Fermionic determinant of the massive Schwinger model

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A representation for the fermionic determinant of the massive Schwinger model, or two-dimensional QED (QED_2), is obtained that makes a clean separation between the Schwinger model and its massive counterpart. From this it is shown that the index theorem for QED_2 follows from gauge invariance, that the Schwinger model's contribution to the determinant is canceled in the weak-field limit, and that the determinant vanishes when the field strength is sufficiently strong to form a zero-energy bound state.

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Quantum electrodynamics in two-dimensional spacetime (QED_2), otherwise known as the massive Schwinger model, is defined in Euclidean space by the action

$$S[A,\overline{\Psi},\Psi]=\tfrac{1}{2}\int d^2x\, B^2+\int d^2x\, \overline{\Psi}(\not{D}+m)\Psi \ , \qquad (1)$$

where $\not{D}=\gamma\cdot(-i\nabla-e\,\mathbf{A})$ and $B=F_{01}=\partial_0 A_1-\partial_1 A_0$. Our designation of F_{01} as a magnetic field is consistent with regarding S as the Hamiltonian for a charged, massive fermion confined to a plane in the presence of a static magnetic field perpendicular to the plane. For definiteness we set $\gamma_0=-i\sigma_1$, $\gamma_1=-i\sigma_2$, where $\sigma_{1,2}$ are the Pauli matrices. The model is superrenormalizable, requiring no infinite renormalization other than a trivial renormalization of the zero-point energy. Hence e and m are finite parameters.

The case when $m=0$, known as the Schwinger model [1], is exactly soluble and has become an important tool for gaining insight into gauge field theories. It continues to generate enormous interest with some fifty papers per annum connected with the model and variations of it. The literature for the case $m\neq 0$ is sparse, the classic references remaining those in [2]. It is not thought to be exactly soluble. As might be suspected by our interpretation of the massive model's action, its fermionic determinant determines (after integrating over the fermion mass) the one-loop effective action for QED_4 in the presence of smooth, polynomial-bounded, unidirectional, static magnetic fields with a fast decrease at infinity [3]. Therefore, QED_2 contains information on physics in four dimensions and should not be regarded as just a model.

In this Brief Report we wish to consider QED_2's gauge-invariant fermionic determinant. It will appear in the computation of the theory's n-point functions as a result of integration over the fermionic degrees of freedom. The first problem is to make sense out of the formal expressions

$$\det{}^2(1-S\not{A})=\frac{\det[(\mathbf{p}-\mathbf{A})^2-\sigma_3 B+m^2]}{\det[p^2+m^2]} \ , \qquad (2)$$

on a Euclidean manifold. Here S is the free (Euclidean) fermion propagator, and e has been absorbed into A_μ. There are several ways to define determinants of Dirac operators [4], but one of these definitions seems more suited than others to grasp the known simplifications presented by QED_2, namely, the "propertime regularization" definition [5]. It defines the determinant as

$$\ln\det(\not{D}^\dagger\not{D}+m^2)=-\int_\epsilon^\infty\frac{ds}{s}\mathrm{Tr}[\exp(-s\not{D}^\dagger\not{D})]e^{-sm^2} \ , \qquad (3)$$

where ϵ is an ultraviolet cutoff which, due to superrenormalizability, can be set to zero later. Because we will always assume $m^2>0$, we feel assured that potential infrared divergences due to the zero modes of $\not{D}^\dagger\not{D}$ are regulated.

The above definition of the determinant respects gauge invariance. Therefore we should be able to calculate in the Lorentz gauge $\partial_\mu A_\mu=0$ which, in two dimensions, allows us to set $A_\mu=\epsilon_{\mu\nu}\partial_\nu\phi$, with $B=-\partial^2\phi$ and $\not{A}=i\sigma_3\not{\partial}\phi$. The antisymmetric tensor $\epsilon_{\mu\nu}$ is normalized as $\epsilon_{01}=1$. Following Alvarez [5], we consider the operator

$$\not{D}_t=-i\not{\partial}-t\not{A}=-ie^{-t\sigma_3\phi}\not{\partial}e^{-t\sigma_3\phi} \ , \qquad (4)$$

where t is a real parameter. Differentiating with respect to t,

$$\dot{\not{D}}=-\sigma_3\phi\not{D}_t-\not{D}_t\sigma_3\phi \ , \qquad (5)$$

we calculate

$$\begin{aligned}\frac{d}{dt}\ln\det(\not{D}_t^\dagger\not{D}_t+m^2)&=4\int_\epsilon^\infty ds\,\mathrm{Tr}(\sigma_3\phi\not{D}_t^2e^{s\not{D}_t^2})e^{-sm^2}\\&=-4\,\mathrm{Tr}(\sigma_3\phi e^{\epsilon\not{D}_t^2})e^{-\epsilon m^2}\\&\quad+4m^2\int_\epsilon^\infty ds\,\mathrm{Tr}(\sigma_3\phi e^{s\not{D}_t^2})e^{-sm^2} \ .\end{aligned} \qquad (6)$$

