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Gauss law commutators in anomalous gauge theories from a geometrical point of view

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A new method of deriving the commutation relations for Gauss law operators is studied in anomalous gauge theories. We use the method of coadjoint orbits extensively. Instead of dealing with the Gauss law operators explicitly, we consider the Becchi-Rouet-Stora-Tyutin (BRST) operator on a coadjoint orbit associated with anomalous gauge theories. We derive a basic equation satisfied by the BRST operator. The equation for the BRST operator plays a fundamental role in our formulation and it precisely reproduces the Schwinger terms in commutation relations for the Gauss law operators.

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I. INTRODUCTION

In recent years there have been increased attempts to study quantum field theories from a geometric point of view. In 1970, Kostant [1] and Souriau [2] found out the method of geometric quantization, which allows us to quantize classical fields without referring to any particular coordinate system. As a consequence, the method enables us to describe quantum field theories in coordinate-free, geometric language.

Bowick and Rajeev [3] formulated a classical nonperturbative bosonic string field theory on the basis of a general symplectic geometry. They succeeded in quantizing closed bosonic strings by application of geometric quantization, showing that $\mathrm{Diff}S^1/S^1$ is a homogeneous Kähler manifold, which is one of the typical symplectic manifolds [4].

A symplectic manifold [5] is an even-dimensional manifold M with a symplectic structure ω which is a nondegenerate and closed two-form on M. In particular, if a field theory has a symmetry governed by a Lie group, then the field theory is constructed on the basis of a group manifold, which is endowed with a symplectic structure.

As is well known, typical symplectic manifolds are the Kähler manifold, adjoint and coadjoint orbits [1,5,6], as well as cotangent bundles. Among others, coadjoint orbits have been studied [7,8] recently in dealing with some kind of field theories such as quantized string theories, the nonlinear σ model with the Wess-Zumino-Witten term and two-dimensional (2D) gravity theory, which are strongly subject to infinite-dimensional Lie groups.

On the other hand, we have also gained important geometrical insight into quantum field theories from the study of anomalies [9]. Zumino [10] and Stora [11] have revealed anomalies inherent in chiral gauge theories from the point of view of the cohomology theory of Lie groups. Faddeev and Shatashvili [12] studied the commutation relations of Gauss law operators in non-Albelian gauge theories by making use of the geometrically expedient property of the Chern-Pontryagin density and obtained anomalous Schwinger terms in the commutation relations.

A perturbative derivation of Schwinger terms in the commutation relations of Gauss law operators has been carried out in Refs. [13,14] for Yang-Mills theory with chiral fermions. A derivation by means of path integrals has been done in Ref. [15]. Anomalous Gauss law commutators in the chirally gauged Wess-Zumino-Witten model have been evaluated in Refs. [16,17]. There have been proposed some other interesting ways of deriving the Gauss law commutators such as those based on the quantum phase holonomy [18], current algebras [19], and on a more unified framework [20–22].

In this work we present a new method of deriving the commutation relations for Gauss law operators in anomalous gauge theories. The aim of this work is to explore some other possible ways of studying quantum field theories geometrically. We use the method of coadjoint orbits extensively. Instead of dealing with the Gauss law operators explicitly, we consider the Becchi-Rouet-Stora-Tyutin (BRST) operator [23] on a coadjoint orbit associated with anomalous gauge theories. We derive a basic equation satisfied by the BRST operator. We will see that the equation for the BRST operator precisely reproduces the commutation relations for the Gauss law operators and that the BRST transformation [24] of field variables plays an important role in our formalism.

The rest of this paper is organized as follows. In Sec. II, we give a brief review of the method of coadjoint orbits, by introducing a canonical coordinate system on a coadjoint orbit. We lay special emphasis upon the role of the BRST operator which is obtained as a solution of the fundamental equation on the coadjoint orbit. In Sec. III, we apply the method to an effective chiral gauge theory and derive anomalous commutation relations for Gauss law operators. We consider the chirally gauged Wess-Zumino-Witten model in Sec. IV, where we also derive the Schwinger terms in the Gauss law commutators. Section V is devoted to clarifying remarks, in which we discuss the difference between a similar work studied in Ref. [25] and ours.

