Scaling exponent of multiplicity fluctuation in phase transition

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It is suggested that the study of the multiplicity fluctuations of hadrons produced in high-energy heavy-ion collisions can be used as a means to detect evidence of a quark-hadron phase transition. In analogy with the photocount problem at the onset of lasing in nonlinear optics, we use the coherent-state description in the framework of the Ginzberg-Landau theory. It is shown that the normalized factorial moments of the multiplicity distribution exhibit a complicated behavior as functions of the resolution scale and parameters that describe the phase transition. However, there is a scaling behavior that is universal, and a scaling exponent can be determined that is independent of the details of the phase transition.

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I. INTRODUCTION

Although it has been known for a long time that fluctuations are large for statistical systems near their critical points, the idea that multiplicity fluctuation of hadrons produced in high-energy heavy-ion collisions can be used as a measure of whether a quark-gluon system has undergone a phase transition has only recently been suggested [1,2]. At the present stage of our investigation along this line it is not certain whether there are complications that may render the signature ambiguous, since the hadronization process in soft QCD interaction is far from being well understood, let alone the detailed properties of phase transition (PT). In the framework of a Monte Carlo simulation (ECCO [3]) that can reproduce the data of *pp* collisions [4] on intermittency [5], it has been shown that conventional hadron-hadron and nucleus-nucleus collisions at high energies without PT yield quantitatively different results on multiplicity fluctuation [6,7], when compared to what is theoretically predicted for the case with PT [2]. Thus it is promising that this line of investigation may provide a useful diagnostic tool to detect the formation of a quark-gluon plasma, when the experiments on heavy-ion collision can be carried out at higher energies. In this paper we elaborate further the investigation initiated in Ref. [2] and examine in detail the scaling behavior of the factorial moments of the multiplicity distribution when the system undergoes a second-order PT.

It is useful to point out that, whereas the theory and experiment for a quark-hadron PT are still in their infancy, there is another area of physics where the study of multiplicity fluctuation in a similar type of PT is already in its mature age. That is the problem of photon production at the threshold of lasing. It has been known for a long time that the threshold behavior in nonlinear optics is a problem concerning the onset of instability of certain modes of oscillation in the optical system and that the symmetry-changing instability of stationary nonequilibrium states is intimately related to the general phenomena of a second-order PT [8]. Indeed, the stationary and time-dependent solutions of the Fokker-Planck equation for lasers have been thoroughly investigated, and, of course, in such a mature subject the agreement between theory and experiment has been well established [9]. In laser physics the pump parameter is under experimental control so it is possible to tune the laser system to different points around the critical point. The photocount problem in nonlinear optics may therefore serve as a guide to the study of the hadron production problem in heavy-ion collisions, in which there is no experimental control of the temperature or any other variable specifying the proximity to PT—provided, of course, that the analogy can be shown to be a helpful guide.

Conversely, the study of scaling properties associated with intermittency [5] in hadronic and nuclear collisions may provide insight into problems not yet fully explored in optical instability. More specifically, we shall derive a scaling behavior in this paper with a universal exponent that has been overlooked in laser physics, and should be checked experimentally. A confirmation of the numerical value of the exponent would not only cement the theoretical basis on which such a scaling exponent is predicted, but also offer realistic evidence for what one should expect in heavy-ion collisions if our description of the quark-hadron PT that shares a common basis with that of lasing transition is correct.

II. GINZBURG-LANDAU THEORY OF SECOND-ORDER PHASE TRANSITION

For quark-hadron PT there are two types to consider: confinement and chiral transitions. Lattice gauge calculations indicate that for two flavors the PT is most likely of the second order for both types at the same critical temperature T_c [10,11]. When strange quarks are included, it may become a weak first-order PT [12]. In this paper we shall confine our consideration to the case of the second-order PT. The problem of pion production becomes particularly simple for $f=2$ chiral-symmetry breaking when framed in the context of the σ model [13], as far as the excitation of the pion mode is concerned. However, the σ model does not provide any information on the multiplicity of pions produced in a collision, which involves physics at a scale larger than what the microscopic theory of chiral transition can readily describe. The same is true with the treatment of the confinement problem in lattice gauge theory.

Since the principal observables of the remnants of PT in heavy-ion collisions are hadron multiplicities and correlations, it is essential that we have a framework in which such observables can be calculated. It does not mean that we must abandon the microscopic theory, but it suggests that a somewhat more macroscopic description may be more efficient.

The situation reflects well the opposite scenario in superconductivity, where before the advent of the BCS theory [14] there was first the Ginzburg-Landau (GL) theory [15], which captured the essence of the superconducting transition and the macroscopic nature of the superconducting state. In that theory a pseudo wave function $\psi(\mathbf{r})$ is introduced to serve as a complex order parameter, and $|\psi(\mathbf{r})|^2$ is related to the local density of superconducting electrons. Although $\psi(\mathbf{r})$ was later related to the gap parameter in the BCS theory, that connection is not of urgent concern to us now, if our immediate aim is to find a phenomenological framework suitable for describing multiplicity fluctuations. The fact that the GL theory can adequately describe the phenomenological features of type-II superconductivity (which shows a second-order PT) and that $|\psi(\mathbf{r})|^2$ can be related to some aspect of the observables relevant to our problem is more important to our search for an appropriate formalism.

