Relation between real-time and imaginary-time formalisms of finite-temperature quantum field theory

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Perturbative structures of finite-temperature quantum field theory are studied at a fundamental level. It is shown that the real-time formalism based on the Keldysh expansion and the imaginary-time formalism are related by a unitary transformation, and therefore are physically equivalent provided both of them are expressed in retarded form. On the other hand it is impossible for the thermo field dynamics to be reformulated in terms of retarded functions in conformity with the imaginary-time formalism due to the lack of unitary equivalence with the Keldysh and the imaginary-time formalisms which arises from its choice of path in the complex-time plane.

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It has been the opinion of many authors in studying finite-temperature quantum field theory that the imaginary-time formalism (ITF) [1], the thermo field dynamics (TFD) [2], and the real-time formalism based on the Keldysh expansion or the closed-time-path Green's function (CTPGF) theory [3,4] all belong to one equivalent class [5,6]. The recent discovery of a qualitative difference in forms of the one-loop β function between TFD and the ITF [7] has raised the question: What is the exact relation between TFD and the ITF? It is argued that the analytical continuation of the ITF results in a retarded product of fields, while TFD considers a time-ordered one, and thus there are no fundamental differences in physics between them and they must have the same physical content as a whole [8], despite apparent differences in β functions and in one-loop self-energy corrections [9]. Along this line an attempt is made currently to reformulate TFD in terms of retarded functions by a summation of graphs or a rearrangement of the Dyson series [10]. Indeed, for bosonic self-energy, a retarded expression in agreement with ITF for general cases does appear. However, the problem of whether or not such a sum could arise from a first-principles derivation is still open. On the other hand, the calculation of self-energy graphs with the CTPGF approach yields exactly the same results as in ITF either for simple models or for more realistic ones such as QED [11] without resort to artificial manipulations. The purpose of the present paper is to show that there exists indeed a unitary equivalence between the CTPGF method and the ITF, while TFD lacks the unitary equivalence with ITF and consequently it cannot be reduced to a retarded representation in conformity with ITF.

For a general discussion, we start by reexamining the argument in Ref. [5] on the equivalence problem among various formalisms of finite-temperature quantum field theory. The statistical average for a thermal equilibrium system described by Hamiltonian H can be generally expressed as

$$G(t_1,\ldots,t_n)=\mathrm{Tr}\{e^{-\beta H}A(t_1)\cdots A(t_n)\}/\mathrm{Tr}\{e^{-\beta H}\}\ , \tag{1}$$

where $A(t)$ is the Heisenberg operator. The interaction picture can be introduced by defining

$$A_i(t)=R(t)A_S R^{-1}(t)=R(t)A(0)R^{-1}(t) \tag{2}$$

with the unitary operator R to be determined by the equation of motion

$$i\frac{\partial}{\partial t}R(t)=-R(t)H_0 \tag{3}$$

which follows from the definition of the interaction picture, where the subscripts i and S indicate the interaction and Schrödinger pictures, respectively, and H_0 is the free part of H. Here we let the interaction picture be determined by initial conditions to be taken, instead of being fixed at the beginning as in Ref. [5]. The general solution of Eq. (3) is

$$R(t)=e^{iHt_0}e^{iH_0(t-t_0)} \tag{4}$$

implying that the interaction picture and the Heisenberg picture coincide at $t=t_0$. Introducing the evolution operator

$$U(t_2,t_1)=R(t_2)e^{-iH(t_2-t_1)}R(t_1)^{-1} \tag{5}$$

which, by use of (3), can be cast into

$$U(t_2,t_1)=T\exp\left\{-i\int_{t_1}^{t_2}H_i'(t)dt\right\}\ , \tag{6}$$

where H' is the interaction part of H and $H_i'(t)=R(t)H'R^{-1}(t)$. Let the time variable in (1) be continued to the complex-time plane and rewrite (1) in the interaction picture to give

$$G(z_1,z_2,\ldots,z_n)=\mathrm{Tr}\{R(\tau_0)R^{-1}(\tau_0-i\beta)U(\tau_0-i\beta,z_1)A_i(z_1)U(z_1,z_2)\cdots \times A_i(z_n)U(z_n,\tau_0)\}/\mathrm{Tr}\{R(\tau_0)R^{-1}(\tau_0-i\beta)U(\tau_0-i\beta,\tau_0)\}\ , \tag{7}$$