II. SYMPLECTIC GEOMETRY OF COADJOINT ORBITS

In this section, we make a brief survey of the theory of coadjoint orbits in the symplectic geometry of Lie groups, laying special emphasis upon the fundamental equation (2.23) and its solution, which we will use in subsequent sections. A coadjoint orbit is a manifold constructed by a coadjoint action of a connected Lie group on a dual space of the Lie algebra. Every orbit of the coadjoint action of a Lie group possesses a symplectic structure, and the orbit is even dimensional. In fact, a coadjoint orbit is a symplectic manifold on which one can define a nondegenerate, closed symplectic two-form ω.

Let G be a connected finite-dimensional Lie group, $\mathcal{G}=T_eG$ its Lie algebra, and $\mathcal{G}^*$ the dual space of the Lie algebra. The adjoint action of G and that of $\mathcal{G}$ on $\mathcal{G}$ are defined, respectively, by

$$\mathrm{Ad}(g)Y=gYg^{-1} , \tag{2.1a}$$

$$\mathrm{ad}(X)Y=[X,Y] , \tag{2.1b}$$

where $X,Y\in\mathcal{G}$, $g\in G$, and $[.,.]$ is the commutator in the Lie algebra. We define the coadjoint action of G on $\mathcal{G}^*,\mathrm{Ad}^*:G\times\mathcal{G}^*\to\mathcal{G}^*$ and the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$, $\mathrm{ad}^*:\mathcal{G}\times\mathcal{G}^*\to\mathcal{G}^*$, respectively, by

$$[\mathrm{Ad}^*(x)\alpha](Y)=\alpha[\mathrm{Ad}(x)^{-1}Y]=\alpha(x^{-1}Yx) , \tag{2.2a}$$

$$[\mathrm{ad}^*(X)\alpha](Y)=-\alpha[\mathrm{ad}(X)Y]=-\alpha([X,Y]) . \tag{2.2b}$$

It should be noted that ad^* is a representation, the coadjoint representation, of the Lie algebra and satisfies the commutation relation

$$[\mathrm{ad}^*(X),\mathrm{ad}^*(Y)]=\mathrm{ad}^*([X,Y]) , \tag{2.3}$$

which immediately follows from (2.2b).

The coadjoint orbit M_α through a fixed point $\alpha^{(0)}\in\mathcal{G}^*$ is a symplectic manifold which is given by $M_\alpha=\{\mathrm{Ad}^*(g)\alpha^{(0)}|g\in G\}$ with a definite symplectic structure. The coadjoint orbit is isomorphic to the coset space G/H, where H is the isotropy subgroup of the fixed point $\alpha^{(0)}$. The tangent space to M_α at α is given by a set of vector fields $\{X_\alpha^*|X\in\mathcal{G}\}$. Let $\{X_iX_j,\ldots\}$ be a basis of the Lie algebra $\mathcal{G}$ with commutation relations

$$[X_i,X_j]=C_{ij}^kX_k \quad (i,j,k\in\mathbb{Z}) , \tag{2.4}$$

where C_{ij}^k are structure constants, and let $\{e^i,e^j,\ldots\}$ be a basis of the dual space. The relation between the two bases is given by the orthogonality condition

$$e^i(X_j)=\delta_j^i , \tag{2.5}$$

implying the duality of the two bases.

It should be noted that a point on a coadjoint orbit is expressed in terms of coordinates $\{\alpha_i,\alpha_j,\ldots\}$ with respect to the dual basis as

$$\alpha=\sum\alpha_ie^i . \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\alpha(X_j)=\alpha_j . \tag{2.7}$$

It is worthwhile to note that the tangent vector field X_i^* is expressed in terms of the coordinates on the coadjoint orbit as

$$X_i^*=\sum_{j,k}C_{ij}^k\alpha_k\frac{\partial}{\partial\alpha_j} , \tag{2.8}$$

which satisfies the commutation relations (2.4) and the tangent vector on the coadjoint orbit is given by

$$X_i^*\alpha=\sum_{j,k}C_{ij}^k\alpha_ke^j . \tag{2.9}$$

A natural G-invariant symplectic structure on M_α is defined by introducing the symplectic two-form ω_α at α as

$$\begin{aligned}\omega_\alpha(X_\alpha^*,Y_\alpha^*)&=\mathbf{i}(Y_\alpha^*)\mathbf{i}(X_\alpha^*)\omega_\alpha\\&=\alpha([X,Y])\end{aligned} \tag{2.10}$$