In nonlinear optics the GL theory has long been known to be directly relevant to the problem of counting phonon multiplicity at the threshold of lasing [8]. In laser physics the coherent state $|\phi\rangle$ provides an essential theoretical basis, where $|\phi\rangle$ is the eigenstate of the annihilation operator $\hat{a}(t)$, i.e.,

$$\hat{a}(t)|\phi\rangle=\phi(t)|\phi\rangle \ . \tag{2.1}$$

Here $\phi(t)$ is a complex function of time t, in terms of which the average photon multiplicity in a time interval T at t is

$$\begin{aligned}\bar{n}(T,t)&=\left\langle\phi\left|\int_t^{t+T}dt'\hat{a}^\dagger(t')\hat{a}(t')\right|\phi\right\rangle\\&=\int_t^{t+T}dt'|\phi(t')|^2 \ .\end{aligned} \tag{2.2}$$

Thus the rate of photon emission is

$$\rho(t)=|\phi(t)|^2 \ . \tag{2.3}$$

It plays the same role as $|\psi(\mathbf{r})|^2$ for the superconducting electrons. In the usual formulation of the coherent state of a stationary system [16], $|\phi\rangle$ can be expressed in the photon-number representation as

$$|\phi\rangle=\sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}e^{-|\phi|^2/2}|n\rangle \ , \tag{2.4}$$

so the probability of having n photons in a pure coherent state is Poissonian:

$$|\langle n|\phi\rangle|^2=\frac{|\phi|^{2n}}{n!}e^{-|\phi|^2} \ . \tag{2.5}$$

For a time-dependent lasing system, since $|\phi(t)|^2$ is identified with the rate of photon emission, as indicated in (2.3), the distribution of photon multiplicity at t in the interval T is

$$P_n^0(T,t)=\frac{1}{n!}\bar{n}(T,t)^n e^{-\bar{n}(T,t)} \ , \tag{2.6}$$

where $\bar{n}(T,t)$ is given by (2.2). Since the convolution of Poissonian distributions remains Poissonian, we can increase T in (2.2), and (2.6) remains valid.

In the absence of a theory that can describe hadron production in heavy-ion collisions with a PT, we adapt the laser formalism for our problem and treat hadron production (mostly pions) as photon emission. Instead of time we consider rapidity (and its generalization) to be discussed below. In view of the success that the GL theory has in describing superconducting and lasing transitions, we adopt that theory also for a macroscopic treatment of the quark-hadron PT. The order parameter will be in exact analogy with $\psi(\mathbf{r})$ and $\phi(t)$ in the former two cases. We shall also suspend the usual and legitimate concern about the GL theory being a mean-field theory, i.e., as such it is not ordinarily reliable in predicting the critical exponents for any statistical system whose dimension is less than the critical dimension (which for the Ising system is 4) [17]. Our reasons are as follows. (a) It works well for the superconducting and lasing transitions and may also be so for the quark-hadron PT, since the thermodynamical limit of infinite particle multiplicity is not taken, at least in the latter two cases. (b) We shall not calculate the usual critical exponents but study instead a scaling exponent in connection with intermittency, not necessarily at the critical point. (c) The implications of the GL theory for this problem should be investigated in any case whether or not its validity can be confirmed later by a microscopic theory. Even if the critical exponents turn out to be incorrect, it would still be of interest to see whether the scaling exponent remains correct. (d) The critical exponents for quark-hadron PT are not likely ever to be measured, but the scaling exponent is a property of the factorial moments that are directly measurable, as we shall explain below.

In order to pose our problem without being encumbered by all the complications of the realistic heavy-ion collisions, let us assume that for large enough nuclei at high enough energy a cylindrical volume of quark-gluon plasma is formed at some temperature T, expanding longitudinally at relativistic speed. Let us further assume that relativistic hydrodynamics can be applied to the system and that the local thermodynamical quantities are boost invariant in the longitudinal direction in accordance to the similarity flow described by Bjorken [18]. Thus the longitudinal coordinate of the system is the spatial rapidity

$$\eta=\frac{1}{2}\ln\frac{t+z}{t-z} \ , \tag{2.7}$$

which is to be identified with the momentum rapidity

$$y=\frac{1}{2}\ln\frac{p_0+p_3}{p_0-p_3} \ . \tag{2.8}$$

Setting $\eta = y$ enables us to relate the coordinate variable of an expanding system undergoing PT to the kinematical variable, in terms of which the produced hadrons are measured. The azimuthal angle φ is the same in the coordinate and momentum spaces. Thus if the transverse momentum p_T is integrated over, corresponding to considering slices of the cylinder, the energy or entropy density of the physical system in the (η,φ) space can alternatively be formulated in the (y,φ) space. If one wants to include the third dimension, one could add the radial r, or the transverse p_T, in their respective spaces with the understanding that there is a nontrivial relationship between r and p_T that depends on the hydrodynamics of the problem [19]. We shall use z to denote (η,φ,r) collectively, as well as (y,φ,p_T), although strict identification is implied only for the first two pairs.

Instead of $\phi(t)$ in (2.2) we shall use $\phi(z)$ as the coherent-state variable of our system. If the system is in a pure coherent state that leads to hadronization, the corresponding average hadron multiplicity in a volume V is

$$\bar{n}_V = \int_V dz |\phi(z)|^2 \ . \tag{2.9}$$

We have used dz to denote an elementary volume in the d-dimensional space of (η,φ,r) or any projected subspace thereof. In referring to the cell volume V in the space of observable variables, one should choose a space of cumulative variables $(\tilde{y},\tilde{\varphi},\tilde{p}_T)$ [20] in which the hadrons are uniformly distributed so that $\bar{n}_V$ in (2.9) depends only on the cell size and not on where the cell is located. Since our final result will not depend on the details of such choices, we proceed formally on the basis that the z space of any chosen dimension d is uniform.

