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Proof of the convergence of the linear δ expansion: Zero dimensions

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The convergence of the linear δ expansion is studied in the context of the integral $I := \int_{-\infty}^{\infty} e^{-gx^4} dx$, which corresponds to massless φ^4 theory in 0 dimensions. The method consists of rewriting the exponent as $-\delta g x^4 - \lambda(1-\delta)x^2$ and expanding in powers of δ. The arbitrary parameter λ is fixed by the principle of minimal sensitivity, $\partial I_K(\lambda)/\partial\lambda = 0$, where I_K is the expansion truncated at order K with δ set equal to 1. This has a solution $\bar{\lambda}_K$ only for K odd, when it gives very good numerical results. We are able to show analytically, using saddle-point methods, that the sequence of approximants $I_K(\bar{\lambda}_K)$ is convergent, the error decreasing exponentially with K, even though for *fixed* λ the series expansion is a divergent alternating series.

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I. INTRODUCTION

In almost all physically interesting theories, analytic progress depends on perturbative approximants which form a divergent asymptotic series. Such a series is clearly only useful in the minority of cases where the effective expansion parameter is small. In massless Yang-Mills theory there is no natural small expansion parameter (except in the large N limit) in the low-energy regime, and one must rely on numerical techniques. Moreover, the divergent asymptotic series obtained by conventional perturbation theory is usually not even Borel summable, so that in interesting field theories we really have no idea, even in principle, how to reconstruct the full theory from a weak-coupling perturbative expansion. Even in simple toy integrals defining nonresummable asymptotic series, the required modifications of the Borel method [1] are awkward and extremely complicated. In this paper we present a rigorous proof that for such integrals the optimized linear δ expansion can completely cure the problems of perturbation theory, even in the non-Borel-summable case.

The linear δ expansion has been used in a variety of situations, with good numerical results in most cases [2–8]. The method involves the introduction of an artificial parameter δ which does not appear in the original problem. In the language of field theory we define a new action S_δ, which interpolates between the theory we hope to solve, with action S, and another theory, with action S_0, which is soluble and reflects as much as possible of the physics of S.

In the linear expansion method S_δ is defined as

$$S_\delta := \lambda(1-\delta)S_0 + \delta S \,, \tag{1}$$

with $S_{\delta=0} = S_0$ and $S_{\delta=1} = S$.

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Any desired quantity is evaluated as a perturbation series in δ, which is set equal to 1 at the end of the calculation. At the moment λ is arbitrary.

In fact we might expect that quantities in the δ expansion would lose their λ dependence when δ was set equal to 1. This *would* be the case if the evaluation were exact, but of course, any truncated expansion is necessarily approximate and does retain some λ dependence, even after $\delta = 1$ has been imposed. This apparent ambiguity can be turned to our advantage, as a means of optimizing the expansion. Namely we impose the *principle of minimal sensitivity* (PMS) [9], which states that λ should be chosen as a stationary point of the truncated expansion, reflecting at least locally the expected lack of dependence of the results on λ.

Up until now, there has been no guarantee, by way of a proof, that the linear δ expansion converges; although in practice it seems to do so in a wide variety of contexts. In this paper we prove, in the simplest case of φ^4 in $d=0$, that the linear δ expansion does indeed converge. The PMS condition plays a crucial role here (cf. [10]): in fact for any *fixed* λ the perturbation series ultimately diverges, but the imposition of the PMS makes λ a function $\bar{\lambda}_K$ of the order K. Denoting by R_K the expansion for the quantity R truncated at odd order K, it then turns out that the sequence of approximants $R_K(\bar{\lambda}_K)$ is a convergent sequence bounded above by R. Moreover, the saddle-point techniques used to establish these results are capable of extension to the path integrals of quantum mechanics and field theory. Work on these generalizations is in progress (see following paper).

In Sec. II we discuss the general features and the numerical evaluation of the δ expansion for the 1-dimensional integral which corresponds to the vacuum generating functional of the 0-dimensional theory, and then in Sec. III we develop a proof of the convergence which takes advantage of some properties of the truncated exponential and relies heavily on saddle-point integration at large order. The details of the saddle-point calculations are given in the Appendix.

