Effective potential and stability of the rigid membrane

E. Elizalde* and S. D. Odintsov†
Department E.C.M., Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain
(Received 11 May 1992)

The calculation of the effective potential for fixed-end and toroidal rigid p-branes is performed in the one-loop as well as in the $1/d$ approximations. The analysis of the involved ζ functions (of inhomogeneous Epstein type) which appear in the process of regularization is done in full detail. Asymptotic formulas (allowing only for exponentially decreasing errors of order $\leq 10^{-3}$) are found which carry all the dependences on the basic parameters of the theory explicitly. The behavior of the effective potential (specified to the membrane case $p = 2$) is investigated, and the extrema of this effective potential are obtained.

PACS number(s): 11.17.+y, 03.70.+k, 11.10.Gh

I. INTRODUCTION

The theory of the rigid string [1,2] is interesting because of the number of applications it has to quantum chromodynamics (see [1–3] and references therein) and to statistical physics. Using the same idea, it is not difficult to construct the action for the rigid membrane (or p-brane). During the last few years, there has been some activity in the study of quantum extended objects (such as membranes, see Ref. [4] for a review). However, already, the semiclassical quantization of such a highly nonlinear system as a membrane is a very difficult task [5–7]. Nevertheless, some interesting issues, such as the study of the Casimir energy, the large-d approximation, and the tachyon problem can be addressed already at the semiclassical level [8,9].

In the present paper, we study the Casimir energy and the static potential for the rigid p-brane (at the classical level, this theory has been considered in Ref. [10]), specifying afterwards our results to the membrane case ($p = 2$). We shall start from the following action, which is multiplicatively renormalized only in the string case ($p = 1$):

$$S = \int d^{p+1}\xi \, \sqrt{g} \left(k + \frac{1}{2\rho^2} \left[\Delta(g) X^i\right]^2 \right), \tag{1}$$

where $g_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^i$, $\alpha = 0, 1, \ldots, p$, $i = 1, 2, \ldots, d$, and $\Delta(g) = g^{-1/2}\partial_\alpha g^{1/2} g^{\alpha\beta}\partial_\beta$, the constant k is the analogue of the usual string tension, and $1/\rho^2$ is the coupling constant corresponding to the rigid term.

Let us note that the p-brane is an interacting system, without a free part in the action. Hence, one must start from some classical solution for the ground state, and then study the quantum fluctuations on such background. In this framework we can understand how the tachyon appears (if that is the case), if the background is stable, and also address some other issues. Owing to the fact that string theory can be obtained as some compactification of the membrane [11], we can also expect to find in this way some new features of string physics.

The paper is organized as follows. In Sec. II we calculate the potential corresponding to two cases: fixed-end and periodic boundary conditions. In Sec. III we obtain the static potential, that is, the effective potential in the limit of large spacetime dimensionality. Owing to the difficulty of the exact expressions, a saddle-point analysis is carried out in Sec. IV. In Sec. V we provide a short summary of very useful mathematical results on the inhomogeneous Epstein-type ζ functions and apply them to the expressions that appear in the process of regularization. Finally, in Sec. VI we study the general case and provide some discussion of the results obtained.

II. CALCULATION OF THE POTENTIAL

We consider as the background the classical solutions of the field equations [8,9] (which are the same for the rigid as for the usual p-brane)

$$X^0_{\rm cl} = \xi_0, \quad X^\perp_{\rm cl} = 0, \quad X^{d-1}_{\rm cl} = \xi_1, \ldots, X^{d-p}_{\rm cl} = \xi_p, \tag{2}$$

with $X^\perp_{\rm cl} = (X^1, \ldots, X^{d-p-1})$ and

$$(\xi_1, \ldots, \xi_p) \in \mathcal{R} \equiv [0, a_1] \times \cdots \times [0, a_p].$$

We also use the axial gauge of Ref. [8], i.e.,

$$X^0 = X^0_{\rm cl}, \quad X^{d-1} = X^{d-1}_{\rm cl}, \ldots, X^{d-p} = X^{d-p}_{\rm cl}, \tag{3}$$

where the Faddeev-Popov ghosts are absent.

