Studying the continuum limit of the Ising model

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Different continuum limits of the Ising model in dimensions 2, 3, and 4 are investigated numerically. The data indicate that triviality occurs for $D=4$ and fails for $D<4$ in each limit.

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The triviality of quantum field theories in four dimensions (4D) that are not asymptotically free in perturbation theory has been advocated for many years now. Both theoretical and numerical studies have been concentrated on the $\lambda\phi^4$ model, relevant to the Weinberg-Salam model. Not only is there general agreement that the model becomes a free field theory in the continuum limit, but several groups have translated that information into upper bounds on the mass of the Higgs particle [1]. A different point of view has been expressed by Branchina *et al.* [2], who contend that in the phase with spontaneous symmetry breaking a nontrivial continuum limit exists. In this work we will present the results of what we believe are the most extensive numerical studies regarding the continuum limit of the Ising model in 2D, 3D, and 4D; they indicate that whereas several nontrivial limits can be constructed for $D<4$, all continuum limits in 4D are trivial.

Let us briefly review the issue of triviality. In the Feynman path-integral approach to quantization, the functional integral is undefined without introducing space-time discretization (except for the Gaussian case). Consequently, for the $\lambda\phi^4$ model, the functional integral is defined via the expression

$$Z=\left[\prod_{i\in\Lambda}\int_{-\infty}^{\infty}d\phi_i\right]\exp\left\{-\sum a^D\left[\frac{1}{2}z\left[\frac{\phi_i-\phi_j}{a}\right]^2+\frac{m^2}{2}z\phi_i^2+\frac{\lambda}{4}z^2\phi_i^4\right]\right\}. \quad (1)$$

Here $\Lambda\subset\mathbf{Z}^D$, a is the lattice spacing, m and λ are the bare parameters, and z is the wave-function renormalization. In the usual parlance of particle physics, the continuum limit is achieved by letting $a\to 0$, while adjusting suitably the bare parameters m and λ and the wave-function renormalization constant z in such a way that the Green's functions of ϕ_x approach well-defined limits. The lattice spacing a is not a dimensionless quantity; hence, a better specification of the limit $a\to 0$ is needed. It comes from observing that Eq. (1) can also be regarded as the partition function of a certain lattice model. This model depends upon two parameters; the others can be scaled out. In general its truncated correlation functions decay exponentially at large (lattice) distances. However, for suitably chosen values of the two relevant parameters, the correlation length may become infinite. This situation could be regarded as having let the lattice spacing go to zero, since the correlation length describes the physically meaningful distances. Therefore constructing the continuum limit of a quantum field theory is equivalent to studying the critical behavior of a certain statistical-mechanics model. We are emphasizing this point because most papers on the subject refer to the continuum limit as $a\to 0$ and in fact a can obviously be scaled away, so it should be regarded as 1.

To complete this brief review of the triviality problem, let us assume first that the parameters in Eq. (1) are such that $\langle\phi\rangle=0$ and the correlation length is finite. We are interested in the behavior of the Green's functions of the lattice model at distances which are asymptotic in lattice and in physical units. The central limit theorem guarantees that at such distances the $2n$-point ($n\ge 2$) Green's function becomes a sum of products of two-point functions. Consequently, the renormalization-group invariant introduced by Binder [3],

$$U_L(\{\phi\})=3-\frac{\sum_{i_1,i_2,i_3,i_4}\langle\phi_{i_1}\phi_{i_2}\phi_{i_3}\phi_{i_4}\rangle}{\sum_{i_1,i_2,i_3,i_4}\langle\phi_{i_1}\phi_{i_2}\rangle\langle\phi_{i_3}\phi_{i_4}\rangle}, \quad (2)$$

must vanish as $1/L^D$ (L is the linear size of the lattice) as long as the correlation length ξ_∞ is finite and $\langle\phi\rangle=0$. Dimensional analysis then suggests that a renormalization-group invariant which does measure the non-Gaussian character of the continuum limit is

$$g_R^{(4)}=\lim_{L\to\infty}U_L(\{\phi\})(L/\xi_L)^D. \quad (3)$$

The quantity $g_R^{(4)}$ plays the role of a renormalized coupling constant at zero momentum and describes the non-Gaussian character of the symmetric, massive phase. Namely, if, as one adjusts m and λ to approach some critical point, $g_R^{(4)}$ approaches a nonzero value, then a nontrivial massive, symmetric continuum limit exists (according to Newman's theorem [4], the larger-point Green's functions are also non-Gaussian.)