II. NUMERICAL EVALUATION

Evaluating the path integral in 0 dimensions for φ^4 theory amounts to evaluating the 1-dimensional integral

$$I := \int_{-\infty}^{+\infty} dx\, e^{-gx^4-\mu^2x^2} , \tag{2}$$

where g is the coupling and μ the mass. In what follows we shall restrict our attention to the massless case $\mu=0$: the cases $\mu^2=\pm 1$ are dealt with in Ref. [11].

The first step in the linear expansion method is to modify the exponent (action) to

$$\begin{aligned} S_\delta &:= -\delta g x^4-(1-\delta)\lambda x^2 \\ &= -\lambda x^2+\delta(\lambda x^2-gx^4) , \end{aligned} \tag{3}$$

giving the integral

$$\begin{aligned} I_\delta &:= \int_{-\infty}^{+\infty} dx\, e^{-\lambda x^2+\delta(\lambda x^2-gx^4)} \\ &= \int_{-\infty}^{+\infty} dx\, e^{-\lambda x^2} \sum_{n=0}^{\infty} \frac{\delta^n}{n!}(\lambda-gx^2)^n x^{2n} \\ &= \left[\frac{\pi}{\lambda}\right]^{1/2} \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \sum_{r=0}^{n} (-g)^r \begin{bmatrix} n \\ r \end{bmatrix} \\ &\qquad \times \frac{\lambda^{-2r}}{2^{r+n}}[2(r+n)-1]!! , \end{aligned} \tag{4}$$

where we have expanded out the $(\lambda-gx^2)^n$ terms, and performed the integrations using

$$\int_{-\infty}^{+\infty} dx\, x^{2m} e^{-\lambda x^2} = \left[\frac{\pi}{\lambda}\right]^{1/2} \frac{(2m-1)!!}{(2\lambda)^m} . \tag{5}$$

To find a truncated expansion for I, we simply terminate the sum over n at the chosen order K, and set $\delta=1$. The most economical way to compute the coefficients is recursively, building up from lower orders.

The integral (2) can be evaluated in terms of a Γ function by the substitution $y=gx^4$, which gives $I=\Gamma(\frac{1}{4})/2g^{1/4}=1.812\,81/g^{1/4}$. We are immediately aware of a $g^{-1/4}$ dependence which makes an ordinary perturbation expansion in the coupling constant impossible. However, by virtue of the PMS condition, the linear δ expansion is able to reproduce this nonanalytic behavior in g. Let us just show how this happens at $O(\delta)$:

$$I_1(\delta) = \left[\frac{\pi}{\lambda}\right]^{1/2} \left[1+\delta\left[\frac{1}{2}-\frac{3g}{4\lambda^2}\right]\right] . \tag{6}$$

We set $\delta=1$ in the truncated expansion,

$$I_1 = \frac{3\sqrt{\pi}}{4}(2\lambda^{-1/2}-g\lambda^{-5/2}) , \tag{7}$$

and impose $\partial I_1/\partial\lambda=0$. This fixes $\lambda=\bar{\lambda}_1=\sqrt{(5g/2)}$, which when substituted into (7) gives $\bar{I}_1=\frac{6}{5}(2\pi^2/5g)^{1/4}$. So indeed, at $O(\delta)$ and in fact at all orders, the PMS gives rise to the nonperturbative $g^{-1/4}$ dependence. Solutions to the PMS equations only exist for odd orders in δ, but this is no real problem, and we obtain $\bar{I}_9=1.811\,87/g^{1/4}$, accurate to 0.1%. Knowing that the g dependence is correctly reproduced, we set $g=1$ in subsequent numerical comparisons.

TABLE I. The coefficients of the δ expansion applied to ϕ^4 theory in 0 dimensions for $\lambda=1$ and $\lambda=4$. $\lambda=4$ is the value that satisfies the PMS condition at $O(\delta^{11})$.