with $X,Y \bmod \mathcal{G}_H$, which is nothing but the definition of the symplectic structure given by Kirillov and Kostant [1,7]. Here, $\mathbf{i}(X)\omega$ denotes an interior product between X and ω. The symplectic structure ω_α can be explicitly written in terms of the one-forms $\{y^i,y^j,\ldots\}$ dual to the vector fields $\{X_i^*,X_j^*,\ldots\}$ as follows:

$$\omega_\alpha=\tfrac{1}{2}\alpha_{ij}y^i\wedge y^j , \tag{2.11}$$

where

$$\alpha_{ij}=\alpha([X_i,X_j])=C_{ij}^k\alpha_k . \tag{2.12}$$

The one-forms y^j satisfy the equations

$$d\alpha_i=\alpha_{ij}y^j , \tag{2.13}$$

$$dy^k=-\tfrac{1}{2}C_{ij}^ky^i\wedge y^j , \tag{2.14}$$

owing to the fact that the symplectic form (2.11) is closed; $d\omega_\alpha=0$.

Substituting (2.14) back into (2.11), one obtains a simple expression for (2.11):

$$\omega_\alpha=-\alpha_kdy^k=-\tfrac{1}{2}d\alpha_k\wedge y^k . \tag{2.15}$$

The symplectic two-form ω_α is closed and nondegenerate at all points of the coadjoint orbit and (2.11) is, in fact, G invariant, since owing to the Jacobi identity one has, for arbitrary $\xi\in\mathcal{G}$,

$$\begin{aligned}\delta_\xi\omega_\alpha(X_\alpha^*,Y_\alpha^*)&=[\mathrm{ad}^*(\xi)\alpha]([X,Y])+\alpha([[\xi,X],Y])\\&\quad+\alpha([X,[\xi,Y]])\\&=\alpha([[X,Y],\xi])+\alpha([[\xi,X],Y])\\&\quad+\alpha([[Y,\xi],X])=0 .\end{aligned} \tag{2.16}$$

Here δ_ξ denotes infinitesimal transformation on the orbit by ξ. As a matter of fact δ_X $(X\in\mathcal{G})$ is nothing but the Lie derivative with respect to X:

$$\begin{aligned}\delta_X\alpha&=\mathcal{L}_X\alpha=\mathrm{ad}^*(X)\alpha ,\\\delta_XY&=\mathcal{L}_XY=\mathrm{ad}(X)Y=[X,Y] .\end{aligned} \tag{2.17}$$

Now it is worthwhile to consider the situation where the infinitesimal transformation on a coadjoint orbit is generated by an underlying generator Ω, which takes values in $\mathcal{G}$. For simplicity, we denote $\delta_\Omega=\delta$. Our basic equations are

$$\delta\alpha=\mathrm{ad}^*(\Omega)\alpha\ ,$$
$$\delta X=\mathrm{ad}(\Omega)X=[\Omega,Y]\ , \quad (2.18)$$
$$\delta y^k=-\tfrac{1}{2}C^k_{ij}y^i\wedge y^j\ ,$$

and

$$\delta\alpha_i=\alpha([\Omega,X_i])=\alpha_{ij}y^j\ . \quad (2.19)$$

Here one should note the differences between $\delta\alpha_i$, $\delta(\alpha_i)$ and $(\delta\alpha)_i$, i.e.,

$$\delta(\alpha_i)=0\ ,$$
$$\delta\alpha_i=\alpha(\delta X_i)=\alpha([\Omega,X_i])\ , \quad (2.20)$$
$$(\delta\alpha)_i=[\mathrm{ad}^*(\Omega)\alpha](X_i)=\alpha([X_i,\Omega])=-\delta\alpha_i\ .$$

One notices from (2.19) that Ω is a one-form taking the value in $\mathcal{G}$ and, in fact, the solution to (2.19) is found to be

$$\Omega=X_iy^i\ . \quad (2.21)$$

Taking infinitesimal transformations of both sides of this expression, one obtains

$$\delta\Omega=\delta X_iy^i+X_i\delta y^i$$
$$=[\Omega,X_j]y^i-\tfrac{1}{2}C^k_{ij}X_ky^i\wedge y^j\ , \quad (2.22)$$

and we finally arrive at the equation for Ω:

$$\delta\Omega=\Omega^2=\tfrac{1}{2}\{\Omega,\Omega\}\ , \quad (2.23)$$

which we will use in subsequent sections. This is the fundamental equation for Ω, so that the equation should hold even if there exist central extensions in the algebra. Expression (2.21) is the simplest form for Ω; namely, we can take any representation for generators of the Lie algebra $\mathcal{G}$ instead of generators X_i themselves. In fact, we will take Gauss law operators in place of the X_i's in subsequent sections. Also, it should be noted that solution (2.21) is not unique, since we may add terms which commute with generators and/or their representations.