We shall consider the toroidal rigid p-brane which has the boundary conditions

$$X^\perp(0, \xi_1, \ldots, \xi_p) = X^\perp(T, \xi_1, \ldots, \xi_p) = 0 \tag{4}$$

and

*Electronic address: eli@ebubecm1.bitnet
†On leave from Department of Mathematics and Physics, Pedagogical Institute, 634041 Tomsk, Russia.

$$X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) = X^{\perp}(\xi_0, a_1, \xi_2, \ldots, \xi_p),$$
$$\vdots \tag{5}$$
$$X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0) = X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p).$$

For the fixed-end boundary conditions, Eq. (4) is exactly the same, while Eqs. (5) are replaced by (of Dirichlet type)

$$X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) = \cdots = X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0),$$
$$X^{\perp}(\xi_0, a_1, \xi_2, \ldots, \xi_p) = \cdots = X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p). \tag{6}$$

The effective potential is given by

$$V = -\lim_{T\to\infty} \frac{1}{T} \ln \int \mathcal{D}X^{\perp} \exp(-S). \tag{7}$$

Restricting ourselves to the one-loop approximation, we need only consider the terms which are quadratic in the quantum fields (this applies to the usual membrane and p-brane; see [8,9]).

Integrating out $X^{\perp}$ and using boundary conditions to read off the resulting Tr $\ln \Delta$ (see [8,9,12]), we get

$$V_{\text{fixed end}} = k\prod_{i=1}^{p} a_i + \frac{d-p-1}{2}\left[\sum_{n_1,\ldots,n_p=1}^{\infty}\left(\frac{\pi^2 n_1^2}{a_1^2}+\cdots+\frac{\pi^2 n_p^2}{a_p^2}\right)^{1/2} + \sum_{n_1,\ldots,n_p=1}^{\infty}\left(\frac{\pi^2 n_1^2}{a_1^2}+\cdots+\frac{\pi^2 n_p^2}{a_p^2}+k\rho^2\right)^{1/2}\right], \tag{8}$$

and

$$V_{\text{toroidal}} = k\prod_{i=1}^{p} a_i + \frac{d-p-1}{2}\left[\sum_{n_1,\ldots,n_p=-\infty}^{\infty}\left(\frac{4\pi^2 n_1^2}{a_1^2}+\cdots+\frac{4\pi^2 n_p^2}{a_p^2}\right)^{1/2} + \sum_{n_1,\ldots,n_p=-\infty}^{\infty}\left(\frac{4\pi^2 n_1^2}{a_1^2}+\cdots+\frac{4\pi^2 n_p^2}{a_p^2}+k\rho^2\right)^{1/2}\right]. \tag{9}$$

Observe that the contribution from the higher-derivative mode appears in (8) and (9) with a positive sign, as it follows from the path integral. In the second of Refs. [3] this sign has been taken to be negative "by hand." Actually, some arguments based on the linearized approximations were given there in favor of the stability of such quantum field theory. Unfortunately, the expression for the Casimir energy obtained in [3] (second reference) disagrees with the large-d approximation [2].

III. THE LIMIT OF LARGE SPACETIME DIMENSIONALITY

First we calculate the static potential; that is, the effective potential in the limit of large spacetime dimensionality. Such calculation for the usual Nambu-Goto or Eguchi string [13] has been carried out in Ref. [14], and for the rigid string in Ref. [2]. Let us introduce the composite fields $\sigma_{\alpha\beta}$ for $\partial_\alpha X^{\perp}\cdot\partial_\beta X^{\perp}$, and constrain $\sigma_{\alpha\beta} = \partial_\alpha X^{\perp}\cdot\partial_\beta X^{\perp}$ by introducing Lagrange multipliers $\lambda^{\alpha\beta}$:

$$Z = \int \mathcal{D}X^{\perp}\mathcal{D}\sigma\mathcal{D}\lambda\, \exp\left\{-k\int d^{p+1}\xi\, [\det(\delta_{\alpha\beta}+\sigma_{\alpha\beta})]^{1/2} + \frac{1}{2\rho^2}\int d^{p+1}\xi\Delta_0 X^{\perp}\cdot\Delta_0 X^{\perp} - \frac{k}{2}\int d^{p+1}\xi\lambda^{\alpha\beta}\left(\partial_\alpha X^{\perp}\cdot\partial_\beta X^{\perp}-\sigma_{\alpha\beta}\right)\right\}, \tag{10}$$