	$\lambda=1$	$\lambda=4$
c_1	−0.4	0.4016
c_2	3.1	0.2398
c_3	−27.1	0.1405
c_4	336.6	0.0767
c_5	−5498.6	0.0385
c_6	111 471.8	0.0177
c_7	−2 700 994.1	0.0075
c_8	76 166 358.6	0.0029
c_9		0.0010
c_{10}		0.0003
c_{11}		0.0001

We can understand the operation of the PMS process better if we know the form of the divergence that it has to overcome. It is easy to establish, either by saddle-point integration or via recursion relations, that the coefficients c_n in $I_K(\delta)=\sum_{n=0}^{K} c_n\delta^n$ behave as

$$c_n = (-1)^n g^n e^{-n} 4^n n^{n-1/2}/\lambda^{2n+1/2} \tag{8}$$

for large n. For fixed λ, for example $\lambda=1$, the c_n coefficients alternate in sign and grow as n^n. However, when λ is set to $\bar{\lambda}_K$, the PMS value, the divergent behavior of the numerator of (8) is tamed by the denominator, as is shown in Table I. At $\lambda=4$, the PMS value appropriate to $K=11$, the coefficients c_1 to c_{11} are all of the same sign and decrease in such a way that c_{11} is only 0.005% of the total answer.

The general behavior of I_K for K odd as a function of λ is shown in Fig. 1. For small λ the integral is large and negative, reflecting the divergent behavior of the last coefficient c_K. It then increases with λ, with a single maximum at $\bar{\lambda}_K$. As will be shown in the next section, the position of this maximum grows like $\sqrt{K}$. Beyond $\bar{\lambda}$

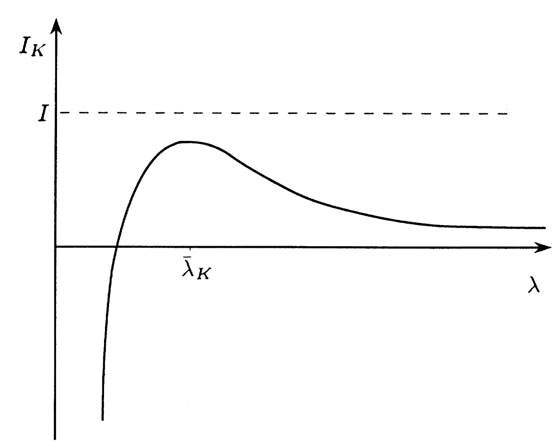

FIG. 1. Schematic behavior of the truncated series I_K (K odd) as a function of λ.

the integral decreases because of the increasing damping from the factor $e^{-\lambda x^2}$, eventually tending to zero like $1/\sqrt{\lambda}$.

III. PROOF OF CONVERGENCE

We shall shortly perform a similar saddle-point integration to find $\bar{\lambda}$, the value of λ picked out by the PMS, but first we prove that for odd K, $I_K(\lambda)$ is bounded from above by the true value I. We can rewrite I_K as $\int_{-\infty}^{\infty}\exp(-gx^4)\theta_K(z)dx$ where $\theta_K(z)=e^{-z}\{e^z\}_K$ with $z=\lambda x^2-gx^4$ and $\{f\}_K$ represents the truncation of the Taylor expansion of f at Kth order. If we can prove that $\theta_K(z)<1$ for all values of x and λ then we have shown that $I^{(K)}$ is always less than I. The crucial point is that the derivative of θ_K is rather simple, involving only the last term of the series, namely

$$\frac{d\theta_K}{dz}=-\frac{z^K}{K!}e^{-z}, \tag{9}$$

with $\theta_K(0)=1$, so by integration from $z=0$ it is clear that $\theta_K(z)<1$, for K odd. For K even, the inequality is only true for z positive, as illustrated in Fig. 2.

It is not possible to evaluate I_K analytically for large K because the sum includes terms of low orders: $n=0$ to K. However, derivatives with respect to λ exhibit the same sort of cancellations which occurred in $d\theta_K/dz$, depending only on one or two of the higher-order coefficients c_n^p. These involve an extra factor of x^{2p} relative to the c_n and are defined as

$$c_n^p:=\frac{1}{n!}\int_{-\infty}^{+\infty}dx\, e^{-\lambda x^2}(\lambda-gx^2)^n x^{2(n+p)} . \tag{10}$$

They are evaluated by saddle-point methods in the Appendix. There we also use the asymptotic expression for c_K to estimate the fastest apparent convergence (FAC) value of λ, where this coefficient vanishes.

