PHYSICAL REVIEW D VOLUME 47, NUMBER 6 15 MARCH 1993

Convergence proof for optimized δ expansion: Anharmonic oscillator

A. Duncan
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

H. F. Jones
Physics Department, Imperial College, London SW7 2BZ, United Kingdom
(Received 13 July 1992)

A recent proof of the convergence of the optimized δ expansion for one-dimensional non-Gaussian integrals is extended to the finite-temperature partition function of the quantum anharmonic oscillator. The convergence is exponentially fast, with the remainder falling as $e^{-cN^{2/3}}$ at order N in the expansion, independently of the size of the coupling or the sign of the mass term. In particular, the approach gives a convergent resummation procedure for the double-well (non-Borel-summable) case.

PACS number(s): 11.10.Jj, 11.15.Tk

I. INTRODUCTION

Conventional perturbation expansions in renormalizable quantum field theories, in which the Green's functions or S-matrix elements of the theory are formally expanded in powers of a physically defined coupling, typically yield well-defined (term-by-term) asymptotic, rather than Taylor, series [1]. Such series are only useful in those fortunate cases where the effective expansion parameter is small. Moreover, in many (indeed, most) cases of physical interest, the perturbative series is not even Borel summable, and one lacks a precise procedure, even in principle, for reconstructing the Green's functions of the theory to arbitrary accuracy from purely perturbative information.

In four-dimensional Yang-Mills theory, for example, the perturbative series is not Borel summable [2], and the low-energy physics is effectively strongly coupled. Indeed, a pure SU(N) gauge theory has *no* dimensionless parameters available in terms of which mass ratios (say) might sensibly be expanded, other than the heretofore intractable $1/N$ expansion employing the dimension of the gauge group itself. The relation of the perturbative series to the "full" theory is very tenuous indeed in this case.

There have been many suggestions [3–21] that the convergence of perturbation theory may be improved by an optimizing procedure in which the partition of the action into "free" and "interacting" parts is made to depend on some set of auxiliary parameters, and the results obtained by expanding to finite order in this "floating" perturbation theory are optimized (one hopes) by fixing the auxiliary parameters at a point where the result is least sensitive to them ["principle of minimum sensitivity" (PMS)], or at a point where the next term in the series vanishes ["fastest apparent convergence" (FAC)].

Despite the distinctly alchemical flavor of such proceedings, there has recently appeared strong evidence that optimized perturbation theory may indeed lead to a rigorously convergent series of approximants, even in cases with (a) strong coupling, and (b) where the conventional perturbative expansions would be non-Borel summable. In particular, the convergence of the optimized linear δ expansion has been rigorously established [14] for the one-dimensional integral which serves as the archetype for studying the asymptotics of saddle-point expansions around Gaussian theories:

$$Z(\mu,g,\lambda;\delta)=\int_0^\infty dx\, e^{-(\mu+g\lambda)x^2-g\delta(x^4-\lambda x^2)} .$$

At $\delta=1$, we recover the non-Gaussian integral which we wish to evaluate. The error R_N incurred by evaluating the expansion of this integral to order δ^N at $\delta=1$, evaluated at the (global) maximum in the auxiliary parameter λ, was shown in [14] to satisfy

$$R_N < CN^{1/4}e^{-0.663N} , \quad N\rightarrow\infty ,$$

irrespective of the size of g, or the sign of μ. For $\mu<0$, a conventional saddle-point evaluation of Z leads to a non-Borel-summable series. Evidently, the optimized expansion is capable of handling simultaneously the problems of strong coupling and non-Borel behavior.

In this paper, we extend the results of [14] to a proof of convergence of an optimized δ expansion (in some sense, using the simplest imaginable interpolation of the action) for the finite-temperature partition function of the quantum anharmonic oscillator. In this case the error falls, modulo power prefactors, like $R_N\simeq e^{-CN^{2/3}}$ for large N. The methods used here extend straightforwardly to Euclidean two- or three-dimensional ϕ^4 $[(\phi)^4_{2,3}]$ field theories in a finite space-time box. We begin by reviewing in Sec. II the convergence proof for the optimized expansion of the toy integral defined above, as many features of the calculation reappear in the analysis of the functional integral defining the quantum-mechanical partition function. In Sec. III, the general structure of the optimized expansion in quantum (field) theories is outlined and various useful general formulas are given, in particular the condition for PMS extrema and explicit formulas for the remainder term R_N. In Secs. IV and V the contributions to the remainder term R_N for the anharmonic oscillator from the weak- and strong-field regions, respectively, of the functional integral are bounded. In particular, we find that the strong-field contribution is dominated by an instanton configuration of exactly the type known to determine the large-order behavior [15] of *conventional* (nonoptimized) perturbation theory. The PMS scaling for λ is determined analytically in Sec. VI and confirmed by explicit high-order calculations in Sec. VII. In Sec. VIII we discuss in greater detail the nonuniformity of the convergence for $\beta\to\infty$, and show that the optimized expansion for Z necessarily fails in this region. Sec. IX presents some evidence that optimized perturbation theory still leads to convergent results for large β when carried out for connected quantities (i.e., for $W\equiv\ln Z$), although the appropriate choice of PMS extremum appears far more delicate in this case. The rigorous basis for this convergence is not yet clear to us. Finally, in Sec. X we summarize our results and indicate some promising avenues for further work.

II. AN INSTRUCTIVE TOY INTEGRAL

The divergent character of conventional perturbation theory is clearly illustrated by the famous example of the non-Gaussian one-dimensional integral

$$Z(\mu,g)\equiv\int_0^\infty dx\, e^{-\mu x^2-gx^4} \tag{1}$$

(which can be regarded as a 0-dimensional ϕ^4 field theory). For $\mu>0$, the formal expansion of $Z(\mu,g)$ in powers of the "coupling" g leads to a divergent asymptotic series, with coefficients c_k behaving at large k effectively as $(-1)^k k!$ This series is resummable by the Borel technique. On the other hand, for $\mu<0$, a formal perturbative expansion must be performed by expanding around a minimum of the double-well potential $V(x)\equiv-|\mu|x^2+gx^4$, leading to a non-Borel-summable series with coefficients behaving like $k!$ (without the oscillating sign factor) at large k. The rather intricate contour deformations required to reconstruct the full function $Z(\mu,g)$ from the saddle-point expansions have been discussed by Zinn-Justin [16].

In a recent paper, we have shown [14] that the optimized δ expansion leads to a rapidly convergent evaluation of $Z(\mu,g)$ for either sign of μ and any size coupling g. Many qualitative features of this calculation survive in the functional integral case. In this section we review, using a somewhat streamlined argument, the results of [14].

One begins by extending the desired integral to a one-parameter family interpolating between (1) and a Gaussian integral:

$$Z(\mu,g,\lambda;\delta)\equiv\int_0^\infty dx\, e^{-(\mu+g\lambda)x^2-g\delta(x^4-\lambda x^2)} \equiv\sum_{n=0}^{N}c_n\delta^n+R_N\ . \tag{2}$$

Evidently, $Z(\mu,g,\lambda;1)=Z(\mu,g)$. The partial approximants at $\delta=1$,

$$Z_N\equiv\sum_{n=0}^{N}c_n\ , \tag{3}$$

will be shown to converge to $Z(\mu,g)$ exponentially fast as $N\to\infty$, provided λ varies appropriately with N. The scaling of λ which yields the optimal convergence will be shown in Sec. III to correspond to the condition [6] of minimal sensitivity (PMS) of Z_N with respect to the auxiliary parameter λ.