III. ANOMALOUS GAUSS LAW COMMUTATORS

In this section, we apply the method of coadjoint orbits to deriving the Gauss law commutators in anomalous gauge theories. We will make extensive use of the BRST operator Ω and show that Ω plays an essential role even for BRST noninvariant actions.

Let us start with an elementary consideration of the variational principle applied to the action functional

$$S=\int d^4x\,\mathcal{L}(\phi^\alpha,\partial_\mu\phi^\alpha)\quad(\alpha=1,\ldots,n,\ \mu=0\ ,\ldots,3)\ , \quad (3.1)$$

where $\mathcal{L}$ is the Lagrangian describing our dynamical system and ϕ^α are localized fields. Taking account of the Euler-Lagrange equations

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\alpha}\right]-\frac{\partial\mathcal{L}}{\partial\phi^\alpha}=0 \quad (3.2)$$

for ϕ^α, one obtains the variation of S to be

$$\delta S=\int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\alpha}\delta\phi^\alpha\right]\ . \quad (3.3)$$

Here we have taken $\delta x^\nu=0$ ($\nu=0,\ldots,3$). Also taking the boundary conditions appropriately we are lead to a descent equation

$$\delta S=-d\Omega\ , \quad (3.4)$$

where

$$\Omega=-\int d^3x\ \pi_\alpha\delta\phi^\alpha \quad (3.5)$$

with $\pi_\alpha=\partial\mathcal{L}/\partial\dot\phi^\alpha$ being the canonical momentum conjugate to ϕ^α.

The simplest nontrivial Lagrangian for Yang-Mills fields without spinor fields is given by

$$\mathcal{L}_{\mathrm{YM}}=-\tfrac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_g\ . \quad (3.6)$$

Here $F_{\mu\nu}=F^a_{\mu\nu}T_a$ with $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf_{abc}A^b_\mu A^c_\nu$, and the gauge group is taken to be SU(N), generators of which satisfy the commutation relations

$$[T_a,T_b]=if_{abc}T_c \quad (3.7)$$

with f_{abc} being structure constants and $\mathrm{Tr}(T_aT_b)=\tfrac{1}{2}\delta_{ab}$. The Lagrangian (3.6) is invariant under the BRST transformation by virtue of the fact that the first term on the right-hand side of (3.6) is BRST closed whereas the ghost term $\mathcal{L}_g$ is BRST exact. In this case we have

$$\Omega_0=i\int d^3x\{\pi^a_i\delta A^a_i-i\delta(\xi^a\eta^a)\}\ , \quad (3.8)$$

where δ is the BRST transformation and the exact term on the right-hand side of this expression comes from the ghost part of (3.6): η^a is the ghost field and ξ^a is the antighost field both of which are Grassmann variables and the Poisson brackets among them read

$$\{\xi^a,\eta^b\}=-\delta^{ab}\ . \quad (3.9)$$

We have not taken account of the gauge-fixing term, which might give additional exact terms to (3.8). As a matter of fact, we have taken the noncovariant gauge $A_0=0$. However, the gauge-fixing term does not play an essential role in the present formulation and we take $\delta\xi^a=0$ for the sake of simplicity.

By virtue of the BRST transformation

$$\delta A^a_\mu(x)=i[D_\mu\eta(x)]^a\ ,$$
$$\delta\eta^a(x)=-\frac{i}{2}gf_{abc}\eta^b(x)\eta^c(x)\ , \quad (3.10)$$

(3.8) is rewritten as

$$\Omega_0=\int d^3x\left[(D_i\pi_i)^a\eta^a+\delta\xi^a\eta^a+\frac{i}{2}gf_{abc}\xi^a\eta^b\eta^c\right]\ . \quad (3.11)$$