In terms of the c_n^p,

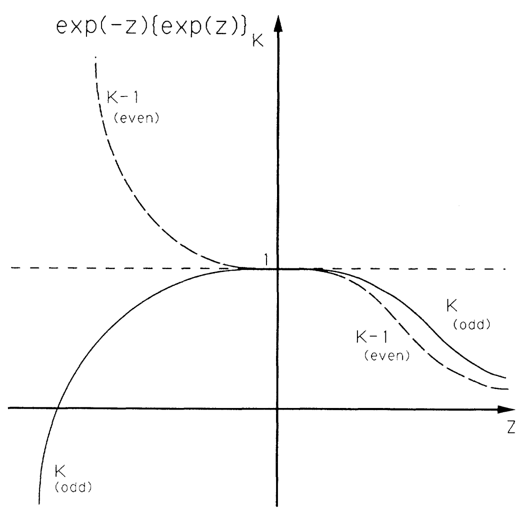

FIG. 2. Sketch of $\theta_N(z)=e^{-z}\{e^z\}_N$ for $N=K$ (odd) and $N=K-1$ (even).

$$\begin{aligned} I_K' &= -c_K^1 , \\ I_K'' &= c_K^2 - c_{K-1}^2 . \end{aligned} \tag{11}$$

We observe that, after integrating by parts,

$$I_K''=-(3/2\lambda)I_K'-(2g/\lambda)c_{K-1}^3 . \tag{12}$$

Equation (12) shows that $(\partial/\partial\lambda)(\lambda^{3/2}I_K')<0$ for K odd. Hence in this case there is only one maximum in the $I_K(\lambda)$ curve. For even K there is no stationary point at all.

A. Estimate of the PMS value $\bar{\lambda}_K$

According to Eq. (11), $\bar{\lambda}_K$ is the solution to $c_K^1(\lambda)=0$. This does not differ greatly from the FAC condition, whose solution is given in the Appendix. In terms of β, defined by $\sinh\beta=\alpha_K=2Kg/\lambda^2$, the solution is again close to β_0, about which we may perturb: $\bar{\beta}=\beta_0+\epsilon$. In particular [cf. (A19)],

$$\begin{aligned} (-f_+/\tilde{\alpha}+\tilde{\beta})_K &= -(1+1/\sinh\beta_0) \\ &\quad +\epsilon\cosh\beta_0/(\cosh\beta_0-1) , \\ (-f_-/\tilde{\alpha}-\tilde{\beta})_K &= -(1+1/\sinh\beta_0) \\ &\quad -\epsilon\cosh\beta_0/(\cosh\beta_0+1) . \end{aligned} \tag{13}$$

To the order that we are working, we can use the $p=0$ expressions for $\partial^2\phi/\partial x^2$, as $|fh^2/(f^2+h^2)|_\pm^p=|fh^2/(f^2+h^2)|_\pm^{p=0}+O(1/K)$. So eventually we have an expression similar to (A22):

$$e^{\bar{\beta}}\tanh\tfrac{1}{2}\bar{\beta}=\exp[-4K\epsilon\beta_0^2-2p(\beta_0-\ln\tanh\tfrac{1}{2}\beta_0)] , \tag{14}$$

giving

$$\begin{aligned} \epsilon &\approx -\frac{\beta_0+\ln\tanh\frac{1}{2}\beta_0+2p(\beta_0-\ln\tanh\frac{1}{2}\beta_0)}{4K\beta_0^2} \\ &= -\frac{0.733\,190\,168}{K} \end{aligned} \tag{15}$$

for $p=1$. Substituting in $\lambda=\sqrt{(2Kg/\sinh\beta)}$ we find that the predictions for $\bar{\lambda}$ are good, though not as good as those for $\lambda_{\rm FAC}$ because the magnitude of $\epsilon_{\rm PMS}$ is greater than the magnitude of $\epsilon_{\rm FAC}$; for example $\bar{\lambda}_{K=45}$ is correct to two decimal places (0.01% error).