Two other continuum limits could be constructed. We could approach the critical point from the phase in which $\langle\phi\rangle$ is nonzero. For that purpose it is convenient to define shifted fields

$$\tilde{\phi}_i=\phi_i-\langle\phi\rangle. \quad (4)$$

Again the central limit theorem guarantees that $U_L(\{\tilde{\phi}\})$ vanishes as long as ξ_∞ is finite and that

$$\tilde{g}_R^{(4)}=\lim_{L\to\infty} U_L(\{\tilde{\phi}\})(L/\xi_L)^D \quad (5)$$

exists. The question of triviality concerns the limiting value of $\tilde{g}_R^{(4)}$ as the bare parameters are adjusted to reach a critical point from the phase exhibiting symmetry breaking. In such a phase though, one is not restricted to investigate $2n$-point couplings and an equally good indicator of triviality is provided by the three-point coupling

$$\tilde{g}_R^{(3)}=\frac{\sum_{i_1,i_2,i_3}\langle\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\tilde{\phi}_{i_3}\rangle_C}{\sum_{i_1,i_2}\langle\tilde{\phi}_{i_1}\tilde{\phi}_{i_2}\rangle^{3/2}}(L/\xi_L)^{D/2}, \quad (6)$$

which should tend to a nonzero value as the critical point is approached, if the continuum limit is nontrivial. Finally, one can attempt to construct a massless continuum limit by adjusting the bare parameters to some critical value and studying the limiting value of $g_R^{(4)}$ and $U_L(\{\phi\})$ as $L\to\infty$ (see below).

In this paper we report numerical results on the quantities discussed above. However, for computational reasons, we investigated only the Ising model, which can be regarded as a certain limiting value of the model defined in Eq. (1). Therefore, we studied only the approach to one particular critical point on the critical line of the lattice $\lambda\phi^4$ model. It is generally believed that all critical points on that line have identical continuum properties; however, this fact remains unproven.

Before presenting our results, we would like to recall what is rigorously known about the problems we are addressing.

(1) Aizenman [5] proved that, for the 2D Ising model, $g_R^{(4)}\neq 0$.

(2) Aizenman [5] and Fröhlich [6] separately proved that $g_R^{(4)}=0$ for $D>4$.

(3) Aizenman and Graham [7] proved that in 4D $g_R^{(4)}=0$ if there is a logarithmic correction to the mean-field divergence of the susceptibility. This fact was established by Hara and Tasaki [8] for λ sufficiently small.

(4) Gawedzki and Kupiainen [9] proved that the massless continuum limit in 4D is trivial for λ sufficiently small.

(5) A nontrivial continuum limit in 2D and 3D has been constructed [10] but not for the Ising model (strong coupling), the case investigated here.

The numerical study consisted in using the Monte Carlo procedure to compute normalized expectation values. We used the Wolff [11] version of the Swendsen-Wang [12] cluster method. The number of measurements depended on D and L. As an example, for the case of 4D, $\beta=0.1490$ and $L=30$, we had five different bins, each of which consisted of 4 500 000 clusters. For thermalization, 100 000 clusters were generated for each bin.

We employed always periodic boundary conditions. Consequently, at low temperature, the magnetization was defined as the time-averaged value of the absolute magnitude of the lattice spin. This procedure produces the correct magnetization provided a sufficiently large lattice is used. We defined the correlation length ξ_L as

$$\xi_L=\frac{1}{2\sin(\pi/L)}\left[\frac{\chi}{\chi_1}-1\right]^{1/2}, \quad (7)$$

where

TABLE I. ξ_L and $g_R^{(4)}$ in dimensions 2, 3, and 4 in the symmetric phase. In 2D, we took two different values of L/ξ_L. $L/\xi_L\simeq 6$ is the limit above which the thermodynamic limit is reached. The data in 4D show $g_R^{(4)}$ drops very rapidly as $\beta_c\simeq 0.1497$ is approached.