The calculation is considerably simplified by use of the contour representations:

$$Z_N=\int_0^\infty dx\, e^{-(\mu+g\lambda)x^2}\oint_{C_0}\frac{dz}{2\pi i}\frac{1}{z^{N+1}}\frac{1-z^{N+1}}{1-z}e^{-gz(x^4-\lambda x^2)}\ , \tag{4}$$

$$Z(\mu,g)=\int_0^\infty dx\, e^{-(\mu+g\lambda)x^2}\oint_{C_1}\frac{dz}{2\pi i}\frac{1}{z-1}e^{-gz(x^4-\lambda x^2)}\ , \tag{5}$$

$$R_N\equiv Z(\mu,g)-Z_N=\int_0^\infty dx\, e^{-(\mu+g\lambda)x^2}\oint_{C_{01}}\frac{dz}{2\pi i}\frac{1}{z^{N+1}}\frac{1}{z-1}e^{-gz(x^4-\lambda x^2)}\ . \tag{6}$$

In (4)–(6), the contours C_0,C_1 refer to small loops around the origin and $z=1$, while the contour C_{01} enfolds both the poles at $z=0$ and $z=1$ (see Fig. 1). Differentiating (4) with respect to λ, we find the PMS condition

$$0=\int_0^\infty dx\, x^2(x^4-\lambda x^2)^N e^{-(\mu+g\lambda)x^2} \tag{7}$$

which has a real root in λ for odd N only. For large odd N, the integral in (7) is dominated by two saddle points, one in the region $x^2/\lambda\equiv u<1$ and the other in the region $u>1$, which give canceling contributions. Defining $\alpha\equiv(g\lambda^2+\mu\lambda)/N$, one finds [14] that cancellation of the leading saddle-point contribution occurs for α satisfying

$$2\sqrt{1+\alpha^2/4}=\ln\left[\frac{\sqrt{1+\alpha^2/4}+1}{\sqrt{1+\alpha^2/4}-1}\right]\Longrightarrow\alpha=1.3254\ldots \tag{8}$$

implying $\lambda\simeq\sqrt{N}$ for large N. (In the notation of [14], $\alpha=2/\sinh\beta$.)

The remainder term R_N, given by the expression (6), can be estimated by finding the joint saddle points in z and $u\equiv x^2/\lambda$. Changing variables from x to u,

$$R_N=\frac{\sqrt{\lambda}}{2}\int_0^\infty\frac{du}{\sqrt{u}}\oint_{C_{01}}\frac{dz}{2\pi iz(z-1)}e^{-NS_{\text{eff}}(u,z)} \tag{9}$$

where

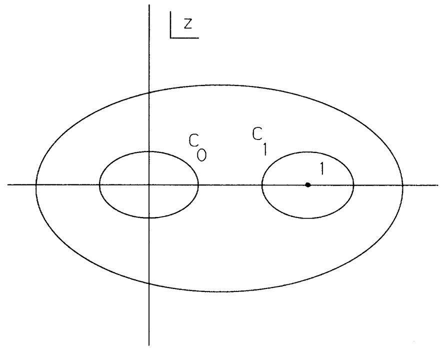

FIG. 1. Contours used for the representation of Z_N, Z, and R_N [Eqs. (4)–(6)].

$$S_{\rm eff}(u,z)\equiv\alpha u+\alpha_0 zu(u-1)+\ln(z)\ ,$$
$$\alpha_0\equiv\frac{g\lambda^2}{N}=\alpha-\frac{\mu\lambda}{N}\ . \tag{10}$$

Note that irrespective of the sign of μ, $\alpha=\alpha_0[1+O(1/\sqrt{N})]$ for large N, scaling λ as $\sqrt{N}$. Thus, if this scaling leads to convergence at large N, it will do so in the non-Borel-summable case ($\mu<0$) as well as for $\mu>0$.

The saddle-point condition for z reduces to

$$z_0=\frac{1}{\alpha_0 u(1-u)}\ . \tag{11}$$

In the region $u<1$, $z_0>4/\alpha_0>1$ and the contour C_{01} may be deformed on the right side to pick up the contribution

$$A_N\simeq\frac{\sqrt{\lambda}}{2}\int_0^1\frac{du}{\sqrt{u}}\frac{1}{z_0-1}\frac{1}{\sqrt{2\pi N}}e^{-N\{\alpha u-1-\ln[\alpha_0 u(1-u)]\}}\ . \tag{12}$$

The saddle point in u of the integral (12) is identical to the lower saddle in the PMS integral (7). Setting $\alpha=\alpha_0$ at large N, this saddle occurs at

$$u=u_<(\alpha_0)=\frac{1}{2}+\frac{1}{\alpha_0}(1-\sqrt{1+\alpha_0^2/4})=0.349\ldots \tag{13}$$

with

$$S_A(u_<)\equiv\alpha_0 u_<-1-\ln[\alpha_0 u_<(l-u_<)]=0.663\ldots \tag{14}$$

leading to the final result

$$A_N\simeq C_A N^{-3/4}e^{-0.663N}\ . \tag{15}$$

Similarly, for $u>1$, the saddle point in z occurs at

$$z_0=-\frac{1}{\alpha_0 u(u-1)}<0\ , \tag{16}$$

so the contour C_{01} should now be deformed on the left of the origin to pass through this saddle point. The result is a contribution of exactly the same form as (15):

$$B_N\simeq C_B N^{-3/4}e^{-0.663N}\ . \tag{17}$$

The identity of the exponential factors in (15) and (17) should come as no surprise: the PMS condition precisely implies the equality of the effective actions at the relevant saddle points in the "A" and "B" regions.

In most of the rest of this paper, we shall return to a purely real expression for the remainder R_N, to avoid issues of complex deformations in functional integrals, which make it difficult to derive rigorous bounds. Namely, one easily verifies, for odd N (by repeated integration by parts), the identity

$$R_N(f)\equiv e^{-f}-\sum_{n=0}^{N}\frac{(-f)^n}{n!}=e^{-f}\int_0^{|f|}e^{\xi\,{\rm sgn}(f)}\frac{\xi^N}{N!}d\xi\ . \tag{18}$$

III. GENERAL STRUCTURE OF THE OPTIMIZED δ EXPANSION

The basic idea of the optimized linear δ-expansion approach is to generate approximants to the functional integral defining a quantum (field) theory by expanding in an artificial parameter [17] which interpolates between a soluble (i.e., Gaussian) model and the full physical theory of interest. Thus, one writes an extended action, containing the parameter δ, as

$$S_\delta=\delta S+(1-\delta)S_0(\lambda) \tag{19}$$

where $S\equiv\int\mathcal{L}$ is the full (interacting) action functional, corresponding to $\delta=1$, and $S_0(\lambda)$ is an arbitrary (it might even be nonlocal) Gaussian functional of the fields, depending on some set of auxiliary parameters, here denoted collectively by λ, not present in the original theory. The convergence of the sequence of approximants obtained by expanding in δ depends crucially on the optimization performed via these parameters, which must be scaled appropriately as we work to higher and higher order. Thus, convergence is achieved by arranging a sliding separation between the "free" and "interacting" parts of the action.

Denoting the fields on a Euclidean time slice generically by x, and Euclidean time by τ, the finite-temperature partition function of the δ-extended theory is given by the functional integral:

$$Z_\delta=\frac{1}{Z_0}\int_{x(\beta)=x(0)}\mathcal{D}x\,\exp\left[-\int_0^\beta[\delta\mathcal{L}(x)+(1-\delta)\mathcal{L}_0(x;\lambda)]d\tau\right]$$
$$\equiv\sum_{n=0}^{N}\delta^n c_n+R_N\ . \tag{20}$$

The functional integral in (20) has been normalized relative to a free theory. The Nth-order approximant to the partition function is given by setting $\delta=1$ in the partial sum:

$$Z_N \equiv \sum_{n=0}^{N} c_n \ . \tag{21}$$

Of course, both Z_N and R_N (at $\delta=1$) depend on λ for any finite N. In analogy with (4) we find

$$Z_N=\frac{1}{Z_0}\int \mathcal{D}x \oint \frac{dz}{2\pi i}\frac{1}{z^{N+1}}\frac{1-z^{N+1}}{1-z}e^{-zS-(1-z)S_0} \tag{22}$$

so that the PMS condition of minimal sensitivity to λ implies

$$\frac{\partial Z_N}{\partial\lambda}=\frac{1}{Z_0}\int \mathcal{D}x\frac{\partial S_0(x,\lambda)}{\partial\lambda}(S-S_0)^N e^{-S_0}=0 \ . \tag{23}$$

It is frequently the case that $\partial S_0/\partial\lambda$ is a positive (or negative) definite functional, so the PMS condition can only be satisfied for N odd, and requires a balancing of the contributions to the functional integral (23) from the regions $S_0>S$ (called henceforth the "weak-field" regime) and $S_0<S$ ("strong-field regime").