B. Estimate of the error E_K

This estimate is greatly facilitated by the use of the identity (cf. Ref. [10])

$$\theta_K(z)=1-\frac{1}{K!}\int_0^{|z|}dw\, w^K e^{-w(\mathrm{sgn}z)} . \tag{16}$$

It is true for K odd, and is proved by integration by parts. In applying this identity to the error,

$$E_K:=I-I_K(\bar{\lambda}_K) , \tag{17}$$

the integration range divides naturally into the two regions $0<x<\sqrt{(\lambda/g)}$ and $x>\sqrt{(\lambda/g)}$, leading to

$$E_K=A_K+B_K \tag{18}$$

where

$$A_K=\frac{2}{K!}\int_0^{(\lambda/g)^{1/2}}dx\, e^{-gx^4}\int_0^{\lambda x^2-gx^4}dw\, w^K e^{-w} \tag{19}$$

and

$$B_K=\frac{2}{K!}\int_{(\lambda/g)^{1/2}}^{\infty}dx\, e^{-gx^4}\int_0^{gx^4-\lambda x^2}dw\, w^K e^{w} . \tag{20}$$

In the first of these, the maximum of the second integrand occurs at $w=K$. However, the upper limit $\lambda x^2-gx^4<\lambda^2/4g$. With $\lambda=\sqrt{(2Kg/\alpha)}$, this is K/α, which is below the position of the maximum. So within the integration range, the integrand is an increasing function of w, and may be bounded by $(\lambda x^2-gx^4)^K\exp(gx^4-\lambda x^2)$. Thus

$$A_K<\frac{2}{K!}\int_0^{(\lambda/g)^{1/2}}dx\, e^{-\lambda x^2}(\lambda x^2-gx^4)^{K+1} . \tag{21}$$

In the second contribution to E_K, the integrand is again an increasing function, so

$$B_K<\frac{2}{K!}\int_{(\lambda/g)^{1/2}}^{\infty}dx\, e^{-\lambda x^2}(\lambda x^2-gx^4)^{K+1} , \tag{22}$$

using again the fact that K is odd.

Altogether, then,

$$E_K<\frac{2}{K!}\int_0^{\infty}dx\, e^{-\lambda x^2}(\lambda x^2-gx^4)^{K+1}$$
$$=(K+1)c_{K+1} . \tag{23}$$

Referring to Eq. (A32), with $p=0$, and using Stirling's theorem for the factorial we find that

$$E_K<CK^{1/4}e^{-K/\alpha_0}=CK^{1/4}e^{-0.663K} . \tag{24}$$

Thus the error decreases exponentially fast at large (odd) K.

IV. CONCLUSIONS

We have been able to show in this toy model that the δ expansion in conjunction with the PMS prescription leads to a highly convergent sequence of approximants. It is worth commenting that the proof goes through essentially unaltered for $\mu^2\neq 0$, even for $\mu^2<0$, the broken symmetry (and non-Borel-summable) case. The PMS prescription does not in fact differ asymptotically from other possible criteria for fixing λ, such as the FAC criterion: what is crucial is that λ must not be kept fixed, but must be allowed to grow like $\sqrt{K}$.

Clearly the next goal is to attempt to extend these results to higher-dimensional theories, starting with the anharmonic oscillator in quantum mechanics [2]. The saddle-point techniques we have employed are well suited to such a generalization.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we evaluate the generalized coefficients defined in Eq. (10) for large n, using saddle-point techniques:

$$c_n^p:=\frac{1}{n!}\int_{-\infty}^{+\infty}dx\, e^{-\lambda x^2}(\lambda-gx^2)^n x^{2(n+p)}$$
$$=\frac{1}{n!}\int_{-\infty}^{+\infty}dx\, e^{\phi} , \tag{A1}$$

where, with $y=x^2$,

$$\phi=-\lambda y+n\ln(\lambda-gy)+(n+p)\ln y . \tag{A2}$$

Accordingly,

$$\frac{\partial\phi}{\partial y}=-\lambda-\frac{ng}{\lambda-gy}+\frac{n+p}{y} , \tag{A3}$$

and

$$\frac{\partial^2\phi}{\partial y^2}=-\frac{ng^2}{(\lambda-gy)^2}-\frac{n+p}{y^2} , \tag{A4}$$

with $\partial^2\phi/\partial x^2=4y\partial^2\phi/\partial y^2$ when $\partial\phi/\partial x=0$.