This is nothing but the BRST operator without spinor fields. One finds from $\delta\Omega_0=0$ the BRST transformation

$$\delta G^a(x)=gf_{abc}G^b(x)\eta^c(x)\ , \tag{3.12}$$

where $G^a(x)=(D_i\pi_i)^a$, and $G^a(x)=0$ is the Gauss law constraint. It follows from $\{\Omega_0,\Omega_0\}=0$ that

$$[G^a(x),G^b(x')]=igf_{abc}G^c(x)\delta^3(x-x')\ . \tag{3.13}$$

It is known that the chiral SU(N) gauge theory is anomalous for $N\geq 3$. Similar anomalous features are observed in the gauged nonlinear σ model with the Wess-Zumino-Witten term $\mathcal{L}_{\mathrm{WZW}}$. By making use of the cohomology theory of Lie groups, Faddeev [12] pointed out that the infinitesimal one-cocycle ω_1 descending from the Chern-Pontryagin density is the Lagrangian density $\mathcal{L}_W$ which makes the theory anomalous. The BRST transformation of $\mathcal{L}_W$ gives rise to a two-cocycle of the gauge group on $\mathbb{R}^3$: explicitly one obtains

$$\delta\int\mathcal{L}_W d^4x=-\frac{g}{48\pi^2}\epsilon^{ijk}\int d^3x\ \eta^a\mathrm{Tr}[T_a(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)]\ . \tag{3.14}$$

Consequently one has

$$\Omega=\int d^3x\left[G^a\eta^a+i\delta\xi^a\eta^a+\frac{i}{2}gf_{abc}\xi^a\eta^b\eta^c\right]-\frac{g}{48\pi^2}\epsilon^{ijk}\int d^3x\ \mathrm{Tr}[\eta(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)]\ . \tag{3.15}$$

Evidently $\delta\Omega$ does not vanish; in other words, the BRST operator is not left invariant under the BRST transformation. However, the BRST operator Ω is the generator of the BRST transformation; in fact, one obtains

$$\delta\eta^a=\{\eta^a,\Omega\}=-\frac{i}{2}gf_{abc}\eta^b\eta^c\ ,\qquad \delta A_k^a=\{A_k^a,\Omega\}=i(D_k\eta)^a\ . \tag{3.16}$$

Also one finds from

$$\delta\xi^a=\{\xi^a,\Omega\}=0 \tag{3.17}$$

that

$$G^a+igf_{bca}\xi^b\eta^c=\frac{g}{48\pi^2}\epsilon^{ijk}\mathrm{Tr}[\eta(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)]\ . \tag{3.18}$$

The Poisson brackets of Ω with itself is to give the BRST transformation of Ω itself. Precisely, one has

$$\delta\Omega=\tfrac{1}{2}\{\Omega,\Omega\}\ . \tag{3.19}$$

The Poisson brackets of the BRST operator with itself is evaluated as

$$\{\Omega,\Omega\}=\int d^3x\ d^3y\{[H^a(x),H^b(y)]\eta^a\eta^b-igf_{abc}H^a(x)\eta^b\eta^c\delta^3(x-y)\}\ , \tag{3.20}$$

where

$$H^a=G^a-\frac{ig}{48\pi^2}\epsilon^{ijk}\mathrm{Tr}[T_a(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)]\ . \tag{3.21}$$

As a consequence, one obtains from (3.19) and (3.20) that

$$[H^a(x),H^b(y)]_{x_0=y_0}=igf_{abc}H^a(x)\delta^3(x-y)+\frac{g^2}{48\pi^2}\epsilon^{ijk}\{\mathrm{Tr}[T_a\partial_i(A_jT_bA_k)]+if_{abc}\mathrm{Tr}[T_c(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)]\}\delta^3(x-y)\ , \tag{3.22}$$

which is nothing but the anomalous Gauss law commutation relations obtained in [13–17].

IV. THE GAUGED WESS-ZUMINO-WITTEN MODEL

In this section, we consider the chirally gauged Wess-Zumino-Witten (WZW) model in order to reinforce the formalism developed in the preceding section. The Lagrangian density $\mathcal{L}$ of the gauged WZW model is given by [16,21]

$$\mathcal{L}=\mathcal{L}_\sigma+\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{WZW}}\ , \tag{4.1}$$

where

$$\mathcal{L}_\sigma=-\frac{1}{f^2}\mathrm{Tr}[(W_\mu+A_\mu)(W^\mu+A^\mu)]\ , \tag{4.2}$$