We may note here in passing that the large N estimates made below rely on saddle-point calculations which are dominated by the exponential and the $(S-S_0)^N$ factor in (23): in particular, the $\partial S_0/\partial\lambda$ term is subdominant. It follows that the PMS condition for N odd is asymptotically equivalent to the condition of fastest apparent convergence (FAC) for $N-1$ (even):

$$c_N=\frac{1}{Z_0}\frac{1}{N!}\int \mathcal{D}x(S-S_0)^N e^{-S_0}=0 \ .$$

The variational significance of the PMS condition (23) may be more fully appreciated by using the remainder formula (18), which for odd N implies

$$R_N=A_N+B_N \ ,$$
$$A_N=\frac{1}{Z_0}\int \mathcal{D}x\,\theta(S_0-S)e^{-S(x)}\frac{1}{N!}\int_0^{S_0-S} e^{-\xi}\xi^N d\xi \ ,$$
$$B_N=\frac{1}{Z_0}\int \mathcal{D}x\,\theta(S-S_0)e^{-S(x)}\frac{1}{N!}\int_0^{S-S_0} e^{\xi}\xi^N d\xi \ . \tag{24}$$

The crucial point is that both A_N and B_N above and, hence, R_N, are *positive* for N odd. It follows that for arbitrary λ the partial approximant Z_N is bounded from above by the exact Z:

$$Z_N(\lambda)\le Z \ , \quad N \text{ odd} \ . \tag{25}$$

There are two immediate consequences of this positivity.

(1) At any given odd order in the δ expansion, the most accurate estimate of the exact partition function is achieved by picking the global maximum (in λ) for Z_N. Assuming that this does not occur at an end point of the allowed range for λ, such a point will satisfy the PMS condition (23).

(2) Any scaling of λ with N [not necessarily that implied by the PMS condition (23)] which leads to

$$R_N\to 0 \ , \quad N\to\infty \tag{26}$$

establishes, *a fortiori*, the convergence of the $Z_N(\lambda)$ approximants when evaluated at the PMS point corresponding to a global maximum in λ.

In the following two sections we shall show, by explicit bounding of A_N, B_N, that such a scaling exists for the partition function of the anharmonic oscillator (for arbitrary coupling and sign of the quadratic term). The scaling we find will be shown in Sec. VI to be essentially identical to that implied by PMS. Specifically, we shall be interested in the quantum-mechanical system defined by the Hamiltonian

$$H=\tfrac{1}{2}(p^2+m^2x^2)+gx^4 \ . \tag{27}$$

The partition function for this system at finite temperature $1/\beta$ is given by evaluating at $\delta=1$ the functional integral

$$Z_\delta=\frac{1}{Z_0}\int_{x(0)=x(\beta)}\mathcal{D}x\,\exp\left[-\int_0^\beta[\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}(m^2+2g\lambda)x^2+\delta g(x^4-\lambda x^2)]d\tau\right] \tag{28}$$

corresponding to the choice

$$S_0=\int_0^\beta[\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}(m^2+2g\lambda)x^2]d\tau \ ,$$
$$S=\int_0^\beta(\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}m^2x^2+gx^4)d\tau \tag{29}$$

in the preceding discussion. In (29), m^2 may be taken positive or negative. In the latter case, we have a double-well potential, tunneling phenomena, and a non-Borel-summable theory [16]. Nevertheless, we shall see that the optimized δ-expansion method handles this case with equal ease.

IV. ESTIMATE OF REMAINDER: WEAK-FIELD REGIME

The weak-field regime in the path integral (20) for the partition function of the anharmonic oscillator is defined by the inequality

$$\lambda\int_0^\beta x^2 d\tau \ge \int_0^\beta x^4 d\tau \ . \tag{30}$$

At order N in the δ expansion (20), the contribution to the remainder from this regime is given by (24):

$$A_N \equiv \frac{1}{Z_0} \int_A \mathcal{D}x \exp\left[-\int_0^\beta (\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}m^2x^2 + gx^4)d\tau\right] \int_0^{g\int(\lambda x^2 - x^4)d\tau} \frac{\xi^N}{N!} e^{-\xi} d\xi \tag{31}$$

where the notation $\int_A \mathcal{D}x \cdots$ denotes the restriction of the functional integral to the weak-field regime characterized by (30), and we have normalized the anharmonic partition function by dividing by the simple-harmonic partition function

$$Z_0 \equiv \int \mathcal{D}x \exp\left[-\int_0^\beta (\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}|m^2|x^2)d\tau\right] \tag{32}$$

[note: m^2 may be negative in (31)].

Making the change of variable

$$\xi = g\sigma \int_0^\beta (\lambda x^2 - x^4)d\tau \ , \quad 0 \le \sigma \le 1 \ , \tag{33}$$

one finds

$$A_N = \frac{1}{Z_0} \int_0^1 d\sigma \int_A \mathcal{D}x \int g(\lambda x^2 - x^4)d\tau \frac{1}{N!} \left[g\sigma \int (\lambda x^2 - x^4)d\tau\right]^N$$
$$\times \exp\left[-\int d\tau[\tfrac{1}{2}(\dot{x}^2 + m^2x^2) + g(1-\sigma)x^4]\right] \exp\left[-g\sigma\lambda \int x^2 d\tau\right]$$

leading to the bound

$$A_N < \frac{1}{Z_0} \int_0^1 d\sigma \int_A \mathcal{D}x \int g(\lambda x^2 - x^4)d\tau \frac{1}{N!} \left[g\sigma \int (\lambda x^2 - x^4)d\tau\right]^N \exp\left[-\int d\tau(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}m^2x^2)\right] \exp\left[-g\sigma\lambda \int x^2 d\tau\right] . \tag{34}$$

Using Stirling's formula for large N, we find

$$A_N < \frac{g}{\sqrt{2\pi N} Z_0} \int_0^1 d\sigma \int_A \mathcal{D}x \int (\lambda x^2 - x^4)d\tau \exp\left[-\int (\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}|m^2|x^2)d\tau - NS_A(x,\sigma)\right] \tag{35}$$

with

$$S_A(x,\sigma) \equiv \left[\frac{g\lambda\sigma}{N} + \frac{m^2}{N}\theta(-m^2)\right] \int_0^\beta x^2(\tau)d\tau - \ln\left[\frac{g\sigma}{N} \int (\lambda x^2 - x^4)d\tau\right] - 1 \ . \tag{36}$$

Define

$$\int_0^\beta x^2(\tau)d\tau \equiv \beta\lambda U \ . \tag{37}$$

A Cauchy-Schwarz identity implies

$$\int_0^\beta x^4(\tau)d\tau \ge \frac{1}{\beta}\left[\int_0^\beta x^2 d\tau\right]^2 = \beta\lambda^2 U^2 \ , \tag{38}$$

so the weak-field regime corresponds to the range $0 \le U \le 1$. Moreover

$$g \int (\lambda x^2 - x^4)d\tau = gB\lambda^2 U(1-U) \le \frac{g\beta\lambda^2}{4} \tag{39}$$

and

$$S_A(x,\sigma) \ge \left[\frac{g\lambda^2\beta}{N}\sigma + \frac{m^2}{N}\beta\lambda\theta(-m^2)\right] U - \ln\left[\frac{g\lambda^2\beta\sigma}{N} U(1-U)\right] - 1$$
$$\equiv \tilde{\alpha} U - \ln[\alpha U(1-U)] - 1 \tag{40}$$

where

$$\alpha \equiv \frac{g\lambda^2\beta}{N}\sigma \ ,$$
$$\tilde{\alpha} \equiv \alpha + \frac{m^2\beta\lambda}{N}\theta(-m^2) \ . \tag{41}$$