Let us first deal with the simpler case $p=0$: the expressions for $p\neq 0$ can be treated perturbatively in p/n. For $p=0$ the two solutions to $\partial\phi/\partial y=0$ can be written as

$$y_\pm=\frac{\lambda}{2g}f_\pm \tag{A5}$$

where

$$f_\pm=1+\alpha\pm(1+\alpha^2)^{1/2} \tag{A6}$$

and

$$\alpha=2ng/\lambda^2 . \tag{A7}$$

It is also useful to define another quantity $h_\pm$ as

$$h_\pm=1-\alpha\mp(1+\alpha^2)^{1/2} \tag{A8}$$

so that

$$\frac{\lambda}{g}-y_\pm=\frac{\lambda}{2g}h_\pm . \tag{A9}$$

In terms of f and h the second derivative of ϕ at the stationary point is

$$\frac{\partial^2\phi}{\partial x^2}=-\frac{8gn}{\lambda}\frac{f^2+h^2}{fh^2} \tag{A10}$$

while

$$\phi=-\frac{n}{\alpha}f+n\ln(n/2\alpha)+n\ln fh . \tag{A11}$$

The form of (A6) and (A8) suggest the definition

$$\sinh\beta=\alpha \tag{A12}$$

in terms of which

$$f_\pm=1\pm e^{\pm\beta} ,$$
$$h_\pm=1\mp e^{\pm\beta} , \tag{A13}$$

so that

$$f_{\pm}h_{\pm}=\mp 2\alpha e^{\pm\beta}\ . \tag{A14}$$

There are two solutions to the equation $\partial\phi/\partial y=0$ and consequently four solutions to $\partial\phi/\partial x=0$, so we can approximate our integrand by four Gaussians. As the integral is symmetric in x, we can just consider the two Gaussians in the positive x region and multiply the result by two. For odd K, the integral c_K consists of one Gaussian above the x axis, from the y_- solution, and one below, from the y_+ solution. The saddle-point formula for the integral is

$$\int_0^\infty dx\, e^{\phi(x)}=\sum_i\left[\frac{-2\pi}{\phi''(x_i)}\right]^{1/2}e^{\phi(x_i)}\ , \tag{A15}$$

where the sum over i represents the sum over solutions to $\partial\phi/\partial x=0$.

We now use these expressions to estimate the optimized λ chosen by the criterion of fastest apparent convergence (FAC), whereby λ is chosen in such a way that the last (Kth) coefficient is set to zero. While this is not of direct relevance to the main calculation of Sec. III, it exemplifies the method and serves to introduce some additional notation.

According to the FAC criterion, we wish to solve

$$c_K(\lambda)=0\ . \tag{A16}$$

To satisfy this we must simply ensure that the area of the Gaussian below the axis is equal to the area of that above:

$$\left[\left[\frac{fh^2}{f^2+h^2}\right]^{1/2}\exp[K(-f/\alpha+\ln|fh|)]\right]_+ =\left[\left[\frac{fh^2}{f^2+h^2}\right]^{1/2}\exp[K(-f/\alpha+\ln|fh|)]\right]_-\ . \tag{A17}$$

However, $|fh/(f^2+h^2)|_+=|fh/(f^2+h^2)|_-$, so this reduces to

$$[|h|^{1/2}\exp[K(-f/\alpha+\ln|fh|)]]_+ =[|h|^{1/2}\exp[K(-f/\alpha+\ln|fh|)]]_- \tag{A18}$$

or

$$|h_-|^{1/2}\exp[K(-f_-/\alpha-\beta)] =|h_+|^{1/2}\exp[K(-f_+/\alpha+\beta)]\ . \tag{A19}$$

Thus (A16) reduces to

$$(e^{\beta}\tanh\tfrac{1}{2}\beta)^{1/2}=\exp[2K(\coth\beta-\beta)]\ . \tag{A20}$$

