lett. B **239**, 1 (1990).
[27] W. Jaus and G. Rasche, Nucl. Phys. **A143**, 202 (1970).
[28] W. Jaus, Phys. Lett. **40**, 616 (1972).
[29] A. Sirlin and R. Zucchini, Phys. Rev. Lett. **57**, 1994 (1986).
[30] W. Jaus and G. Rasche, Phys. Rev. D **35**, 3420 (1987).
[31] A. Sirlin, Phys. Rev. D **35**, 3423 (1987).
[32] J. C. Hardy *et al.*, Nucl. Phys. **A509**, 429 (1990).
[33] O. Nachtmann, Z. Phys. **215**, 505 (1968).
[34] R. Christian and H. Kühnelt, Acta Phys. Austriaca **49**, 229 (1978).
[35] V. K. Grigor'ev *et al.*, Yad. Fiz. **6**, 329 (1967) [Sov. J. Nucl. Phys. **6**, 239 (1967)].
[36] Chr. Stratowa, R. Dobrozemsky, and P. Weinzierl, Phys. Rev. D **18**, 3970 (1978).
[37] J. Byrne, Rep. Prog. Phys. **45**, 115 (1982).
[38] A. Garcia and M. Maya, Phys. Rev. D **17**, 1376 (1978).
[39] K. Tóth, K. Szegö, and A. Margaritis, Phys. Rev. D **33**, 3306 (1986).