The function $\tilde{\alpha}U - \ln[U(1-U)]$ is minimized in the range $0 \le U \le 1$ at

$$U=\frac{1}{2}+\frac{1}{\tilde{\alpha}}(1-\sqrt{1+\tilde{\alpha}^2/4})=u_<(\tilde{\alpha})$$

[cf. (13)], so we have

$$S_A(x,\sigma)\geq F(\tilde{\alpha})-\ln(\alpha/2) \tag{42}$$

with

$$F(\tilde{\alpha})\equiv\tilde{\alpha}/2-\sqrt{1+\tilde{\alpha}^2/4}+\ln(1+\sqrt{1+\tilde{\alpha}^2/4})\ . \tag{43}$$

We shall see below (cf. Secs. V and VI) that the natural scaling needed for control of the strong-field contribution to the remainder at large N is to take λ to grow as $N^{2/3}$ for large N (indeed, this is essentially the growth required by the PMS condition, as we shall see analytically in Sec. VI, and numerically in Sec. VII). It is then clear that for large N, *and irrespective of the sign of* m^2,

$$\frac{\partial}{\partial\alpha}[F(\tilde{\alpha})-\ln(\alpha/2)]<0 \tag{44}$$

so that the integral over σ in (35) is dominated by the value attained at $\sigma=1$. This leads to the further bound

$$A_N<\frac{g\beta\lambda^2}{4\sqrt{2\pi N}Z_0}\int_A\mathcal{D}x\,\exp\left[-\int(\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}|m^2|x^2)d\tau\right]\exp\left[-N[F(\tilde{\alpha}_0)-\ln(\alpha_0/2)]\right] \tag{45}$$

where

$$\alpha_0\equiv\frac{g\lambda^2\beta}{N}\ ,\qquad \tilde{\alpha}_0\equiv\alpha_0+\frac{m^2\beta\lambda}{N}\theta(-m^2)\ . \tag{46}$$

As the functional integral in the numerator of (45) is bounded above by Z_0, we have simply

$$A_N<\frac{g\beta\lambda^2}{4\sqrt{2\pi N}}\exp\{-N[F(\tilde{\alpha}_0)-\ln(\alpha_0/2)]\}\ . \tag{47}$$

For large N, $\tilde{\alpha}_0\simeq\alpha_0[1+O(N^{-2/3})]$ and one finds

$$F(\alpha_0)-\ln(\alpha_0/2)\simeq\frac{1}{\alpha_0}\ ,\quad N\to\infty \tag{48}$$

and, finally

$$A_N<\frac{g\beta\lambda^2}{4\sqrt{2\pi N}}\exp(-N/\alpha_0) =\frac{g\beta\lambda^2}{4\sqrt{2\pi N}}\exp\left[-\frac{1}{g\beta}\left[\frac{N}{\lambda}\right]^2\right]\ . \tag{49}$$

If, as assumed above, $\lambda\simeq N^{2/3}$, we find A_N vanishing at large N like $N^{5/6}e^{-CN^{2/3}}$. Note that the convergence with N is *lost for* β *large.* This problem returns in field theory with greater force, where β becomes βV, with V the volume of the system. The point is simply that with the choice of interpolation (28), the weak-field contribution is necessarily dominated by constant field configurations, so the effective action (for A_N) becomes volume dependent. We will return to the issue of large β convergence and, more generally, the issue of uniform convergence of connected parts for large space-time volume, in Secs. VIII and IX.

V. ESTIMATE OF REMAINDER: STRONG-FIELD REGIME

The contribution of the Nth-order remainder from the strong-field regime, defined by the condition $\int x^4d\tau\geq\lambda\int x^2d\tau$ (indicated henceforth in the functional integration by the notation $\int_B\mathcal{D}x\cdots$), is given by

$$B_N\equiv\frac{1}{Z_0}\int_B\mathcal{D}x\,\exp\left[-\int_0^\beta[\tfrac{1}{2}(\dot{x}^2+m^2x^2)+gx^4]d\tau\right]\int_0^{g\int(x^4-\lambda x^2)d\tau}\frac{\xi^n}{N!}d^\xi d\xi \tag{50}$$

with Z_0 defined as in (32). With the change of variable

$$\xi \equiv g\sigma \int (x^4-\lambda x^2)d\tau \tag{51}$$

we have

$$B_N=\frac{g}{N!Z_0}\int_B \mathcal{D}x \exp\left[-\int_0^\beta [\tfrac{1}{2}(\dot{x}^2+m^2x^2)+gx^4]d\tau\right]\left[\int (x^4-\lambda x^2)d\tau\right]^{N+1}\int_0^1 d\sigma\, \sigma^N e^{g\sigma\int (x^4-\lambda x^2)d\tau}$$

$$\le \frac{g}{\sqrt{2\pi N}Z_0}\int_B \mathcal{D}x \int (x^4-\lambda x^2)d\tau \exp\left[-\int_0^\beta [\tfrac{1}{2}(\dot{x}^2+m^2x^2)+g\lambda x^2]d\tau+N\ln\left[\frac{g}{N}\int (x^4-\lambda x^2)d\tau\right]+N\right]$$

$$=\frac{g}{\sqrt{2\pi N}Z_0}\int_B \mathcal{D}x \int (x^4-\lambda x^2)d\tau\, e^{-NS_B(x)}\ , \tag{52}$$

where the effective action controlling the contribution in this regime takes the form

$$S_B(x)\equiv \frac{1}{2N}\int_0^\beta \dot{x}^2 d\tau+\frac{m^2+2g\lambda}{2N}\int_0^\beta x^2 d\tau-\ln\left[\frac{g}{N}\int_0^\beta (x^4-\lambda x^2)d\tau\right]-1\ . \tag{53}$$

We shall see below that λ should be taken to grow with a power of N (actually, as $N^{2/3}$), so that once again the sign of the mass term will turn out to be irrelevant to the convergence (as $g\lambda \gg m^2$ for large N). The action functional $S_B(x)$ is bounded below. To see this, it is convenient to make the change of variables

$$t\equiv \sqrt{2g\lambda}\tau\ ,\quad x(\tau)\equiv \sqrt{\lambda}u(t)\ . \tag{54}$$

The action $S_B(x)$ now becomes, in terms of $u(t)$,

$$S_B(u)=\gamma \int_{-\beta\sqrt{g\lambda/2}}^{\beta\sqrt{g\lambda/2}}\left[\left[\frac{du}{dt}\right]^2+\left[1+\frac{m^2}{2g\lambda}\right]u^2\right]dt-\ln\left[\gamma\int [u^2(u^2-1)]dt\right]-1 \tag{55}$$

with the combination $\gamma \equiv \sqrt{g\lambda^3/2N^2}$. For large N, we may neglect the vanishing term $m^2/2g\lambda$ and apply the Sobolev inequality [18]

$$\left\{\int \left[\left[\frac{du}{dt}\right]^2+u^2\right]dt\right\}^2 \ge R\int u^4 dt \tag{56}$$

(with R a constant) which implies that

$$S_B>\gamma\int \left[\left[\frac{du}{dt}\right]^2+u^2\right]dt-\ln\left[\gamma\int u^4 dt\right]-1$$

$$>\gamma X-\ln\left[\frac{\gamma X^2}{R}\right]-1\ , \tag{57}$$

which is bounded below for $X\ge 0$. Moreover, the global minimum lies in the strict interior of the integration region $\int_B \mathcal{D}x \cdots$, as S_B manifestly goes to $+\infty$ at the boundary. The global minimum thus corresponds to a local extremum of S_B, i.e., to a solution of

$$-\frac{d^2u}{dt^2}+\left[1+\frac{1}{\gamma I}\right]u-\frac{2}{\gamma I}u^3=0 \tag{58}$$

where

$$I\equiv \int_{-\beta\sqrt{g\lambda/2}}^{\beta\sqrt{g\lambda/2}} u^2(u^2-1)dt\ ,$$

and subject to periodicity of $u(t)$ on the altered range

$$u(\beta\sqrt{g\lambda/2})=u(-\beta\sqrt{g\lambda/2})\ . \tag{59}$$

Note that with $\lambda\to\infty$ the range of the integration in t goes to infinity with N, even for fixed β.