where τ_0 is introduced as the starting point of the time evolution of the system in the interaction picture. Choosing an appropriate counter c in the complex-time plane starting from τ_0 and ending at $\tau_0-i\beta$, (7) is reduced to

$$G(z_1,\ldots,z_n)=\mathrm{Tr}\{\rho T_c[A_i(z_1)\cdots A_i(z_n)U(\tau_0-i\beta,\tau_0)]\}/\mathrm{Tr}\{\rho U(\tau_0-i\beta,\tau_0)\}\ , \tag{8}$$

where ρ is given by

$$\rho=R(\tau_0)R^{-1}(\tau_0-i\beta) \tag{9}$$

and T_c is a contour-ordering operator. Mathematically, t_0 in (4) and τ_0 in (9) may be assigned arbitrarily, while physically, they are to be chosen to ensure that ρ must be a density operator and the partition function $\mathrm{Tr}\{e^{-\beta H}\}$ remains unchanged. The familiar one is to choose $t_0=\tau_0=0$. This initial condition leads to the conventional interaction picture: $\rho_1=e^{-\beta H_0}$ and

$$U_1(t_2,t_1)=e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1}\ . \tag{10}$$

Then the imaginary-time or Matsubara formalism follows after Wick rotation:

$$G_m(\tau_1,\ldots,\tau_n)=\mathrm{Tr}\left\{\rho_1 T\left[A_i(\tau_1)\cdots A_i(\tau_n)\exp\left[-\int_0^\beta H_i'(\tau)d\tau\right]\right]\right\}_c\ , \tag{11}$$

where the subscript c denotes that only the connected part is retained in the expansion of (11), and $A_i(\tau)=e^{H\tau}A(0)e^{-H\tau}$. In Ref. [5] the choice is $\tau_0=-\infty$ and $t_0=0$ with the contour shown in Fig. 1. By applying the adiabatic hypothesis they arrive at

$$G(t_1,\ldots,t_n)=\mathrm{Tr}\{\rho_1 T_c[A_i(t_1)\cdots A_i(t_n)S_c]\}/\mathrm{Tr}\{\rho_1 S_c\}\ , \tag{12}$$

where S_c is given by

$$S_c=T_c\exp\left\{-i\int_{-\infty}^{\infty}[H_i'(t)-H_i'(t-i\sigma)]dt\right\}\ . \tag{13}$$

From (12) and (13) a first-principles formulation of TFD has been derived [5,6]. However, (12) is only a mathematical object, because the partition function of the studied system has been changed, i.e.,

$$\mathrm{Tr}\{e^{-\beta H_0}S_c\}\neq\mathrm{Tr}\{e^{-\beta H}\}\ . \tag{14}$$

Therefore (11) and (12) are not unitarily equivalent (see also below), contrary to their conclusion that TFD and ITF belong to one equivalent class. The alternative choice of initial condition physically permissible is $t_0=\tau_0=-\infty$. It can be seen by considering the limit

$$\lim_{t\to-\infty}e^{iHt_0}e^{iH_0(t-t_0)}=\lim_{t\to-\infty}U_1(0,t_0)e^{iH_0t}=e^{iHt}U_1(0,-\infty)\ , \tag{15}$$

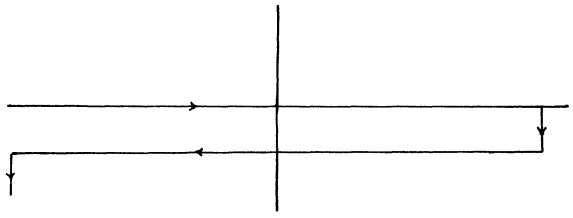

FIG. 1. The path chosen in thermo field dynamics.