In the GL theory the free energy of the system is

$$F[\phi] = \int dz \, [a|\phi(z)|^2 + b|\phi(z)|^4 + c|\partial\phi/\partial z|^2] \ , \tag{2.10}$$

where the integration is over all volume of the system under consideration. The parameters b and c may be regarded as constants in the neighborhood of the critical temperature T_c, and are positive, but a changes sign as

$$a = a_1(T - T_c) \tag{2.11}$$

with $a_1 > 0$. The values of a_1, b, and c are not known in the quark-hadron PT. Our aim is to derive a feature of the PT that is relatively independent of these parameters. The nonkinetic part of (2.10), i.e., the first two terms on the right-hand side (RHS), has its minimum at $|\phi|^2 = 0$ for $T > T_c$ until a becomes negative at $T < T_c$, whence the minimum occurs at

$$|\phi_0|^2 = -a/2b > 0 \ . \tag{2.12}$$

If the system can be adequately described by the coherent state at ϕ_0, then the average hadron density is

$$\rho_0 = \bar{n}_V / V = -a/2b \ , \tag{2.13}$$

provided, of course, $a < 0$. This is the hadron phase with $T < T_c$. For $T > T_c$, the minimum of the GL potential being at $|\phi|^2 = 0$, (2.13) does not apply, and there are no hadrons. That is the quark phase, for which we have $\rho_0 = 0$. The dependence of ρ_0 on a in the neighborhood to T_c is shown in Fig. 1 by the dashed line.

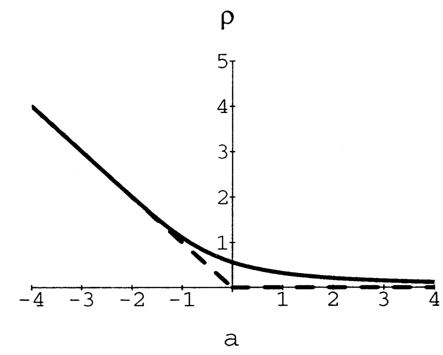

FIG. 1. Hadron number density ρ in units of $(\delta^d b)^{-1/2}$ plotted against the parameter a in units of $2(b\delta^d)^{1/2}$.

In reality a thermal system need not remain at the potential minimum. The free energy (2.10) specifies how the system fluctuates from ϕ_0. More specifically, when hadrons are produced, the hadron multiplicity distribution is not like (2.6) for a pure state ϕ_0 but is given by the functional integral

$$P_n = Z^{-1} \int \mathcal{D}\phi P_n^0 e^{-F[\phi]} \ , \tag{2.14}$$

where

$$Z = \int \mathcal{D}\phi e^{-F[\phi]} \ , \tag{2.15}$$

$$P_n^0 = \frac{1}{n!} \left[\int_V dz |\phi(z)|^2 \right]^n \exp\left[-\int_V dz |\phi(z)|^2 \right] \ . \tag{2.16}$$

Thus the probability of having a large n in V is controlled by how much ϕ can deviate from ϕ_0 as specified by the thermodynamical factor $e^{-F[\phi]}$; it is no longer the Poissonian tail of P_n^0. In fact, even the average hadron density is now different.

In the notation

$$\langle n \rangle = \sum_n n P_n \quad \text{and} \quad \rho = \langle n \rangle / V \ , \tag{2.17}$$

one can show that, in the case of uniform $\phi(z)$ [see (3.18) below],

$$\rho = \frac{-a}{2b} + \frac{e^{-Va^2/4b}}{(\pi b V)^{1/2}\{1 + \mathrm{erf}[-(a/2)(V/b)^{1/2}]\}} \ . \tag{2.18}$$

A sketch of its dependence on a is shown by the solid line in Fig. 1. Evidently, the break at $a = 0$ (dashed line) is now smoothed out, implying that the hadron multiplicity is not strictly zero even at $T > T_c$ (i.e. $a > 0$). However, since $\langle n \rangle \propto \sqrt{V}$ in that region, it can readily be distinguished from the fully developed hadron phase, for which $\langle n \rangle \propto V$, in the large V limit. This blurring of the transition region for photocount at lasing threshold was expected and observed long ago [21]. We should not be surprised if a similar continuous PT occurs for hadron production. On the other hand, it should serve notice on us to be careful about how small the cell volume V can be set, if we want to use ρ as the order parameter. That is the subject we consider next.

III. INTERMITTENCY

An effective way to investigate the nature of the multiplicity fluctuation in high-energy collisions is to examine the dependence of the normalized factorial moments F_q on the bin width δ in rapidity [5,22], where

$$F_q = \frac{\langle n(n-1)\cdots(n-q+1)\rangle}{\langle n\rangle^q} . \tag{3.1}$$

Here n is the number of hadrons detected in δ in an event, and the averages are taken over all events. This is the vertical analysis at one bin. To improve statistics in practice, F_q may be averaged (horizontally) over many bins. The multiplicity fluctuation is said to exhibit intermittency behavior if, as δ is decreased, there exists a range of δ in which F_q has a power-law behavior

$$F_q \propto \delta^{-\varphi_q} , \tag{3.2}$$

where φ_q is referred to as the intermittency index. In a large variety of experiments involving e^+e^-, μp, pp, pA, and AA collisions, the behavior (3.2) has been observed [23].