The extremal condition (58) is precisely the equation of motion for the instanton configuration known [16] to determine large-order behavior of conventional perturbation theory for the anharmonic oscillator. For $\lambda\to\infty$, the solution of (58) approaches a finite action configuration centered at an arbitrary time t_0:

$$u(t)=\sqrt{1+\gamma I}\frac{1}{\cosh\left[\left[1+\frac{1}{\gamma I}\right]^{1/2}(t-t_0)\right]} \tag{60}$$

where $-\beta\sqrt{g\lambda/2}\le t_0\le \beta\sqrt{g\lambda/2}$. Reinserting this solution in the definition of I, we obtain a consistency condition (for λ large) for I:

$$I=\int dt\,u^2(u^2-1)$$

$$=\frac{1}{\left[1+\frac{1}{\gamma I}\right]^{1/2}}[\tfrac{4}{3}(1+\gamma I)^2-2(1+\gamma I)] \tag{61}$$

or

$$\left[\frac{I}{\gamma}\right]^{1/2}=\tfrac{2}{3}(2\gamma I-1)\sqrt{1+\gamma I}\ , \tag{62}$$

which determines I in terms of γ. There are two positive roots for γI (one for $0<\gamma I<\frac{1}{2}$, the other at $\gamma I>\frac{1}{2}$). The appropriate choice requires an inspection of the action S_B at the saddle point. Inserting the instanton configuration (60) one finds

$$S_{B,\text{cl}}=1+2\gamma\left[1+\frac{1}{\gamma I}\right]^{1/2}-\ln(\gamma I)\equiv S_B(\gamma)\ . \tag{63}$$

For $\gamma>0.186\cdots$ both positive roots (for γI) correspond to $S_B>0$, so $B_N\simeq e^{-NS_B}\to 0$ for large N. The root with $\gamma I>\frac{1}{2}$ always gives the smaller action, and thus the dominant saddle point.

In the next section, we shall determine the scaling of λ with N which corresponds to the PMS condition (23). For present purposes, it is sufficient to note that the scaling $\lambda\propto N^{2/3}$ for large N leads to constant γ and hence (for $\gamma>0.186$) exponentially falling $B_N\simeq e^{-NS_B(\gamma)}$. As we saw in Sec. IV, this scaling is also consistent with an exponentially falling ($\simeq e^{-CN^{2/3}}$) contribution A_N in the weak-field regime, *provided β is held fixed.*

Including the prefactor arising from the Gaussian integration around the instanton saddle points, one finds more precisely, for large N (assuming the scaling $\lambda\simeq N^{2/3}$),

$$B_N\simeq C(m,g)\lambda^{3/2}\beta\sqrt{\lambda}e^{-NS_B(\gamma)}\ . \tag{64}$$

The origin of the various factors in (64) is as follows: (1) The factor $g\int(x^4-\lambda x^2)d\tau$ in (52) gives a contribution $\propto\lambda^{3/2}$ when evaluated at the saddle point; (2) the integration over the range (in t) of instanton positions gives a factor $\propto\beta\sqrt{\lambda}$; (3) an external factor of $1/\sqrt{N}$ in (52) (from Stirling's formula) has been cancelled by a rescaling factor for the single missing mode in the numerator functional integral after the collective coordinate delta function is introduced [16].

With the given scaling, then, the large N behavior is

$$B_N\simeq C(m,g)\beta N^{4/3}e^{-NS_B(\gamma)} \tag{65}$$

and we have exhibited a scaling leading to exponential vanishing of both A_N and B_N at large N. By the arguments of Sec. III [see discussion following (25)], the global PMS maximum of Z_N must converge to the exact partition function even more rapidly.

In this case also, one finds an increase in the remainder with large β; however, the increase is due here to a linear dependence in the prefactor (and hence can be compensated for by logarithmic growth of N with β), in contrast with the situation in the weak-field regime, where the convergence is lost in the exponent, due to the dominance of temporally constant configurations. This suggests that it may be possible to improve the convergence of the δ expansion for the partition function at large β by modifying the choice of S_0 in the interpolation in such a way as to produce temporally confined, finite-action saddle points in both the weak and strong-field sectors of the path integral for the remainder term. With the simplest choice (29) of S_0 adopted in this paper, one has rigorous convergence of the optimized δ expansion for any finite β, but the number of terms needed to achieve a given accuracy rises with β [roughly like $\beta^{3/2}$, cf. (49)].

VI. PMS SCALING

In this section we analyze the precise form of the scaling of λ required to achieve the best possible accuracy in any large odd order N of the δ expansion. Given that the Nth approximant Z_N is bounded above by the exact result Z, the optimal result is obtained by maximizing Z_N with respect to λ. Thus one is led to the PMS condition (23). As remarked in Sec. III, at large N this is essentially equivalent to the FAC condition requiring the Nth coefficient c_N in the expansion of Z to vanish. In order to achieve this, there must be a cancellation between the contributions from the weak-field $S<S_0$ and strong-field $S>S_0$ regimes. These can be estimated by saddle-point methods, and turn out to be closely related to the two contributions to the remainder A_N,B_N evaluated in the preceding two sections.

The weak-field contribution to c_N is given by

$$c_N^A=\frac{1}{N!}\int_A\mathcal{D}x e^{-S_0}(S-S_0)^N$$
$$=-\frac{1}{\sqrt{2\pi N}}\int_A\mathcal{D}x e^{-NS_A(\sigma=1)} \tag{66}$$

where S_A is the effective action given in (36). The subsequent analysis in Sec. III showed that the minimum value of S_A, and hence the dominant contribution to the functional integral, was obtained for a temporally constant field configuration for which

$$S_A=F(\alpha_0)-\ln(2/\alpha_0)\ . \tag{67}$$

For large α_0 (recall that $\alpha_0=g\beta\lambda^2/N$) we have $S_A\simeq 1/\alpha_0$.

It is worth remarking that this result can be obtained from a double saddle-point calculation whereby, in the spirit of Eqs. (4)–(6), the coefficient c_N can be expressed as the contour integral

$$c_N=\int\mathcal{D}x e^{-S_0}\oint_{C_0}\frac{dz}{2\pi i}\frac{1}{z^{N+1}}e^{-z(S-S_0)}\ . \tag{68}$$

The saddle point in z is then given by the positive root of

$$4z=\alpha_0(z^2-1) \tag{69}$$

and S_A can be written entirely in terms of z:

$$S_A=\frac{2}{z+1}-1+\ln z\ . \tag{70}$$

In any event, assuming that λ grows faster than $\sqrt{N}$, the dominant exponential governing the behavior of c_N^A is e^{-N/α_0}, corresponding to a saddle point at $z\simeq 1+2/\alpha_0$.

Turning now to the strong-field contribution to c_N, we have

$$c_N^B=\frac{1}{N!}\int_B\mathcal{D}x e^{-S_0}(S-S_0)^N$$
$$=\frac{1}{\sqrt{2\pi N}}\int_B\mathcal{D}x e^{-NS_B} \tag{71}$$

where S_B is precisely the effective action of (53), the dominant contribution to which comes from the instanton. Again, this can be obtained by a double saddle-point calculation in z and x. The saddle-point condition for z in region B results in $z=-1/(\gamma I)$, so that the consistency condition (62) can be reexpressed as

$$z=-\tfrac{2}{3}\gamma(2+z)\sqrt{1-z} \quad (72)$$

and the expression (63) for the effective action can again be written entirely in terms of z:

$$S_B(\gamma)=\frac{6}{2+z}-2+\ln|z| \;. \quad (73)$$

For the PMS condition to be satisfied, the dominant exponents (67) and (73) must be matched. This implies the scaling assumed earlier, namely $\lambda\simeq N^{2/3}$. Indeed, λ must be taken to increase at least as rapidly as $N^{2/3}$; otherwise γ vanishes for large N, $z\rightarrow 0$, and $S_A\rightarrow-\infty$, cf. (70). Consequently, S_A necessarily vanishes, which implies that we must adjust z (and hence γ) close to a zero of S_B so that the two effective actions match. The desired zero occurs at $z=z_0=-0.243\ldots$, which via (72) gives $\gamma=\gamma_0=0.186$, the value mentioned in Sec. V. But now, a constant value of $\gamma\equiv\sqrt{(g\lambda^3/2N^2)}$ implies λ growing as $N^{2/3}$, and no faster.