Noting that, for small ϵ,

$$\langle x|e^{\epsilon\not{D}_t^2}|x\rangle=\frac{1}{4\pi\epsilon}[1-\epsilon t\sigma_3\partial^2\phi+O(\epsilon^2)] \ , \qquad (7)$$

we obtain our definition of the fermionic determinant:

$$\ln\left[\frac{\det(\not{D}^\dagger\not{D}+m^2)}{\det(p^2+m^2)}\right]^{1/2}=\frac{1}{2\pi}\int d^2x\,\phi\partial^2\phi+2m^2\int_0^1 dt\,\mathrm{Tr}\{[(H_+^{(t)}+m^2)^{-1}-(H_-^{(t)}+m^2)^{-1}]\phi\}\;, \tag{8}$$

where $H_\pm^{(t)}=(\mathbf{P}-t\,\mathbf{A})^2\mp tB$. Note that this definition makes a clean separation between the Schwinger model, the first term, and its massive counterpart. Its perturbative expansion in powers of e is consistent with known results. Thus, it reproduces the $O(e^2)$ result for the vacuum polarization graph:

$$\ln\det=\frac{1}{2\pi}\int\phi\partial^2\phi+2m^2\int d^2x\left\langle x\left|\phi\frac{1}{p^2+m^2}B\frac{1}{p^2+m^2}\right|x\right\rangle$$
$$=-\frac{1}{2\pi}\int\frac{d^2q}{(2\pi)^2}|\hat{B}(q)|^2\int_0^1 dz\frac{z(1-z)}{q^2z(1-z)+m^2}\;, \tag{9}$$

where $\hat{B}$ is the Fourier transform of B. In addition, graphs of $O(e^4)$ and higher vanish order by order in the limit $m^2=0$, in accordance with Schwinger's original result [1].

We have not integrated by parts in the first term of Eq. (8) as is usually done. In the Lorentz gauge the auxiliary potential $\phi(\mathbf{x})=-\int d^2y\,\ln|\mathbf{x}-\mathbf{y}|B(\mathbf{y})/2\pi$ and, assuming that the flux $\Phi=\int d^2x\,B\neq0$, integration by parts is not justified here.

It is by now evident that we are assuming our potentials and fields are sufficiently smooth with enough falloff at infinity so that everything we have done makes mathematical sense. But note: if $\Phi\neq0$, A_μ in the Lorentz gauge behaves like a "winding" field with a $1/|x|$ fall off. This will have some consequence below. It might be objected that since A_μ is to be integrated over, it should be a random field. Our strategy is to first calculate the determinant in an external field in a convenient gauge, the Lorentz gauge, assuming nice potentials, then switching to whichever gauge and potentials are best suited for making sense out of the remaining integration over A_μ. Of course, any gauge-invariant constraints imposed on the determinant required, say, to make it nonvanishing, have to be honored by the functional integral.

Within the Lorentz gauge we still have the freedom to shift ϕ by a constant: $\phi\to\phi+c$. By definition, the determinant depends on A_μ and so is invariant under this shift. Referring to Eq. (8), we have consistency provided

$$e^2\Phi/2\pi=2m^2e\int_0^1 dt\,\mathrm{Tr}[(H_+^{(et)}+m^2)^{-1}-(H_-^{(et)}+m^2)^{-1}]\;, \tag{10}$$

where we have temporarily restored the coupling e. We can get rid of the t integration by setting $\lambda=et$ and differentiating both sides with respect to e. The result is

$$\Phi/2\pi=m^2\mathrm{Tr}[(H_++m^2)^{-1}-(H_-+m^2)^{-1}]\;, \tag{11}$$

where we have again absorbed e into A_μ and B and set $H_\pm=(\mathbf{P}-\mathbf{A})^2\mp B$. But the right-hand side of Eq. (11) is independent of m^2 [6]. One way to see this is to rewrite the right-hand side as