The main emphasis in the line of investigation pursued in Ref. [2] and in this paper is the use of intermittency analysis to explore universal characteristics of quark-hadron PT in the GL theory. Since we have no control over temperature (as in superconductivity or pump parameter as in lasers), we have no way to tune our system to specific points in the phase diagram and examine its behavior as it goes through PT. Moreover, only a limited number of particles are produced in each collision (in the hundreds), so many collisions (in the hundreds of thousands events) are needed to reduce statistical fluctuation. Nearly the only parameter that we can vary, apart from collision energy and nuclear size, is the size of a cell in phase space, in which to observe the multiplicity fluctuations. That is just the central theme of intermittency. However, our goal must go beyond the usual power-law behavior of (3.2), which may or may not be a reality when there is a quark-hadron PT. What we aim for is a universal behavior that is independent of whether (3.2) is fully realized and is insensitive to the details concerning the PT. In this section we examine whether (3.2) is valid in the GL theory.

Defining

$$f_q = \langle n(n-1)\cdots(n-q+1)\rangle = \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} P_n \tag{3.3}$$

we obtain, using (2.14) and (2.16),

$$f_q = Z^{-1} \int \mathcal{D}\phi \left[\int_V dz |\phi(z)|^2 \right]^q e^{-F[\phi]} , \tag{3.4}$$

where V is the volume of the cell in which the factorial moment f_q is measured. We emphasize that f_q is directly measurable. Instead of just the rapidity bin mentioned in connection with (3.1), we now generalize to d dimensions in all or any subspace of $(\tilde{y},\tilde{\varphi},\tilde{p}_T)$ discussed in the previous section. Thus we identify $V=\delta^d$. It follows from (3.1) that

$$F_q = f_q / f_1^q . \tag{3.5}$$

In Ref. [5] it is shown that, if the multiplicity fluctuation is only statistical, there would be no dependence of F_q on δ. That result is trivially obtained here by noting that no dynamical fluctuation means that ϕ does not deviate from ϕ_0, so $f_q = (\delta^d |\phi_0|^2)^q$ and $F_q = 1$. It is therefore clear that the nontrivial property of F_q is a measure of the dynamical fluctuations of ϕ from ϕ_0 as prescribed by the GL free energy $F[\phi]$. We should remark that since F_q is defined in terms of the hadron multiplicity n, it has meaning only in the hadron phase; hence, (3.4) and (3.5) are to be calculated below only for $a<0$.

To perform the functional integration in general with (2.10) for $F[\phi]$ is difficult. In Ref. [2] the simple case of uniform ϕ is considered, which is equivalent to setting $c=0$. The integrations can then be performed exactly, and rather interesting results are obtained. One important result is that F_q does not depend on a, b, δ, and d separately, but on one variable x only, where

$$x = \frac{a^2 \delta^d}{2b} . \tag{3.6}$$

The dependence is shown in Fig. 2. Evidently, for every fixed q, there is a rise region in $\ln F_q$ vs $-\ln x$, and then there is a saturation. If intermittency means (3.2) with $\delta \to 0$, then Fig. 2 implies no intermittency. However, that is a mathematical limit with little physical significance, since in heavy-ion collisions the total multiplicity in any event is finite; thus when δ is small enough, the value of n in a cell would overwhelmingly be only 0 or 1, independent of δ.

Another reason why the $\delta \to 0$ limit should not be taken is that (when $c=0$) we have, from (2.10) and (3.4),

$$f_q = \delta^{dq} I_q / I_0 , \tag{3.7}$$

where

$$I_q = \int_0^\infty d|\phi|^2 |\phi|^{2q} e^{\delta^d(|a||\phi|^2 - b|\phi|^4)} . \tag{3.8}$$

The exponent of the exponential term in (3.8) is propor-

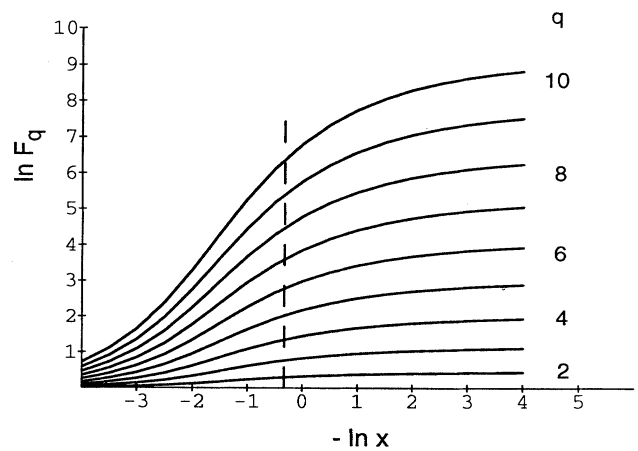

FIG. 2. Log-log plot of F_q vs $1/x$. The region to the right of the dashed line may not be reliable, depending on the parameters a and δ.