More precisely, expanding S_B around γ_0 we find

$$S_B(\gamma)=(\gamma-\gamma_0)S_B'(\gamma_0)+\cdots \;. \quad (74)$$

The matching condition is thus

$$\gamma=\gamma_0+\frac{1}{\alpha_0 S_B'(\gamma_0)} =0.186+\frac{0.1313}{\alpha_0} \;. \quad (75)$$

To summarize, we see that a balance between the two contributions, leading to a solution to the PMS condition, indeed occurs for the scaling $\lambda\simeq N^{2/3}$ ($\gamma\rightarrow$const). The second term in (75) then represents a correction going like $N^{-1/3}$. The approach to the asymptotic scaling of λ is evidently rather slow, as borne out by the numerical results of the following section. In fact, given that $\alpha_0\propto N^{1/3}$, the approach to scaling is more accurately reproduced by avoiding the large α_0 approximation, and instead writing

$$\gamma=\gamma_0+[F(\alpha_0)-\ln(\alpha_0/2)]/S_B'(\gamma_0) \;. \quad (76)$$

VII. COMPARISON WITH NUMERICAL RESULTS

Numerical evaluation of the δ perturbation series can be carried out to very high order by methods developed by Caswell [3] and Killingbeck [5]. The essential element is a recursion relation between the coefficients of the expansion of E and $\langle x^l\rangle$ obtained from the "hypervirial theorem." This latter arises from consideration of the expectation value of the triple commutator $[p,[p,[p,x^{l+1}]]]$, which gives the relation

$$\tfrac{1}{4}l(l^2-1)\langle x^{l-2}\rangle+2(l+1)\langle x^l(E-V)\rangle-\langle x^{l+1}V'\rangle=0 \quad (77)$$

where

$$V=\tfrac{1}{2}(m^2+2g\lambda)x^2+\delta g(x^4-\lambda x^2) \;. \quad (78)$$

When the expansions

$$E=\sum_{n=0}^{\infty}e_n\delta^n \;, \qquad \langle x^q\rangle=\sum_{n=0}^{\infty}a_n^q\delta^n \quad (79)$$

are substituted into (77), there results the recursion relation

$$\tfrac{1}{4}l(l^2-1)a_n^{l-2}+2(l+1)\sum_{r=0}^{n}a_r^l e_{n-r}-(l+2)\mu^2 a_n^{l+2} -2g(l+3)a_{n-1}^{l+4}+2g\lambda(l+2)a_{n-1}^{l+2}=0 \quad (80)$$

where we have defined

$$\mu^2=m^2+2g\lambda \quad (81)$$

the squared mass for the zeroth-order Hamiltonian. A special case of (80), for $n=0$, is

$$\tfrac{1}{4}l(l^2-1)a_0^{l-2}+2(l+1)e_0a_0^l-(l+2)\mu^2a_0^{l+2}=0 \;, \quad (82)$$

from which all the a_0^l can be generated, given that for the rth excited state $e_0=(r+\frac{1}{2})\mu$, $a_0^0=1$, $a_0^2=e_0/\mu^2$.

There are two further ingredients needed to implement the recursion relations. The first is the Feynman-Hellmann theorem, which in our case reads $\partial E/\partial\delta=\langle g(x^4-\lambda x^2)\rangle$, relating the coefficients e_n to those for $\langle x^4\rangle$ and $\langle x^2\rangle$, namely

$$ne_n=g(a_{n-1}^4-\lambda a_{n-1}^2) \;. \quad (83)$$

The second is the ordinary virial theorem, obtained from $[xp,H]$, which gives

$$E=\left\langle V+\tfrac{1}{2}x\frac{dV}{dx}\right\rangle \quad (84)$$

and the relation

$$e_n=\mu^2a_n^2+g(3a_{n-1}^4-2\lambda a_{n-1}^2) \;. \quad (85)$$

The strategy for implementing these recursion relations is as follows. Having generated the a_0^l from (82), and noting that $a_n^0=0$ for $n>0$, we use the Feynman-Hellmann theorem to generate e_1. The virial theorem then gives us a_1^2, which is needed as an initial condition in the hypervirial theorem, (80). Here we recurse on l, generating all the coefficients a_1^l. We are then in a position to calculate e_2 from (83), a_2^2 from (85), and then all the a_2^l from (80). In such a fashion we can bootstrap our way up in both n and l.

The procedure just outlined is perfectly adequate up to $n\simeq 45$, but the coefficients a_n grow rather rapidly, with the end result involving somewhat delicate cancellations between large numbers. It is therefore useful to rescale these coefficients by defining $a_n^{2p}=p!\hat{a}_n^{2p}$ (note that in the recursion relations we use only even powers), which absorbs the major part of the growth. The recursion relations rewritten in terms of the $\hat{a}$ are more numerically stable at large order, and it is these that we have used in our numerical computations (employing quadruple precision on a Vax 8550).

So far we have generated the expansion coefficients for the energy, but these must be translated into an expansion for the partition function

$$Z(\beta)=\sum_r e^{-\beta E_r} \; . \tag{86}$$

This is again most easily accomplished by a recursion relation: if $e^{-\beta E}$ is expanded as $\sum b_n \delta^n$, then

$$n b_n = -\beta \sum_{q=1}^{n} q e_q b_{n-q} \; . \tag{87}$$

In Table I we show the results of the calculation of $Z(\beta)$ up to order $N=75$. The parameters chosen in this particular case were $g=\frac{1}{2}$, $m=1$, and $\beta=2$. The sum over states in (86) was taken up to $r=10$. As discussed in previous sections, for odd N there is a single maximum, at $\lambda=\lambda_N$, which gives the best estimate for Z. At $N=9$ the error is already of the order of 0.01%, and thereafter the sequence of approximants $Z_N(\lambda_N)$ converges rapidly: by order $N=65$ the result is accurate to 17 decimal places. The extreme flatness of $Z_N(\beta)$ as a function of λ after its initial sharp rise is illustrated in Fig. 2, for $N=31$. Here the PMS maximum is at $\lambda=6.868$ and the vertical range is approximately 0.0001.

In the table we have also exhibited the value of $\gamma_N \equiv \lambda_N^{3/2}/2N$ in order to check the PMS scaling analysis of the previous section. As can be seen, it converges rather slowly to the asymptotic value $\gamma_0=0.186$. However, this was actually anticipated in (75), which for the present parameters reads $\gamma_N=\gamma_0+0.1313N/\lambda_N^2$. For example, at $N=65$, the saddle-point prediction for γ_N is 0.262, to be compared with 0.265 obtained numerically. Finally, we can compare the exponents of the two contributions which should balance at the PMS point. Again for $N=65$ we find that $S_B=0.541$ and $S_A=0.531$, using the exact expression of (67).

VIII. LARGE β BEHAVIOR

We saw in Sec. IV that the weak-field contribution to the remainder at order N in the optimized δ expansion,

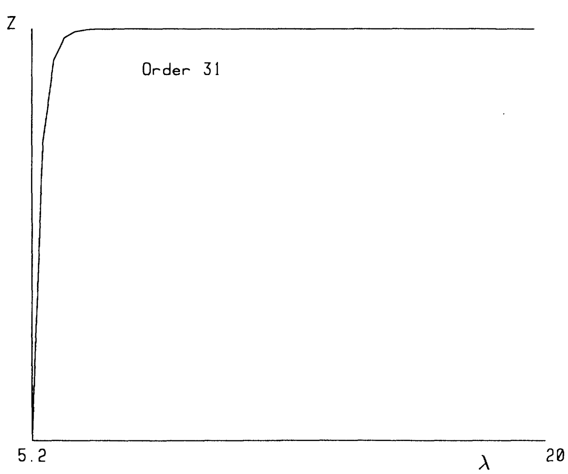

FIG. 2. $Z_N(\beta)$ vs λ for $N=31$. The parameters are $\beta=2$, $g=\frac{1}{2}$, and $m=1$, as in Table I.