$$\mathcal{L}_{\mathrm{WZW}}=-\frac{i}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[(A_\mu\partial_\nu A_\rho+\partial_\mu A_\nu A_\rho+A_\mu A_\nu A_\rho-\tfrac{1}{2}A_\mu W_\nu A_\rho-A_\mu W_\nu W_\rho)W_\sigma]-\frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\phi^\alpha\partial_\nu\phi^\beta\partial_\rho\phi^\gamma\partial_\sigma\phi^\delta\tau_{\alpha\beta\gamma\delta} \tag{4.3}$$

with $W_\mu=\partial_\mu UU^{-1}=W^a_\mu T_a$, and $\mathcal{L}_{\rm YM}$ is defined by (3.6). Here f is a constant and the local field $U=U(x)$ is a map from space-time into the Lie group G. The T_a's are generators of G satisfying the commutation relations (3.7). The local coordinate ϕ^α is defined by

$$\partial_\mu\phi^\alpha=W^a_\mu K^\alpha_a(\phi)\ , \tag{4.4}$$

where $K^\alpha_a(\phi)$ is a component of the Killing vectors defined by $V_a=K^\alpha_a(\phi)\partial/\partial\phi^\alpha$, which satisfy the commutation relations

$$[V_a,V_b]=-f_{abc}V_c\ . \tag{4.5}$$

We can define a set of one-forms W^a dual to the Killing vectors by

$$W^a=d\phi^\alpha K^a_\alpha(\phi)\ . \tag{4.6}$$

The duality condition $W^aV_b=\delta^a_b$ implies that $K^\alpha_a K^b_\alpha=\delta^b_a$ and $K^\alpha_a K^a_\beta=\delta^\alpha_\beta$. The one-forms $W^a, W^b,\dots$ satisfy the Maurer-Cartan equations

$$dW^a=-\tfrac{1}{2}f_{abc}W^b\wedge W^c\ . \tag{4.7}$$

It follows from (4.5) and (4.7) that

$$K^\alpha_a\partial_\alpha K^\beta_b-K^\alpha_b\partial_\alpha K^\beta_a=-f_{abc}K^\beta_c\ , \tag{4.8}$$

$$\partial_\alpha K^a_\beta-\partial_\beta K^a_\alpha=f_{abc}K^b_\alpha K^c_\beta\ , \tag{4.9}$$

where $\partial_\alpha=\partial/\partial\phi^\alpha$.

The Lagrangian density (4.2) can be rewritten in terms of the local coordinates as

$$\mathcal{L}_\sigma=\frac{1}{2f^2}h_{\alpha\beta}\nabla_\mu\phi^\alpha\nabla^\mu\phi^\beta\ , \tag{4.10}$$

where

$$\nabla_\mu\phi^\alpha=\partial_\mu\phi^\alpha+A^a_\mu K^\alpha_a \tag{4.11}$$

and $h_{\alpha\beta}=K^a_\alpha K^a_\beta$ is the metric tensor.

The skew-symmetric tensor $\tau_{\alpha\beta\gamma\delta}$ in the last term on the right-hand side of (4.3) satisfies the differential equation

$$\partial_\alpha\tau_{\beta\gamma\delta\epsilon}+\partial_\beta\tau_{\gamma\delta\epsilon\alpha}+\partial_\gamma\tau_{\delta\epsilon\alpha\beta}+\partial_\delta\tau_{\epsilon\alpha\beta\gamma}+\partial_\epsilon\tau_{\alpha\beta\gamma\delta}=i\,{\rm Tr}[T_aT_bT_cT_dT_e]K^a_\alpha K^b_\beta K^c_\gamma K^d_\delta K^e_\epsilon\ . \tag{4.12}$$

Evidently, $\mathcal{L}_\sigma$ and $\mathcal{L}_{\rm YM}$ are invariant under gauge transformations

$$\begin{aligned}&A_\mu\to A'_\mu=g^{-1}A_\mu g+g^{-1}\partial_\mu g\ ,\\ &W_\mu\to W'_\mu=g^{-1}W_\mu g-g^{-1}\partial_\mu g\ ,\\ &U\to U'=g^{-1}U\quad(g\in G)\ ,\end{aligned} \tag{4.13}$$

whereas, as is well known, $\mathcal{L}_{\rm WZW}$ is not gauge invariant.