tional to the cell volume δ^d instead of the volume over all space, as required in (2.10), because the part of the integral outside the cell is common in both the numerator and denominator of (3.4) and is therefore canceled out, leaving just the part proportional to δ^d in (3.8). It is then clear that when δ becomes very small, $|\phi|$ can become large before the exponential damping suppresses the integrand in (3.8). When that happens, the GL's $F[\phi]$ is no longer valid, since the $|\phi|^6$ term is not included on the grounds that $|\phi|$ is not supposed to be large enough to make it important. A way of estimating the appropriate range of δ in which to examine the properties of f_q is to require that the main contribution to the integral in (3.8) is from the region where $F[\phi]$ is negative, i.e., $0<|\phi|^2<-a/b$, for $-a$ not too close to zero. Thus if we demand that the exponential factor in (3.8) at $|\phi|=|\phi_0|$ to be at least twice as large as at $|\phi|=0$ or $-a/b$, i.e.,

$$e^{\delta^d(|a||\phi_0|^2-b|\phi_0|^4)}>2\ , \tag{3.9}$$

then we get

$$\delta^d>\delta_0^d\equiv(4\ln 2)b/a^2\ . \tag{3.10}$$

This implies

$$x>x_0=2\ln 2\ , \tag{3.11}$$

which corresponds to

$$-\ln x<-\ln x_0=-0.327\ . \tag{3.12}$$

It means that in Fig. 2 we should take more seriously the region to the left of the dashed line. To its right the result on F_q is not necessarily invalid, since at small $-a$ the usual GL expression, (2.10), without a $|\phi|^6$ term can still be adequate for $|\phi|^2$ somewhat larger than $-a/b$. Thus the dashed line in Fig. 2 does not mark the absolute bound of the validity of the GL theory. Depending on the detailed parameters of the problem, the region of validity may extend to the right of the dashed line. However, we shall restrict our attention only to the region on the left that is sure to be reliable. Note that the region of rising F_q is now emphasized as bearing the true consequence of the GL theory.

Another feature to be noticed in the region of rising F_q is that the local slope φ_q increases with $-\ln x$. Thus with δ^d/b kept fixed, the decrease of $|a|$ increases the intermittency index φ_q, reflecting more fluctuation. This is as one should expect for T closer to T_c.

From (3.3) and (3.7) we have

$$\langle n\rangle=f_1=(\delta^d/b)^{1/2}J_1/J_0\ , \tag{3.13}$$

where

$$J_q(\alpha)=\int_0^\infty dt\ t^q e^{2\alpha t-t^2}\ , \tag{3.14}$$

$$\alpha=-\frac{a}{2}\left[\frac{\delta^d}{b}\right]^{1/2}\ . \tag{3.15}$$

Carrying out the integration in (3.14) for $q=0$ and 1, we get

$$J_0(\alpha)=\frac{\sqrt{\pi}}{2}e^{\alpha^2}(1+\operatorname{erf}\alpha)\ , \tag{3.16}$$

$$J_1(\alpha)=\frac{1}{2}+\alpha J_0(\alpha)\ . \tag{3.17}$$

It then follows from (3.13) that

$$\rho=\frac{\langle n\rangle}{\delta^d}=\frac{-a}{2b}+\frac{e^{-\alpha^2}}{(\pi b\delta^2)^{1/2}(1+\operatorname{erf}\alpha)}\ . \tag{3.18}$$

Comparing this with (2.12), we see that it is the last term that accounts for the change in Fig. 1 from the dashed line to the solid line.

To see the effect of the restriction (3.10) on ρ let us rewrite (3.18) in the form

$$\rho=(\delta^d b)^{-1/2}\left[\alpha+\frac{1}{2J_0(\alpha)}\right]\ . \tag{3.19}$$

Since (3.6) and (3.15) imply

$$x=2\alpha^2 \tag{3.20}$$

we have, at $\alpha_0=\sqrt{x_0/2}=0.83$,

$$\frac{1}{2J_0(\alpha_0)}=0.16\ . \tag{3.21}$$

Thus for $x>x_0$, we obtain, from (3.19),

$$(\rho-\rho_0)/\rho_0<0.16/0.83=0.19\ , \tag{3.22}$$

where ρ_0 is given in (2.13). It means that in Fig. 1 the percentage deviation of the solid line from the dashed line should be less than 20%. This gives a rough guide on the region, where ρ can reliably be calculated. However, as we have discussed just below (3.12), the bound (3.10) is not good when $-a$ is small, so the inequality (3.22) need only be regarded as a reference for reliability, but the violation of (3.22) does not necessarily mean unreliability. That is, (3.18) may still be valid for even smaller values of $-a$ than what (3.22) implies. The point is not very important in an actual collision process, where T decreases continuously through T_c, and the bulk of the hadrons are produced far away from the critical point, where ρ becomes large, i.e., in the large negative a region in Fig. 1.

The fact that $F_q(x)$ depends only on x allows us to use the x dependence in Fig. 2 to learn about the δ dependence for various fixed values of a and b. However, since a and b are not fixed, it would be hard to determine the phenomenologically relevant φ_q theoretically. The resolution of this problem is deferred until the next section.

The discussion so far in this section is for the case where the $c|\partial\phi/\partial z|^2$ term in (2.10) can be ignored. While that is exactly the case for the lasing transition problem [8,9], we must estimate its effect for the quark-hadron PT problem. A mean-field type approximation has been considered [24], in which the value of $\phi(z)$ in the neighboring cells surrounding the cell under study is fixed at ϕ_0. It is found that the intermittency curves of $\ln F_q$ vs $-\ln x$ depend on only one parameter γ, in addition to q, where

$$\gamma=\pi\left[\frac{cd}{|a|}\right]^{1/2}\left[\frac{a^2}{2b}\right]^{1/d}\ , \tag{3.23}$$

which is independent of the cell size δ^d. Those curves are generally similar to the ones shown in Fig. 2, but the rise region becomes more extended at higher values of γ. The saturation region is pushed out to much higher values of $-\ln x$, above 15. There are no simple features that can summarize the intermittency behavior. However, when examined differently, as will be discussed in Sec. IV, a universal behavior emerges that does not differ from the $c=0$ case by very much. The reader is referred to Ref. [24] for details.