TABLE I. Approach to asymptotic PMS scaling.

$\gamma_N=\lambda^{3/2}/2N$	N	λ_N	$Z_N(\lambda_N)$
0.533	3	2.170	0.257 495 453 . . .
0.373	9	3.562	0.258 237 1165 . . .
0.321	17	4.921	0.258 239 4050 . . .
0.300	25	6.078	0.258 239 4095 . . .
0.286	35	7.37	0.258 239 409 544 66 . . .
0.277	45	8.54	0.258 239 409 544 711 45 . . .
0.265	65	10.6	0.258 239 409 544 711 65 . . .
0.267	75	11.7	0.258 239 409 544 711 65 . . .

A_N, has a strong dependence on β as a consequence of the dominance in the path integral of temporally constant configurations. The result is a highly nonuniform convergence of the δ expansion with β: in effect, the convergence *apparently* disappears for large β. We say "apparently" because the bound (49) obtained for A_N is strictly speaking only relevant for fixed $\beta, N\to\infty$. It may be, in principle, a wild overestimate of A_N in the region which now interests us, namely N fixed (though $\gg 1$) and $\beta\to\infty$. We shall now see that the "minimal" interpolation based on (28) really does fail us badly in this regime, and not only for Z_N, but more crucially for quantities such as $\lim_{\beta\to\infty}[-(1/\beta)\ln Z_N]$ (=ground-state energy), or $\lim_{\beta\to\infty}[-(1/\beta)(\partial/\partial m^2)\ln Z_N]$ $(=\langle\psi_0|x^2|\psi_0\rangle)$.

For large β, the partition function (28) is dominated by the lowest state, so we have

$$Z_\delta \to e^{-\beta[E_0(\mu^2-2g\lambda\delta,\, g\delta)-E_0(m^2,0)]} \; , \quad \beta\to\infty \; , \tag{88}$$

where $E_0(\eta,\zeta)$ is the ground-state energy of the Hamiltonian

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\eta x^2+\zeta x^4 \tag{89}$$

and, as previously,

$$\mu^2\equiv m^2+2g\lambda \; .$$

It is apparent that the PMS condition (23) can be implemented by appropriate differentiations of Z_δ with respect to δ and μ^2 (the normalizing denominator factor is *not* differentiated):

$$0=\frac{\partial}{\partial\mu^2}\frac{\partial^N}{\partial\delta^N}Z_\delta\bigg|_{\delta=0} \Longrightarrow$$

$$0=\beta^{N+1}\frac{\partial E_0}{\partial\eta}\left[\left[\frac{\partial}{\partial\zeta}-2\lambda\frac{\partial}{\partial\eta}\right]E_0\right]^N\bigg|_{\eta=\mu^2,\zeta=0}+O(\beta^N) \tag{90}$$

which for large β implies

$$\left[\frac{\partial}{\partial\zeta}-2\lambda\frac{\partial}{\partial\eta}\right]E_0(\eta,\zeta)\bigg|_{\eta=\mu^2,\zeta=0}=0 \; . \tag{91}$$

Since $E_0(\eta,\zeta)=\sqrt{\eta}/2+3\zeta/(4\eta)+O(\zeta^2)$, this amounts to

$$\lambda=\frac{3}{2\mu}+O\left[\frac{1}{\beta}\right] , \tag{92}$$

$$\mu^2=m^2+\frac{3g}{\mu} , \tag{93}$$

so that the PMS point in λ becomes independent of N for large β. The cubic equation (93) always has a single positive root with $\mu > m$. It is clear from the structure of the expansion of Z_δ in δ that, for large β,

$$Z_N \simeq P_N(\beta) e^{-\beta[E_0(\mu^2,0)-E_0(m^2,0)]} \tag{94}$$

with μ satisfying (93), and $P_N(\beta)$ a polynomial of degree N in β. Consequently,

$$\lim_{\beta\to\infty}\left[-\frac{1}{\beta}\ln Z_N\right]=E_0(\mu^2,0)-E_0(m^2,0)\ , \tag{95}$$

independent of N for $N \geq 1$. So the δ expansion is "stuck" at the first order in this limit. From the positivity of R_N (cf. Sec. III) we know that

$$E_0(\mu^2,0) > E_0(m^2,g)\ . \tag{96}$$

Similarly, a nonzero remainder persists in $\langle x^2 \rangle$ for arbitrary N once the limit $\beta \to \infty$ is taken to extract the ground-state expectation value.

We therefore conclude that the optimized δ expansion of Z based simply on a floating adjustment of the mass term, which is evidently the simplest interpolation scheme, cannot be used in the zero-temperature limit. Moreover, this example shows the danger of relying on a few low-order calculations in concluding that a PMS procedure is *usefully* convergent: for $\beta \to \infty$ the optimized δ expansion is instantly convergent, *but to the wrong answer.*

It may be possible to resurrect uniform convergence in β of the expansion by a more intricate interpolation scheme, however. In the next section, we turn to the question of convergence of δ expansions carried out directly at the level of connected quantities free of volume factors. We shall see that there is numerical evidence that such expansions also converge.

IX. CONVERGENCE OF δ EXPANSION FOR CONNECTED PARTS

Most applications of the optimized δ expansion to field theory [8–10,19] have dealt directly with connected quantities, with the space-time volume taken *ab initio* to infinity. It would seem that such an approach would be more appropriate, given the difficulties encountered above with the infinite volume limit. Indeed, as we shall see, there is some evidence that convergent results can be obtained from an optimized expansion of $W \equiv \ln Z$. However, the origin and structure of the convergence is much more obscure in this case. In particular, typically several PMS points can be found, some of which apparently do not converge (or at any rate, converge much more slowly) to the correct answer. Moreover, the positivity property, which allowed us to identify the global PMS maximum as necessarily the optimal estimate for Z, fails in the δ expansion for W.

We first describe the situation for the toy model (2). In analogy with (2) and (3), define

$$W(m,g,\lambda;\delta) \equiv \ln[Z(m,g,\lambda;\delta)] = \sum_{n=0}^{N} w_n \delta^n + R_N\ . \tag{97}$$

The partial approximants $W_N \equiv \sum_{n=0}^{N} w_n$ can now be extremized with respect to λ. However, in this case R_N (even for N odd) is not positive definite. We shall describe the situation for $g=m=1.0$, although the qualitative features seem quite generic. By employing the usual (and demonstrably convergent) expansion (3), we find the desired answer (to 12 digits) for W by evaluating Z_{35} and taking the logarithm:

$$W(m,g) = 0.313\,661\,813\,489\ldots,\quad m=g=1.0\ . \tag{98}$$

The δ expansion for W gives 0.2967... at first order, which looks promising. This PMS point (a local maximum) can be tracked through odd orders, but overshoots the correct answer (98) at $N=5$ (giving $W_5=0.313879\ldots$). However, at order $N=11$, a new PMS extremum (a local maximum, the lowest of three extrema for $0<\lambda<10$) appears, giving a more accurate estimate $W_{11}=0.313\,6604\ldots$. When tracked through odd orders, this PMS point overshoots the correct answer at order $N=17$. Once again, a new PMS extremum appears at order $N=27$, giving the very accurate estimate $W_{27}=0.313\,661\,812\,99\ldots$ and so on. To summarize, it appears that a convergent series of approximants may indeed be available, but only by jumping discontinuously to new PMS extrema as these appear from time to time at higher order.