D	β	ξ_L	L/ξ_L	$g_R^{(4)}$
2	0.40	5.96(0.13)	6.04(0.13)	14.0(1.1)
	0.41	8.02(0.26)	5.99(0.19)	14.2(1.4)
	0.42	11.74(0.38)	6.13(0.20)	14.6(1.8)
	0.425	15.76(0.39)	6.03(0.15)	14.2(1.9)
	0.40	5.93(0.09)	4.05(0.6)	12.3(0.4)
	0.41	7.83(0.09)	4.09(0.5)	12.3(0.4)
	0.42	11.63(0.16)	4.13(0.6)	12.5(0.6)
	0.425	15.44(0.19)	4.08(0.5)	12.3(0.6)
3	0.215	4.45(0.02)	6.06(0.03)	26.4(1.1)
	0.217	5.59(0.03)	6.08(0.04)	26.4(1.4)
	0.220	10.89(0.06)	6.06(0.04)	25.1(3.3)
	0.2203	12.39(0.02)	6.05(0.01)	26.2(3.4)
	0.2206	14.53(0.05)	5.99(0.03)	24.1(2.0)
4	0.1460	3.02(0.02)	3.97(0.03)	32.0(1.6)
	0.1480	4.65(0.06)	3.88(0.05)	26.5(1.3)
	0.1490	7.55(0.02)	3.97(0.02)	22.2(0.3)

TABLE II. ξ_L, $\tilde{g}_R^{(3)}$, and $\tilde{g}_R^{(4)}$ in dimensions 2, 3, and 4 in the phase with symmetry breaking. In each dimension, we took two different values of L/ξ_L. We believe that at the larger value the thermodynamic limit has been reached. In 4D, both $\tilde{g}_R^{(3)}$ and $\tilde{g}_R^{(4)}$ drop considerably fast as β_c is approached. Note the sign change in $\tilde{g}_R^{(4)}$ compared to $g_R^{(4)}$ in the symmetric phase.

D	β	ξ	L/ξ_L	$\tilde{g}_R^{(3)}$	$-\tilde{g}_R^{(4)}$
2	0.459	2.88(0.05)	6.60(0.11)	17.1(0.3)	454(19)
	0.455	3.68(0.07)	6.76(0.13)	16.9(0.4)	438(27)
	0.425	5.72(0.15)	6.65(0.17)	16.8(0.4)	432(26)
	0.459	4.11(0.13)	9.73(0.04)	24.6(0.5)	1352(78)
	0.455	5.36(0.05)	9.51(0.09)	24.2(0.4)	1332(67)
	0.450	7.82(0.10)	9.85(0.10)	23.9(0.2)	1349(65)
3	0.2260	3.22(0.01)	4.70(0.02)	12.9(0.2)	256(14)
	0.2235	5.70(0.01)	4.74(0.01)	13.2(0.4)	286(19)
	0.2230	6.86(0.05)	4.67(0.04)	12.7(0.3)	260(12)
	0.2250	3.62(0.01)	7.97(0.01)	16.0(0.2)	768(19)
	0.2240	4.61(0.06)	7.81(0.2)	16.4(0.5)	763(83)
	0.2230	6.37(0.01)	7.86(0.02)	15.6(0.7)	753(86)
4	0.1522	2.53(0.04)	4.82(0.07)	16.4(0.2)	808(9)
	0.1506	4.34(0.09)	4.84(0.1)	12.1(1.2)	436(65)
	0.1504	4.97(0.04)	4.83(0.04)	10.6(0.6)	335(72)
	0.1502	5.90(0.08)	5.08(0.07)	8.0(0.9)	193(59)
	0.154	1.70(0.01)	7.06(0.04)	17.5(0.7)	756(15)
	0.1522	2.39(0.03)	7.11(0.09)	13.1(1.4)	444(60)
	0.151	3.43(0.08)	7.01(0.2)	9.8(0.9)	221(97)

TABLE III. ξ_L and $g_R^{(4)}$ in dimensions 2, 3, and 4 in the massless phase.