At first, we restrict our consideration to the gauge-invariant system given by the Lagrangian density $\mathcal{L}_0=\mathcal{L}_{\rm YM}+\mathcal{L}_\sigma$. The canonical momenta conjugate to A^a_μ and ϕ^α are given, respectively, by

$$\Pi^a_\mu=F^a_{0\mu} \tag{4.14}$$

and

$$\pi_\alpha=\frac{1}{f^2}h_{\alpha\beta}[\dot\phi^\beta+A^a_0K^\beta_a]\ . \tag{4.15}$$

The Gauss law operators are written in terms of the canonical momenta as

$$G^a=\partial_i\Pi^a_i+gf_{abc}A^b_i\Pi^c_i+K^\alpha_a\pi_\alpha\ . \tag{4.16}$$

The equal-time commutation relations read

$$[A^a_i(x),\Pi^b_j(y)]=i\delta_{ij}\delta^{ab}\delta(x-y)\ , \tag{4.17}$$

$$[\phi^\alpha(x),\pi_\beta(y)]=i\delta^\alpha_\beta\delta(x-y)\ . \tag{4.18}$$

The BRST operator Ω_0 is defined as in (3.11):

$$\Omega_0=\int d^3x\left[G^a\eta^a+\delta\xi^a\eta^a+\frac{i}{2}gf_{abc}\xi^a\eta^b\eta^c\right]\ . \tag{4.19}$$

Then, one finds again the BRST transformation (3.12) from $\delta\Omega_0=0$ and the commutation relations (3.13) from $\{\Omega_0,\Omega_0\}=0$. One also finds from (3.12) the BRST transformation for $\pi_a=K^\alpha_a\pi_\alpha$, which is explicitly written as

$$\delta\pi_a(x)=gf_{abc}\pi_b(x)\eta^c(x)\ . \tag{4.20}$$

One can extract the commutation relations for $\pi_a,\pi_b,\dots$, from (3.13):

$$[\pi_a(x),\pi_b(y)]=igf_{abc}\pi_c(x)\delta(x-y)\ . \tag{4.21}$$

We next take into account the WZW term and derive anomalous commutation relations for the Gauss law operators. The BRST transformation of $\mathcal{L}_{\rm WZW}$ is evaluated to be

$$\delta\int\mathcal{L}_{\rm WZW}d^4x=-\frac{g}{48\pi^2}\epsilon^{ijk}\int d^3x\ \eta^a{\rm Tr}[T_a(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k)+T_a(W_i\partial_jA_k+\partial_iA_jW_k-W_iW_jA_k-A_iW_jW_k-W_iW_jW_k)]\ . \tag{4.22}$$

Consequently, one has

$$\Omega=\int d^3x\left[G^a\eta^a+i\delta\xi^a\eta^a+\frac{i}{2}gf_{abc}\xi^a\eta^b\eta^c\right]-\frac{g}{48\pi^2}\epsilon^{ijk}\int d^3x\ {\rm Tr}[\eta(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k+W_i\partial_jA_k+\partial_iA_jW_k-W_iW_jA_k-A_iW_jW_k-W_iW_jW_k)]\ . \tag{4.23}$$

The Poisson brackets of the BRST operator is simply written as

$$\{\Omega,\Omega\}=\int d^3x\, d^3y\{[I^a(x),I^b(y)]\eta^a\eta^b-igf_{abc}I^a(x)\eta^b\eta^c\delta^3(x-y)\}\ , \tag{4.24}$$

where

$$I^a=G^a-\frac{ig}{48\pi^2}\epsilon^{ijk}\mathrm{Tr}[T_a(A_i\partial_j A_k+\partial_i A_j A_k+A_i A_j A_k+W_i\partial_j A_k+\partial_i A_j W_k-W_i W_j A_k-A_i W_j W_k-W_i W_j W_k)]. \tag{4.25}$$

As a consequence, one finally arrives at the result [21]

$$\begin{aligned}[I^a(x),I^b(y)]_{x_0=y_0}=igf_{abc}I^c(x)\delta^3(x-y)&\\ +\frac{g^2}{48\pi^2}\epsilon^{ijk}\{\mathrm{Tr}[T_a\partial_i(A_jT_bA_k+W_jT_bW_k)]&\\ +if_{abc}\mathrm{Tr}[T_c(A_i\partial_jA_k+\partial_iA_jA_k+A_iA_jA_k&\\ +2\partial_iA_jW_k+2W_i\partial_jA_k-A_i\partial_jW_k-\partial_iW_jA_k&\\ -W_iW_jA_k-A_iW_jW_k-W_iW_jW_k)]\}\delta^3(x-y)\ .&\end{aligned} \tag{4.26}$$

The last term on the right-hand side of this expression is derived from $\delta\Omega$ [21] and obviously this reduces to (3.22) if one takes $W=0$.