In the case of the anharmonic oscillator, calculations of $\ln Z$ for large β reduce to evaluation of the ground-state energy $E_0(m,g,\lambda;\delta)$. The situation here is broadly similar to that discussed above for the toy model, at least insofar as the existence of several PMS extrema is concerned. However, in this case, at low orders the most accurate approximants arise from the selection of the first local maximum for *even* orders of the δ expansion, while for low orders a local minimum (interlacing with a related local minimum for even orders) gives rather sluggish convergence. Finally, at order $N=29$, a new local

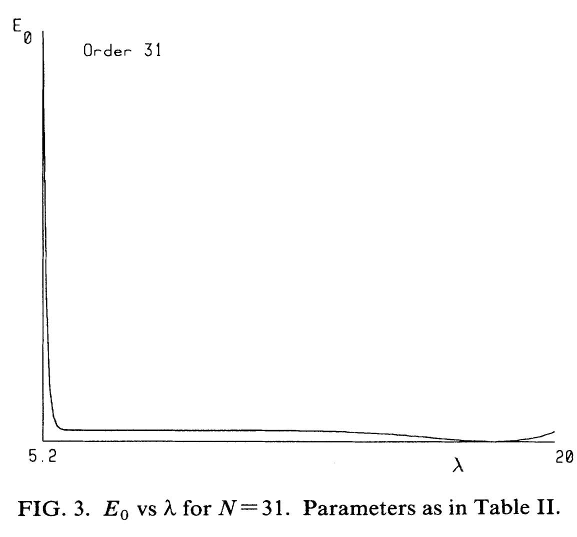

FIG. 3. E_0 vs λ for $N=31$. Parameters as in Table II.

TABLE II. PMS values for ground-state energy of anharmonic oscillator.

Order	$E_{max}(\lambda_{max})$	$E_{min}(\lambda_{min})$	$E_{min}(\lambda_{min})$
9		0.696 171 807 (5.87)	
10	0.696 175 778 (4.18)	0.696 172 083 (6.45)	
11		0.696 172 328 (7.02)	
12	0.696 175 819 (4.80)	0.696 172 542 (7.60)	
19		0.696 173 454 (11.53)	
20	0.696 175 822 (6.98)	0.696 173 533 (12.10)	
29	0.696 175 821 84 (9.8)	0.696 175 8208 (7.0)	0.696 173 982 (17.1)
30	0.696 175 821 80 (10.1)	0.696 174 015 (17.7)	
31	0.696 175 821 76 (10.4)	0.696 175 8208 (7.4)	0.696 174 400 (18.2)

maximum-minimum pair appears, giving a much more accurate approximant. These results (for an anharmonic oscillator with $g=0.5$, $m=1.0$) are summarized in Table II. The dependence of E_0 on λ is shown in Fig. 3, for $N=31$. Note the extreme flatness of the graph between the first minimum and the following maximum. The vertical range is 7×10^{-5}.

In summary, it appears that the δ expansion for ln$Z$ also leads to convergent results with appropriate choice of PMS points. Unfortunately, at present we do not know how to extend the rigorous arguments of Secs. IV–VI (in particular, the estimate of the remainder) to this situation. The loss of positivity of R_N is particularly bothersome, as it removes the clear variational intuition available in the optimized expansion for the full partition function.

X. SUMMARY AND CONCLUSIONS

The bounds derived in Secs. IV and V of this paper establish the convergence of the simplest available version of an optimized δ expansion for the partition function of the anharmonic oscillator. The arguments used here extend straightforwardly to $\phi^4_{2,3}$ theory, *with either sign for the mass term,* where the field theory is regulated both in the infrared (by considering the finite-temperature partition function at finite spatial volume) and the ultraviolet (e.g., by a higher derivative kinetic term, cf. [16]). However, the dominance of spatiotemporally constant configurations in the weak-field region of the functional integral for the remainder implies that the convergence deteriorates in the infinite-volume and/or zero-temperature limit. So this minimal form of interpolating expansion is of limited practical utility for field theory. Nevertheless, the proof given here establishes at least in principle that "perturbative" information (i.e., the evaluation of Green's functions using modified propagators, followed by an optimization procedure) allows the reconstruction of the exact partition function even in the non-Borel-summable case.

There are two possible approaches to finding a systematic and practical nonperturbative procedure for strongly coupled, non-Borel field theories. It may be possible to find an interpolating action which avoids the dominance of constant configurations in weak-field regime, thereby yielding a more uniformly convergent expansion for the partition function directly. Or (as the numerical evidence described in Sec. IX suggests) it may be possible to demonstrate the convergence of an optimized expansion directly at the level of the generating functional for connected diagrams, circumventing infinite-volume factors completely. The latter approach seems physically better motivated, but less susceptible to the analytic arguments presented in this paper. Work on this problem is currently in progress.

In the present paper, which was restricted to dimensions $d\leq1$, the question of renormalization did not arise. There is a potential problem here, insofar as the choice of constants such as λ by the PMS or some similar prescription could possibly conflict with the requirements of the renormalization procedure. However, in continuum theories such a renormalization has so far been successfully carried out in two cases: the Gross-Neveu model in the large N limit [20], and ϕ^4 in four dimensions [21] up to $O(\delta^2)$. In lattice calculations [22] the PMS procedure is implemented on the cutoff theory, where it reproduces the Monte Carlo results rather well. The subsequent renormalization is then the same as for the Monte Carlo calculations.

ACKNOWLEDGMENTS

We are grateful to S. Gandhi, I. Halliday, A. Okopińska, and R. Rivers for helpful discussions. A.D. would like to thank the SERC for financial support.

[1] F. J. Dyson, Phys. Rev. **85**, 861 (1952).
[2] G. 't Hooft, in *The Whys of Subnuclear Physics,* Proceedings of the International School, Erice, Italy, 1977, edited by A. Zichichi, Subnuclear Series Vol. 15 (Plenum, New York, 1979).
[3] W. E. Caswell, Ann. Phys. (N.Y.) **123**, 153 (1979).
[4] I. G. Halliday and P. Suranyi, Phys. Lett. **85B**, 421 (1979); Phys. Rev. D **21**, 1529 (1981).
[5] J. Killingbeck, J. Phys. A **14**, 1005 (1981).
[6] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
[7] B. S. Shaverdyan and A. G. Ushveridze, Phys. Lett. **123B**, 316 (1983).
[8] A. Okopińska, Phys. Rev. D **35**, 1835 (1987).
[9] A. Duncan and M. Moshe, Phys. Lett. B **215**, 352 (1988).

[10] A. Duncan and H. F. Jones, Nucl. Phys. **B320**, 189 (1989).
[11] V. I. Yukalov, J. Math. Phys. **32**, 1235 (1991).
[12] A. Neveu, *Festschrift for A. Martin* (Springer, Berlin, 1991).
[13] A. N. Sissakian and I. L. Solovtsov, Z. Phys. C **54**, 263 (1992).
[14] I. R. C. Buckley, A. Duncan, and H. F. Jones, preceding paper, Phys. Rev. D **47**, 2554 (1993).
[15] L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1977)]; E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D **15**, 154 (1977).
[16] J. Zinn-Justin, in *Recent Advances in Field Theory and Statistical Mechanics,* Proceedings of the Les Houches Summer School, Les Houches, France, 1982, edited by J. B. Zuber and R. Stora, Les Houches Summer School Proceedings Vol. 39 (North-Holland, Amsterdam, 1983), p. 40.
[17] C. M. Bender *et al.*, Phys. Rev. D **37**, 1472 (1988).
[18] V. G. Sigillito, *Explicit a priori Inequalities with Applications to Boundary Value Problems* (Pitman, New York, 1977).
[19] S. K. Gandhi *et al.*, Nucl. Phys. **B359**, 429 (1991).
[20] H. F. Jones and M. Moshe, Phys. Lett. B **234**, 492 (1990).
[21] I. Stancu and P. M. Stevenson, Phys. Rev. D **42**, 2710 (1990).
[22] See, for example, I. R. C. Buckley and H. F. Jones, Phys. Rev. D **45**, 2073 (1992), and references therein.