$$m^2\int_0^\infty ds\,e^{-sm^2}\mathrm{Tr}(e^{-sH_+}-e^{-sH_-})\;,$$

and appeal to the supersymmetry of the operator pair $H_\pm$ [7] so that only the zero modes of $H_\pm$ contribute. This way regulating the trace in Eq. (11) in fact follows from our definition of the determinant [see last term in Eq. (6)] and serves as a reminder of how to deal with any doubt about the trace operation. Thus the gauge invariance leads to the condition

$$\Phi/2\pi=\mathrm{Tr}[P_+(0)-P_-(0)]$$
$$=n_+-n_-+\frac{1}{\pi}\sum_l[\delta_+^l(0)-\delta_-^l(0)]\;, \tag{12}$$

where $P_\pm(0)$ are projection operators into the subspace of zero-energy modes of $H_\pm$; $n_\pm$ denote the number of zero-energy bound states of $H_\pm$, and $\delta_\pm^l(0)$ are the zero-energy phase shifts for scattering by the Hamiltonians $H_\pm$ in a suitable angular momentum basis l. Equation (12) is just the index theorem for a two-dimensional Euclidean manifold [8,9]. By the Aharonov-Casher theorem [7,10] we know that $n_+(n_-)$ is $\{|\Phi|/2\pi\}$, all with $\sigma_3=1$ ($\sigma_3=-1$) if $\Phi>0$ ($\Phi<0$). Here $\{x\}$ denotes the largest integer strictly less than x and $\{0\}=0$. This is our first result, that the index theorem for QED_2 follows from gauge invariance.

Let us now write Eq. (8) in the form

$$\ln\det=-\frac{1}{2\pi}\int d^2x\,\phi B+2\int_0^1 dt\,\mathrm{Tr}\{[P_+^{(t)}(0)-P_-^{(t)}(0)]\phi\}+2m^2\int_0^1 dt\,\mathrm{Tr}'\{[(H_+^{(t)}+m^2)^{-1}-(H_-^{(t)}+m^2)^{-1}]\phi\}\;, \tag{13}$$

where $P_\pm^{(t)}(0)$ are projection operators into the subspace of zero-energy modes of $H_\pm^{(t)}$. The prime on the second trace symbol indicates that zero modes are omitted. Now consider magnetic fields such that $|\Phi|/2\pi\leq1$ so that there are no bound states. According to Musto *et al.* [9] we can write the second term in Eq. (13) as

$$\frac{2}{\pi}\int_0^1 dt\,\mathrm{Tr}\{[\delta_+^{(t)}(0)-\delta_-^{(t)}(0)]\phi\}\;,$$

where the trace is over scattering states, in the limit of zero energy, of the free Hamiltonian H_0 defined by $H_\pm^{(t)}=H_0+V_\pm^{(t)}$. The operators $\delta_\pm^{(t)}$ are calculated from the S matrix $S(\lambda)=\exp[2i\delta(\lambda)]$ as $\lambda\downarrow 0$. Let us assume further that the magnetic field is sufficiently weak to justify the first Born approximation

$$\begin{aligned}\delta_+^{(t)}(0)-\delta_-^{(t)}(0)&=-\pi\delta(H_0)(V_+^{(t)}-V_-^{(t)})\\&=2\pi tB\,\delta(H_0)\ .\end{aligned}\tag{14}$$

The normalized eigenfunctions of H_0 are $\psi_{El}(\mathbf{r})=J_l(kr)e^{il\theta}/\sqrt{4\pi}$. Then

$$\begin{aligned}\frac{2}{\pi}\int_0^1 dt\,\mathrm{Tr}\{[\delta_+^{(t)}(0)-\delta_-^{(t)}(0)]\phi\}&=4\int_0^1 dt\,t\int_{0-}^{\infty}dE\sum_{l=-\infty}^{\infty}\langle El|\delta(H_0)B\phi|El\rangle\\&=\frac{1}{2\pi}\int_{0-}^{\infty}dE\,\delta(E)\int d^2r\sum_{l=-\infty}^{\infty}J_l^2(kr)B(\mathbf{r})\phi(\mathbf{r})\\&=\frac{1}{2\pi}\int d^2x\,B\phi\ ,\end{aligned}\tag{15}$$

where we used the identify $\sum_{l=-\infty}^{\infty}J_l^2(x)=1$. This result cancels the first term in Eq. (13) and is our second result; namely that the Schwinger model's contribution to the determinant of QED_2 is canceled in first Born approximation by a contribution from the zero modes in the massive sector. It may be that our weak-field approximation to the second term in Eq. (13) is exact for $|\Phi|/2\pi\le 1$, but we have not been able to prove this.