Here we discuss another way of considering the $c\neq 0$ case. It is well known [17] that when spatial dependence is introduced into the problem by the kinetic term, the correlation length is

$$\xi=(c/|a|)^{1/2} , \tag{3.24}$$

and, if $b=0$, the correlation function behaves as

$$C(\mathbf{r}_1-\mathbf{r}_2)\propto e^{-|\mathbf{r}_1-\mathbf{r}_2|/\xi}/|\mathbf{r}_1-\mathbf{r}_2| . \tag{3.25}$$

To estimate the effect of this correlation in our problem, let us make the approximation that all points separated by a distance less than ξ are fully correlated, and no correlation otherwise. Thus for the functional integral in (3.4) we discretize the z space into cells of size δ^d, and not only do we let $\phi(z)$ be constant within each cell, labeled ϕ_i for the ith cell, but we also implement our approximation of the dynamical correlation by demanding that all ϕ_i be equal for cells within a block of size ξ^d. Let the number of cells in all space be M, and those in the block be

$$N=(\xi/\delta)^d . \tag{3.26}$$

At the center of the block is the cell under study, labeled j, i.e., it is the jth cell in which the moments f_q are measured. Let the cells in the block be labeled $i\in\{N_j\}$. We then have

$$\begin{aligned} Z&=\int\left[\prod_{i=1}^{M} d^2\phi_i\right]\exp\left\{-\delta^d\left[\sum_{i=1}^{M}(a|\phi_i|^2+b|\phi_i|^4)+\sum_{(i,i')}c\delta^{-2}|\phi_i-\phi_{i'}|^2\right]\right\}\\ &=Z_{i\in\{N_j\}}Z_{i\notin\{N_j\}} , \end{aligned} \tag{3.27}$$

where

$$Z_{i\in\{N_j\}}=\int\left[\prod_{i=1}^{N} d^2\phi_i\right]\left[\prod_{i=1}^{N}S(\phi_i)\right]T(\phi_i,\phi_{i'}) , \tag{3.28}$$

$$S(\phi_i)=e^{\alpha|\phi_i|^2-\beta|\phi_i|^4}, \quad \alpha=\delta^d|a|, \quad \beta=\delta^d b . \tag{3.29}$$

$T(\phi_i,\phi_{i'})$ is the exponential term involving neighboring pairs $(\phi_i,\phi_{i'})$ in the last term of the first line in (3.27). Our approximation is to set

$$T(\phi_i,\phi_{i'})=\delta(\phi_i-\phi_{i'}), \quad i,i'\in\{N_j\} \tag{3.30}$$

so that

$$Z_{i\in\{N_j\}}=\int d^2\phi S^N(\phi) . \tag{3.31}$$

For cells not in the block, $Z_{i\notin\{N_j\}}$ cancels the corresponding integrals in the numerator of (3.4), and will not be of any concern hereafter. We thus get

$$f_q=\frac{\int d^2\phi(\delta^d|\phi|^2)^q S^N(\phi)}{\int d^2\phi S^N(\phi)} . \tag{3.32}$$

This is essentially in the same form as (3.7) and (3.8) for the $c=0$ case, except that the δ^d factor in the exponent of (3.8) is replaced by

$$N\delta^d=\xi^d . \tag{3.33}$$

This means that for $\delta>\xi$ we may ignore the effects of the kinetic term, but when δ is decreased to below ξ, then the effective cell size for the $c\neq 0$ case is equivalent to ξ^d in the $c=0$ case. We may therefore continue to use the results obtained for the $c=0$ case, so long as we do not let δ get smaller than ξ. Or, stated more accurately, when the experimental δ is decreased below ξ, we use the result of the $c=0$ case but with δ replaced by ξ. This is, of course, the physical meaning of having nontrivial correlations with a range of ξ. In terms of the x variable it means that x should not be decreased to below

$$x_\xi=\frac{a^2\xi^d}{2b} , \tag{3.34}$$

so Fig. 2 remains valid for

$$-\ln x<-\ln x_\xi . \tag{3.35}$$

What x_ξ is depends on all the parameters (a,b,c,d), so no general statement can be made about the intermittency index φ_q. However, the important conclusion here is the validity of Fig. 2 for the rising part of F_q. Our final conclusion is again deferred to the next section after we derive the universal property of F_q that is insensitive to the parameters (a,b,c,d), whereupon the precise value of x_ξ for the bound (3.35) becomes immaterial.

IV. SCALING EXPONENT

As we have seen in the previous section, the self-similarity behavior of F_q is not clear cut. There is no extended region of δ, or x, in which one can identify the power-law dependence (3.2) with unambiguous slopes. When the kinetic term in the free energy $F[\phi]$ is included, there is some minor variation on the prediction of the behavior of $F_q(x)$, depending on what approximation is used, but roughly Fig. 2 is still valid for the rising portion. Because there are dependences on all the parameters a, b, c, and d, which are not known in heavy-ion collisions, there is no way to determine the intermittency indices φ_q to be compared with experiment. If this is all that we can learn about F_q from the GL theory, then clearly this avenue of investigation is doomed to fail as a possible way to reveal the signature of quark-hadron PT.

Fortunately, there is a universal scaling behavior, found in Refs. [2] and [24], that saves this approach from failure and turns it into a very promising possibility. In those references it is by numerical computation that the log-log plots of F_q vs F_2 are shown to exhibit approximate linear behavior. Here in this section we pursue an analytical derivation of the scaling behavior as far as possible.