V. CONCLUDING REMARKS

We have introduced the BRST operator Ω through relation (3.4), which is a descent equation appearing in the system given by the diagram

$$\begin{array}{ccccc}\Omega & \xrightarrow{\delta} & \omega & \xrightarrow{\delta} & 0\\ \downarrow d & & \downarrow d & & \\ S & \xrightarrow{\delta} & \delta S & \xrightarrow{\delta} & 0\end{array}\ , \tag{5.1}$$

where ω is the symplectic structure defined in the phase space constructed in terms of the BRST transformation. If the action S is invariant under the BRST transformation, then $\omega=\delta\Omega=0$; namely the nilpotency of the BRST operator is preserved.

As is well known, non-Abelian anomalies appear in field theories containing Weyl fermions interacting with Yang-Mills fields in themselves. From the point of view of path integrals, anomalies originate from the fact that the path-integral measure for Weyl fermions interacting with external gauge fields is not invariant under gauge transformations. Anomalies are observed effectively as cocycle terms appearing in quantum action functionals as a consequence of gauge transformations. Corresponding to the system (5.1) we have the following diagram for a descending system of cocycles:

$$\begin{array}{ccccc}\omega_4^1 & \xrightarrow{\delta} & \delta\omega_4^1 & \xrightarrow{\delta} & 0\\ \downarrow d & & \downarrow d & & \\ \omega_5^0 & \xrightarrow{\delta} & \delta\omega_5^0 & \xrightarrow{\delta} & 0\end{array}\ , \tag{5.2}$$

where

$$\omega_5^0=\mathrm{Tr}[A(dA)^2+\tfrac{3}{5}A^5+\tfrac{3}{2}A^3dA] \tag{5.3}$$

and

$$\omega_4^1=\tfrac{1}{2}\mathrm{Tr}[d\eta(AdA+dAA+A^3)]\ . \tag{5.4}$$

On the other hand, we have shown that the BRST operator Ω can be defined on a coadjoint orbit and it satisfies the Maurer-Cartan equation:

$$\delta\Omega=\tfrac{1}{2}\{\Omega,\Omega\}\ . \tag{5.5}$$

If the action S is not invariant under the BRST transformation, then $\delta\Omega$ does not vanish and it gives us the Schwinger terms in the anomalous Gauss law commutators. It should be emphasized that the derivation of the Schwinger terms is extremely simplified by extensive use of the BRST technique, as we have shown in this work.

Similar consideration as ours has been made by Bar-Moshe, Marinov, and Oz [25] from a different point of view. They started from the anomalous commutation relations for the Gauss law constraints, and they constructed bosonized actions for anomalous gauge theories by making use of the coadjoint-orbit method. Their consideration is based on the descending system of cocycles given by the diagram

$$\begin{array}{ccccc}\omega_3^2 & \xrightarrow{\delta} & \delta\omega_3^2 & \xrightarrow{\delta} & 0\\ \downarrow d & & \downarrow d & & \\ \omega_4^1 & \xrightarrow{\delta} & \delta\omega_4^1 & \xrightarrow{\delta} & 0\end{array}\ , \tag{5.6}$$

where

$$\omega_3^2=\mathrm{Tr}[(d\eta)^2A]\ . \tag{5.7}$$

The BRST operator Ω in this case is written as

$$\Omega=\int d^3x\left[G^a\eta^a+i\delta\xi^a\eta^a+\frac{i}{2}gf_{abc}\xi^a\eta^b\eta^c\right]-\frac{g}{24\pi^2}\int\omega_4^1\ . \tag{5.8}$$

If follows from the descent equation (5.6) that

$$\delta\Omega = \frac{g^2}{24\pi^2}\epsilon^{ijk}\int d^3x\, \mathrm{Tr}[\partial_i\eta\partial_j\eta A_k] \ . \tag{5.9}$$

Consequently, one arrives at the Gauss law commutation relations in the following form:

$$[F^a(x),F^b(y)] = igf_{abc}F^c(x)\delta^3(x-y) + \frac{g^2}{24\pi^2}\mathrm{Tr}[T_aT_bT_c]\epsilon^{ijk}\partial_i A_j^c\partial_k\delta^3(x-y) \ , \tag{5.10}$$

which was originally given by Faddeev [12].

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