where the adiabatic relation [12]

$$HU_1(0,-\infty)=U_1(0,-\infty)H_0 \tag{16}$$

has been employed. Correspondingly, the evolution operator (5) acquires the form

$$U_2(t_2,t_1)=e^{iHt_2}U_1(0,-\infty)e^{-iH(t_2-t_1)}U_1^{-1}(0,-\infty)\times e^{-iHt_1} \tag{17}$$

and the density matrix (9) becomes

$$\rho_2=e^{-\beta H}\ . \tag{18}$$

The operators in (17) and (18) just specify the incoming interaction picture [13]. Since the interaction is switched off on the path $-\infty\to-\infty-i\beta$ one finds

$$U_2(-\infty-i\beta,-\infty)=T_p\exp\left[-i\int_p H_{\mathrm{in}}'(t)dt\right]=S_p\ , \tag{19}$$

where p denotes the Schwinger-Keldysh contour with the understanding that the interaction on the positive time branch $-\infty\to+\infty$ is distinguishable from that on the negative time branch $+\infty\to-\infty$ under the effect of the path ordering operator T_p, and the subscript "in" labels the in operator. With the help of (19), (8) can be written as

$$G_p(t_1,\ldots,t_n)=\mathrm{Tr}\{\rho_2 T_p[A_{\mathrm{in}}(t_1)\cdots A_{\mathrm{in}}(t_n)S_p]\}/\mathrm{Tr}[\rho_2 S_p]\ , \tag{20}$$

which has already been formulated in Refs. [4,13]. It is worth noting that in the CTPGF method the closed-time S matrix has the property

$$S_p=T_p[S^\dagger S]=(\tilde{T}S^\dagger)(TS)=S^\dagger S=1 \tag{21}$$

where $\tilde{T}$ means the anti-time-ordering operator, against the time-ordering operator T, keeping the partition func-

tion unchanged. As a matter of fact, just because of unitarity of the S matrix, as shown in (21), the causality relation is satisfied automatically in the CTPGF [4].

From (10), (16), and (17), the CTPGF and the ITF are related by a unitary transformation

$$U_2(t_2,t_1)=U_1(0,-\infty)U_1(t_2,t_1)U_1^{-1}(0,-\infty) \quad (22)$$

and therefore are physically equivalent if both of them are expressed in retarded form. The above-mentioned coincidence of calculations for individual graphs and even for a sum of graphs [13] becomes now a natural issue.

Now one can see what happened in Ref. [5]: The authors of Ref. [5] did not realize the fact that the interaction picture should be identified as the incoming one, not the usual one, when the limit $\tau_0=-\infty$ is taken and the adiabatic hypothesis is assumed. However, this is not the whole story. The cause leading TFD to the lack of unitarity of the S matrix and hence causality is the choice of the path of Fig. 1. As a consequence, additional factors of $\exp(\pm\beta p_0/2)$ appear in the off-diagonal elements of the TFD propagators, as compared with the CTPGF. The existence of the additional factors prevents TFD from being reduced to a retarded representation in agreement with ITF. In fact, without the imposed removal of these factors, as was done in Ref. [10], not even a single free retarded Green's function can be constructed with the four elements of TFD propagators. The disappearance of the additional factors can only be achieved by setting $\sigma=0$, and at the same time recognizing the interaction picture as the incoming one. But then we arrive again at the CTPGF.

There exists also a parallel discussion on causality based on an extension of the CTPGF to finite temperature with a two-field notation [14] which has some advantages in evaluating the effective action as compared with the conventional single-field notation used in the present paper. It was shown in Ref. [14] that the CTPGF considers the expectation value with only the initial thermal equilibrium being assumed, leading to the in-in zero temperature limit. This point was explained in Ref. [4] and also is justified in the present paper. In fact, Eq. (20) is just an expectation value with the density matrix $\rho_2=\rho_{\text{in}}$. However, TFD concerns the in-out matrix elements of operator products with the in-out boundary conditions being set from the beginning. This difference makes the CTPGF causal and TFD noncausal, in agreement with our analysis.

Now we conclude that the choice of the Schwinger-Keldysh path in the complex-time plane enables the CTPGF to have the unitary equivalence with the ITF and, therefore, they are physically equivalent if both of them are given in a retarded representation. However, the choice of the time path for TFD makes it unitarily inequivalent to ITF and, hence, makes it impossible to be reformulated in terms of retarded functions in agreement with ITF.

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