Let us first summarize our problem. From (3.5) and (3.7) we have

$$F_q = I_q I_1^{-q} I_0^{q-1} \tag{4.1}$$

where I_q, defined in (3.8), differ from J_q, defined in (3.14), only by a multiplicative factor, i.e.,

$$I_q = h(q) J_q \ . \tag{4.2}$$

$h(q)$ is of the form

$$h(q) = c_1 c_2^q \ , \tag{4.3}$$

where c_1 and c_2 are constant parameters in the integrals. Such factors get canceled out in (4.1) so we have

$$F_q = J_q J_1^{-q} J_0^{q-1} \ . \tag{4.4}$$

$J_0(x)$ and $J_1(x)$ are given explicitly in (3.16) and (3.17). By partial integration J_q can be expressed in a recursion formula

$$J_q(\alpha) = \alpha J_{q-1}(\alpha) + \frac{q-1}{2} J_{q-2}(\alpha) \tag{4.5}$$

where α is given in (3.15) and is related to x by

$$\alpha = (x/2)^{1/2} \ . \tag{4.6}$$

We have seen in the previous section that the dependence of F_q on x, thus on α, is complicated, and the dependence on δ cannot be verified experimentally.

Since F_q is directly measurable for any δ, our quest for phenomenological relevance leads us to investigate the relationship among F_q. The recursion relation (4.5) facilitates our inquiry into that. To that end let us define, for $q \geq 1$,

$$L_q = \frac{2\alpha J_q}{q J_{q-1}} \ . \tag{4.7}$$

Then we obtain, from (4.5),

$$\frac{q}{2\alpha^2} L_q = 1 + \frac{1}{L_{q-1}} \ . \tag{4.8}$$

Since (3.14) implies that

$$\frac{dJ_q(\alpha)}{2d\alpha} = J_{q+1}(\alpha) \ , \tag{4.9}$$

it follows from (4.4) that

$$\frac{d \ln F_q(\alpha)}{d\alpha} = K_q(\alpha)/\alpha \ , \tag{4.10}$$

where

$$K_q(\alpha) = (q+1) L_{q+1}(\alpha) - 2q L_2(\alpha) + (q-1) L_1(\alpha) \ . \tag{4.11}$$

From (4.10) we get

$$\frac{d \ln F_q}{d \ln F_2} = \frac{K_q}{K_2} \ . \tag{4.12}$$

The RHS can be determined algebraically in terms of L_1 and L_2 by applying (4.8) recursively; however, its dependence on α is not simple, though straightforward to compute.

Equation (4.12) gives the local slope of $\ln F_q$ vs $\ln F_2$. If it is approximately constant, then we would have the scaling behavior

$$F_q \propto F_2^{\beta_q} \ . \tag{4.13}$$

This behavior describes the relationship between F_q and F_2 over a range of F_2, irrespective of their separate dependences on α. In that sense β_q summarizes the scale invariance property in the global scale. If such a property exists, then the local slope

$$\beta_q(\alpha) = K_q(\alpha)/K_2(\alpha) \tag{4.14}$$

should be approximately independent of α. Emperically, it has been found that (4.13) is approximately satisfied by the data of a wide variety of collision processes [25], all of which are not related to the GL theory discussed here.

In Fig. 3 we show the dependence of $\beta_q(\alpha)$ on α for q ranging from 3 to 10. Evidently, they are remarkably insensitive to the value of α. In a plot of $\ln F_q$ vs $\ln F_2$ the small variations of the local slopes would not be perceptible and a straight-line fit for each q would be excellent. The range of values of α from 0.3 to 2.0 corresponds to $-2 < -\ln x < 1.7$, which covers the rise region in Fig. 2. Independence on α implies that it is immaterial which section of Fig. 2 is taken seriously as the valid consequence of the GL theory. The same β_q values result. In view of our discussion near the end of the previous sec-

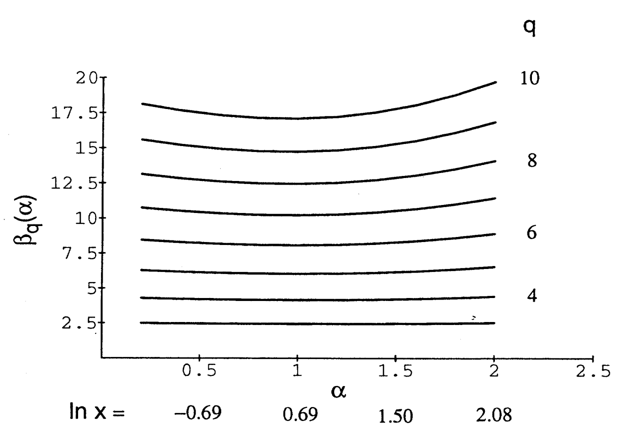

FIG. 3. Scaling behavior of $\beta_q(\alpha)$. The numbers for $\ln x$ correspond to the values of α immediately above.

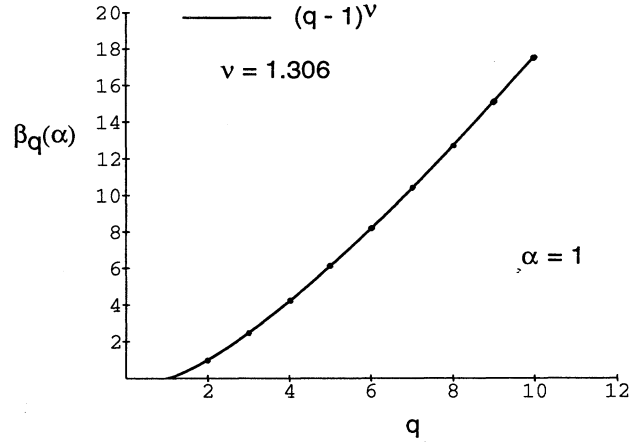

FIG. 4. A fit of $\beta_q(\alpha)$ at $\alpha=1$ by $(q-1)^\nu$ shown in solid line.

tion on the effective block size ξ^d in connection with the $c\neq0$ case and the resultant validity of Fig. 2 for $x>x_\xi$, we can now conclude that the scaling behavior holds true also for the complete GL free energy including the kinetic term, since it is not crucial where $-\ln x_\xi$ is, so long as it is not outside Fig. 2, or $\alpha>3$.