where $\Delta_0 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. Integrating over $X^{\perp}$, we get

$$Z = \int \mathcal{D}\sigma\mathcal{D}\lambda\, \exp(-S_{\text{eff}}), \tag{11}$$

with

$$S_{\text{eff}} = \frac{1}{2}(d-p-1)\text{Tr}\,\ln\left(\frac{1}{\rho^2}\Delta_0^2 + k\partial_\alpha\lambda^{\alpha\beta}\partial_\beta\right) + kTR^p\left[(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}(\sigma_0\lambda_0 + p\sigma_1\lambda_1)\right], \tag{12}$$

where we have chosen $a_1 = \cdots = a_p = R$, and $\sigma_{\alpha\beta} = \text{diag}(\sigma_0, \sigma_1, \ldots, \sigma_1)$, $\lambda_{\alpha\beta} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_1)$ (compare with [8,9], where the case $1/\rho^2 = 0$ was considered).

In this case we obtain

$$\mathrm{Tr}\ \ln\left[\Delta_0^2 + k\rho^2(\lambda_0\partial_0^2 + \lambda_1\boldsymbol{\partial}_x^2)\right] = \mathrm{Tr}\ \ln\left[(\partial_0^2 + \boldsymbol{\partial}_x^2)^2 + k\rho^2(\lambda_0\partial_0^2 + \lambda_1\boldsymbol{\partial}_x^2)\right]$$
$$= \mathrm{Tr}\ \ln\left\{\left[\partial_0^2 + \left(\boldsymbol{\partial}_x^2 + \frac{k\rho^2\lambda_0}{2}\right) - \sqrt{k\rho^2(\lambda_0 - \lambda_1)\boldsymbol{\partial}_x^2 + \frac{k^2\rho^4\lambda_0^2}{4}}\right]\right.$$
$$\left.\times\left[\partial_0^2 + \left(\boldsymbol{\partial}_x^2 + \frac{k\rho^2\lambda_0}{2}\right) + \sqrt{k\rho^2(\lambda_0 - \lambda_1)\boldsymbol{\partial}_x^2 + \frac{k^2\rho^4\lambda_0^2}{4}}\right]\right\}. \tag{13}$$

The spectrum for the boundary conditions (2)–(6) is known. Using this spectrum and evaluating the Tr ln terms by means of analytic regularization for large T, we obtain (see [8,9] for details of this method)

$$S_{\text{eff}} = kTR^p\left\{(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}(\sigma_0\lambda_0 + p\sigma_1\lambda_1)\right.$$
$$+\frac{d-p-1}{2kR^{p+1}}\left[\sum_{\mathbf{n}}\left(\pi^2\mathbf{n}^2 + \frac{k\rho^2\lambda_0R^2}{2} - \sqrt{k\rho^2(\lambda_0-\lambda_1)\mathbf{n}^2R^2 + \frac{k^2\rho^4\lambda_0^2R^4}{4}}\right)^{1/2}\right.$$
$$\left.\left.+\sum_{\mathbf{n}}\left(\pi^2\mathbf{n}^2 + \frac{k\rho^2\lambda_0R^2}{2} + \sqrt{k\rho^2(\lambda_0-\lambda_1)\mathbf{n}^2R^2 + \frac{k^2\rho^4\lambda_0^2R^4}{4}}\right)^{1/2}\right]\right\}, \tag{14}$$

where for the fixed-end p-brane $\mathbf{n}^2 = n_1^2 + \cdots + n_p^2$ and $\sum_{\mathbf{n}}$ means $\sum_{n_1,\ldots,n_p=1}^{\infty}$ as in (8), while for the toroidal p-brane $\mathbf{n}^2 = 4(n_1^2 + \cdots + n_p^2)$ and $\sum_{\mathbf{n}}$ means $\sum_{n_1,\ldots,n_p=-\infty}^{\infty}$ as in (9).