D	β	L	U_L	ξ_L	L/ξ_L	$g_R^{(4)}$
2	$\ln(\sqrt{2}+1)/2$	25	1.832(0.001)	22.66(0.04)	1.10(0)	2.23(0.01)
		75	1.832(0.001)	67.64(0.28)	1.11(0.01)	2.25(0.02)
		125	1.833(0.03)	113.02(0.54)	1.11(0.02)	2.24(0.07)
3	0.221 65	32	1.415(0.002)	20.51(0.03)	1.56(0)	5.37(0.01)
		48	1.406(0.02)	30.73(0.05)	1.56(0)	5.36(0.07)
		72	1.389(0.003)	45.69(0.07)	1.58(0)	5.44(0.03)
4	0.1497	20	1.069(0.016)	13.98(0.10)	1.43(0.01)	4.48(0.18)
		30	1.056(0.016)	21.63(0.13)	1.39(0.01)	3.91(0.17)
		35	1.055(0.029)	25.62(0.35)	1.36(0.02)	3.67(0.31)

$$\chi = \sum_{\mathbf{x}\in\Lambda} \langle \tilde{\phi}_0 \tilde{\phi}_{\mathbf{x}} \rangle_C ,$$

$$\chi_1 = \sum_{\mathbf{x}\in\Lambda} \langle \tilde{\phi}_0 \tilde{\phi}_{\mathbf{x}} \rangle_C \exp\left[i\frac{2\pi}{L}x_1 \right] .$$

Here x_1 is denoted as the first component of $\mathbf{x}$.

We define the massless regime as the regime in which $\xi_\infty/L \gg 1$, $\xi_L \gg 1$, with ξ_L still defined by Eq. (7). Obviously, at β_c, ξ_L does not represent a true correlation length; moreover the limit $L\to\infty$ does not exist (ξ_L increases with L). For this massless regime, we continued to monitor $g_R^{(4)}$ as the renormalized coupling. It is only this latter function which vanishes in 4D. On the contrary, Binder's function U_L seems to converge to nonzero values in all D, in agreement with certain theoretical predictions [13]. Therefore, Binder's function U_L seems to be a good indicator for signaling critical behavior but not triviality. (Theoretically one could modify the definition of Binder's U_L so that it can be used as an indicator of the triviality of a massless limit: namely, let $\beta=\beta_c$ and consider an infinite lattice, partitioned into blocks of size L. Let $\tilde{U}_L$ be the spatial average of Binder's function defined for these blocks. Then in 4D this $\tilde{U}_L$ will vanish as L goes to ∞, indicting the Gaussian nature of the massless continuum limit [14].)

Our numerical findings are recorded in Tables I, II, and III. The difference between the scaling behavior of the Ising model in the superrenormalizable cases 2D and 3D and the renormalizable case 4D leaves little doubt to its trivial continuum limit in 4D. The data also suggest that taking different values of L/ξ_L does not affect these conclusions. Also, we note that there is a sign change in the four-point renormalized coupling in the phase with symmetry breaking. The physical meaning of this is unclear; however, we have verified the correctness of this fact through the low-temperature expansion. We report the result of fitting for $\tilde{g}_R^{(4)}$ with $L/\xi_L \simeq 5$; it is shown that

$$\tilde{g}_R^{(4)} \simeq \frac{C_1}{|C_2+\ln(\beta-\beta_c)|} , \tag{8}$$

with $C_1 \simeq -288.9$, $C_2 \simeq 6.35$, and $\beta_c = 0.1497$. However, it was found that $\tilde{g}_R^{(3)}$ does not fit to an inverse of the logarithmic function. Of course, our data covering only one point of the critical line of the $\lambda\phi^4$ lattice model, do not rule out the conjecture by Branchina *et al.* [2], but make it unlikely. On the contrary, the data confirm nontriviality in $D<4$ and triviality in $D=4$ in every phase of the Ising model.

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