The solution is clearly $\tilde\beta=\beta_0+O(1/K)$, where β_0 is the solution to

$$\beta_0=\coth\beta_0\ , \tag{A21}$$

which is $\beta_0=1.199\,678\,64$. If we define ϵ by $\beta=\beta_0+\epsilon$, then

$$e^{\beta_0}\tanh\tfrac{1}{2}\beta_0=\exp[-4K\epsilon(1+\operatorname{csch}^2\beta_0)]\ , \tag{A22}$$

giving

$$\epsilon\approx-\frac{\beta_0+\ln\tanh\frac{1}{2}\beta_0}{4K\beta_0^2} =-\frac{0.100\,366}{K}\ . \tag{A23}$$

The value of λ can then be retrieved from $\lambda=(2Kg/\sinh\beta)^{1/2}$. The prediction for $\lambda_{\rm FAC}$ from this method is correct to four decimal places when $K=45$ (0.000 15% error), indicating that the assumption that the integrand can be modeled by two Gaussians is a good one. The value of $\lambda_{\rm FAC}$ grows like $\sqrt{K}$, since β tends to the constant value β_0 as $K\to\infty$.

Returning now to the case $p\neq0$, the expressions for f and h are changed to

$$f_\pm=1+\tilde\alpha\pm(1-\gamma+\tilde\alpha^2)^{1/2}\ ,\qquad h_\pm=1-\tilde\alpha\mp(1-\gamma+\tilde\alpha^2)^{1/2}\ , \tag{A24}$$

where $\tilde\alpha=(2n+p)g/\lambda^2$ and $\gamma=2pg/\lambda^2$. The exponent ϕ and its second derivative become

$$\phi=-\frac{n}{\alpha}f+n\ln(n/2\alpha)+n\ln fh+p[\ln(\lambda/2g)+\ln f]\ . \tag{A25}$$

and

$$\frac{\partial^2\phi}{\partial x^2}=-\frac{8g}{\lambda}\frac{nf^2+(n+p)h^2}{fh^2}\ . \tag{A26}$$

It is useful to expand in $\rho\equiv p/n$. After some algebra we find that, to order ρ,

$$f_\pm=1\pm e^{\pm\beta}+\rho(e^{\pm\beta}\mp1)/2\coth\beta \tag{A27}$$

and

$$f_\pm h_\pm=-2\alpha\left[\pm e^{\pm\beta}\pm\rho e^{\pm\beta}\mp\frac{\rho(1+\alpha)}{2(1+\alpha^2)^{1/2}}-\frac{1}{2}\rho\right]\ , \tag{A28}$$

so that

$$\ln|fh|=\ln2\alpha\pm\beta+\rho[1-(\pm1+e^{mp\beta})]/2\cosh\beta\ . \tag{A29}$$

In the expression for ϕ there are cancellations between various ρ terms, so that altogether we have

$$\phi(x_\pm)=n\left[-\frac{1\pm e^{\pm\beta}}{\sinh\beta}+\ln n\pm\beta+\rho\ln f\right]+p\ln(\lambda/2g)\ . \tag{A30}$$

To the order we are working we can evaluate the prefactor at $p=0$, since the difference is of order $(1/K)$. Then

$$\phi''(x_\pm)=-\frac{4\lambda\alpha}{|h_\pm|}\frac{2(1+\alpha^2)^{1/2}}{\alpha} =-\frac{8\lambda}{|h_\pm|}\cosh\beta\ . \tag{A31}$$

The final formula for c_n^p is

$$c_n^p(\lambda)=\frac{2}{n!}\left[\frac{\pi}{4\lambda\cosh\beta_0}\right]^{1/2}\left[\frac{\lambda}{2g}\right]^p n^n\exp\left[-n\frac{1+\sinh\beta_0}{\sinh\beta_0}\right]$$
$$\times\left\{|1+e^{-\beta}|^{1/2}\exp\left[-n\epsilon\frac{\cosh\beta_0}{\cosh\beta_0+1}+p\ln(1-e^{-\beta})\right]+(-1)^n|e^{\beta}-1|^{1/2}\exp\left[n\epsilon\frac{\cosh\beta_0}{\cosh\beta_0-1}+p\ln(1+e^{\beta})\right]\right\}. \tag{A32}$$

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