To summarize Fig. 3, we now point out the amazing fact that for every value of α in the range $0.3<\alpha<2$, $\beta_q(\alpha)$ satisfies very accurately the formula

$$\beta_q(\alpha)=(q-1)^{\nu(\alpha)} . \tag{4.15}$$

For example, at $\alpha=1.0$ the values of β_q are shown by dots in Fig. 4, and the fit by (4.15) is shown by the solid line for $\nu=1.306$. Doing this type of fitting for the whole range of α results in $\nu(\alpha)$, whose dependence on α is shown in Fig. 5. The values fall within the range

$$\nu=1.305\pm0.010 . \tag{4.16}$$

Since the variation of ν throughout the α range is less than 0.8%, we are well justified to regard ν as being constant and refer to it as the scaling exponent.

The value of ν in (4.16) is, within errors, in agreement with that obtained in Ref. [24]: 1.316 ± 0.012. In that

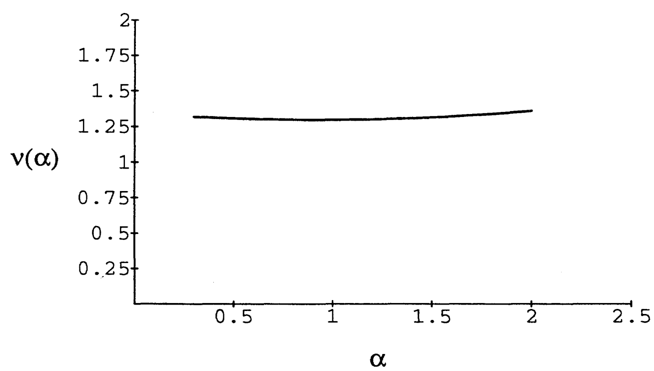

FIG. 5. The value of ν for a range of values of α.

reference the kinetic term is treated by a mean-field type approximation. Here, we are to retain the $c=0$ result as shown in Fig. 5, but the bound $x>x_\xi$ would restrict the range of α from being extended to the low end. Depending on the value of c, the relevant range may well be near $\alpha\sim3$, resulting in a value of ν on the high side of (4.16), in better agreement with 1.316 ± 0.012 quoted above.

For definiteness we may regard

$$\nu=1.31 \tag{4.17}$$

as our final result for the scaling exponent in the GL theory. It is universal in the sense that it is independent of the parameters a, b, c, and d. This universality makes the result phenomenologically significant because of our ignorance about those parameters in heavy-ion collisions. Furthermore, F_q can be readily determined in the experiment. It should be emphasized that the independence of ν on the parameter a does not imply ignorance about PT, for which a changes sign. As we have stressed earlier, the factorial moments in (3.3) and (3.4) are defined only for the hadron phase, for which $a<0$. It is only for $a<0$ that the mathematical properties of F_q are valid and the independence of ν on a follows.

V. CONCLUSION

We have given arguments to support the use of the Ginzburg-Landau theory for studying quark-hadron phase transition. In that framework we have shown that there exists an experimentally feasible way of relating observables on hadron multiplicity fluctuations to a predictable property of the phase transition. It is the scaling exponent ν for the factorial moments F_q. The value of ν is 1.31, independent of the details of the parameters in the GL theory. Since the same theory is known to be applicable to the problem of a photocount at the threshold of lasing, the verification of $\nu=1.31$ in the laser experiment (which is, of course, far easier to perform than heavy-ion collisions) would accomplish the first step of validating the theoretical consideration discussed here.

Existing data on nuclear collisions so far give values in the range $\nu=1.55\pm0.12$ [2,26,27]. That is 2σ away from our prediction, so we may regard the GL theory as being inapplicable to the experiments. A stronger statement according to our criterion is that there is no evidence for phase transition in those experiments.

A larger value of ν does not mean more fluctuations. Since β_q describes the scaling behavior of F_q relative to F_2, larger ν may well be the consequence of smaller intermittency index φ_2. Indeed, most nuclear results on F_2, whether experimental [26,27] or theoretical [6], reveal rough independence on rapidity bin width, although more recent three-dimensional (3D) analyses have yielded nontrivial intermittency behavior [23,28]. For 2D Ising model [29,30] the fluctuation is large near T_c (invalidating a GL treatment) and the conjectured implication for a quark-hadron PT is that the intermittency structure is monofractal [1]; in that case the corresponding value for ν would be 1 [2]. Our present value of $\nu=1.31$ therefore implies less fluctuation than 2D Ising model, but more than what is currently observed in heavy-ion collisions. A decrease of ν in future nuclear experiments would then be really exciting.

In summary the use of hadron multiplicity fluctuations in heavy-ion collisions seems to offer a viable scheme to detect evidences of quark-hadron phase transition. If there is a phase transition, then the quark-gluon system under study must first have been in the state of a quark-gluon plasma.

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