Finally, let us increase the magnetic field to $|\Phi|/2\pi>1$ so that zero-energy bound states of $H_\pm^{(t)}$ begin to appear. These states are of the form [7,11] $\psi^{(t)}(x,y)=f_\pm\exp(\pm t\phi)$, where $f_\pm$ are t-independent polynomials in $x\pm iy$ of degree $<|\Phi|t/2\pi-1$, and ϕ is the auxiliary potential defined above. These zero modes are not in general orthonormal. We define the norm matrix $N_{ij}(t)=(\psi_i^{(t)},\psi_j^{(t)})$ and the projection kernel on the zero-mode L^2 subspace [12]:

$$P^{(t)}(x,y)=\sum_{i,j=1}^{n}\psi_i^{(t)}(x)[N^{-1}(t)]_{ij}(\psi_j^{(t)})^\dagger(y)\ ,\tag{16}$$

with $\mathrm{Tr}P^{(t)}=n\equiv\{|\Phi|t/2\pi\}$. As previously noted, the bound states all have the same chirality, depending on the sign of Φ. Their contribution to the second term in Eq. (13) is, for $|\Phi|/2\pi>1$,

$$\begin{aligned}\pm 2\int_0^1 dt\sum_{i,j=1}^{n}[N^{-1}(t)]_{ij}\int d^2x\,f_if_j^*\phi e^{\pm 2t\phi}=\pm 2\lim_{\epsilon\downarrow 0}\Bigg[&\int_{2\pi(1+\epsilon)/|\Phi|}^{4\pi/|\Phi|}dt N^{-1}(t)\int d^2x\,\phi e^{\pm 2t\phi}\\&+\int_{4\pi(1+\epsilon)/|\Phi|}^{6\pi/|\Phi|}dt\sum_{i,j=1}^{2}[N^{-1}(t)]_{ij}\int d^2x\,f_if_j^*\phi e^{\pm 2t\phi}+\cdots\Bigg]\ .\end{aligned}\tag{17}$$

The above integrals can be expressed in terms of the norms N_{ij} and their derivatives with respect to t after an integration by parts in t. The result is the following zero-energy bound state contribution to the second term in Eq. (13):

$$\lim_{\epsilon\downarrow 0}\ln\left[\frac{N\left[\dfrac{4\pi}{|\Phi|}\right]}{N\left[\dfrac{2\pi(1+\epsilon)}{|\Phi|}\right]}\left|\frac{\det N_{ij}\left[\dfrac{6\pi}{|\Phi|}\right]}{\det N_{ij}\left[\dfrac{4\pi(1+\epsilon)}{|\Phi|}\right]}\right|_{i,j=1}^{2}\cdots\left|\frac{\det N_{ij}(1)}{\det N_{ij}\left[2\pi\left\{\dfrac{|\Phi|}{2\pi}\right\}(1+\epsilon)/|\Phi|\right]}\right|_{i,j=1}^{\{|\Phi|/2\pi\}}\right].$$

The norm of the first bound state, occurring when $2\ge|\Phi|/2\pi>1$, diverges as $\epsilon\downarrow 0$:

$$\begin{aligned}N[2\pi(1+\epsilon)/|\Phi|]&=\int d^2x\,\exp[\pm 4\pi(1+\epsilon)\phi/|\Phi|]\\&=2\pi\int_R^{\infty}dr\,r\,\exp[-2(1+\epsilon)\ln r]+\text{finite at }\epsilon=0\ ,\end{aligned}\tag{18}$$

where R is large compared to the range of B. Hence the logarithm in the expression displayed between Eqs. (17) and (18) becomes minus infinity, thereby causing a zero to appear in the fermionic determinant of QED_2, as seen from Eq. (13). This is our third result. Including more bound states does not improve matters. The problem seems to lie with the slow $1/|x|$ falloff of A_μ in the Lorentz gauge when $\Phi\ne 0$.

We have always kept $m^2>0$. If we take the limit $m^2=0$, it appears from the foregoing that the zero-mass limit of the fermionic determinant of QED_2 does not converge uniformly to that of the Schwinger model, which was calculated with $m^2=0$ *ab initio*. This statement is made subject to the proviso that the $m^2=0$ limit is taken before an expansion in powers of e is made; otherwise, as previously noted, we do indeed regain the Schwinger model's determinant if we take the $m^2=0$ limit order by order.

More questions have been raised here than answered, but our results do indicate that QED_2 remains a rich and relatively unexplored source of physics.

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