IV. A SADDLE-POINT ANALYSIS

The functions that appear on the right-hand side of (14) are rather complicated to analyze. To our knowledge, they have never been considered in the literature and will be the object of a separate investigation. (Note that in the case of the usual Dirac p-brane [8,9] these functions are simply constants.) So we shall have little to say here about the corresponding effective potential, only, for example, that as $R \to \infty$, $V \sim V_{\text{cl}} = kR^p$. Rewriting the expression for the static potential identically as

$$V = kR^p\left[(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}(\sigma_0\lambda_0 + p\sigma_1\lambda_1) + \frac{d-p-1}{2kR^{p+1}}K(k\rho^2R^2, \lambda_0, \lambda_1)\right], \tag{15}$$

we are led to the four saddle-point equations:

$$\lambda_1 = (1+\sigma_1)^{p/2-1}(1+\sigma_0)^{1/2}, \quad \sigma_0 = \frac{d-p-1}{kR^{p+1}}\frac{\partial K(k\rho^2R^2,\lambda_0,\lambda_1)}{\partial\lambda_0},$$
$$\lambda_1 = (1+\sigma_1)^{p/2-1}(1+\sigma_0)^{1/2}, \quad \sigma_1 = \frac{d-p-1}{kpR^{p+1}}\frac{\partial K(k\rho^2R^2,\lambda_0,\lambda_1)}{\partial\lambda_1}. \tag{16}$$

By eliminating from these equations σ_0 and σ_1, we get

$$(\lambda_1\lambda_0)^{p/(p-1)}\lambda_0^{-2} - 1 = \frac{d-p-1}{kR^{p+1}}\frac{\partial K(k\rho^2R^2,\lambda_0,\lambda_1)}{\partial\lambda_0}, \quad (\lambda_1\lambda_0)^{1/(p-1)} - 1 = \frac{d-p-1}{kpR^{p+1}}\frac{\partial K(k\rho^2R^2,\lambda_0,\lambda_1)}{\partial\lambda_1}. \tag{17}$$

In principle, if not analytically, it is of course possible to eliminate (let us say) λ_0 from these two equations by means of a numerical calculation, and to rewrite V in terms of λ_1 only. After doing this, one can study V as a function of R and in terms of this parameter λ_1, in order to see if the tachyon is in the spectrum. We do not do this work here, since our purpose is just to show the possibility, in principle, of calculating the static potential for a rigid string. However, we are ready to look to some interesting limiting case of these equations. Let us assume that $\lambda_0 = \lambda_1 \equiv \lambda$ (such a choice has been taken for the rigid string in [2]). Then

$$V = kR^p\left\{(1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}\lambda(\sigma_0 + p\sigma_1) + \frac{(d-p-1)\pi}{2kR^{p+1}}\left[\sum_{\mathbf{n}}\sqrt{\mathbf{n}^2} + \sum_{\mathbf{n}}[\mathbf{n}^2 + k(\rho/\pi)^2\lambda R^2]^{1/2}\right]\right\}. \tag{18}$$

It follows from the saddle-point equations that $\sigma_0 = \sigma_1$, $\lambda = (1+\sigma)^{(p-1)/2}$, and

$$V = kR^p \left[\lambda^{(p+1)/(p-1)} - \frac{p+1}{2}\lambda(\lambda^{2/(p-1)} - 1) + \frac{(d-p-1)\pi}{2kR^{p+1}} K(k\rho^2\lambda R^2) \right]. \tag{19}$$

Here we have defined

$$K(k\rho^2\lambda R^2) \equiv \sum_{\mathbf{n}} \sqrt{\mathbf{n}^2} + \sum_{\mathbf{n}} [\mathbf{n}^2 + k(\rho/\pi)^2\lambda R^2]^{1/2}, \tag{20}$$

and the last saddle-point equation gives

$$-\frac{p+1}{2}\lambda^{2/(p-1)} + \frac{(d-p-1)\pi}{2kR^{p+1}} \frac{\partial K(k\rho^2\lambda R^2)}{\partial\lambda} = 0. \tag{21}$$

V. EXPLICIT EXPRESSIONS FOR THE INHOMOGENEOUS EPSTEIN FUNCTIONS

The expressions to be regularized are (in general) of the inhomogeneous Epstein form [15]

$$E_p^c(s) \equiv \sum_{n_1,\dots,n_p=1}^{\infty} (n_1^2 + \cdots + n_p^2 + c^2)^{-s}, \tag{22}$$

allowing for $c = 0$. These functions are not easy to deal with (for $p > 1$, the case $p = 1$ is the only one that has been investigated in the literature), and Refs. [15] can be considered pioneering in this respect. There, very general formulas have been derived for the functions:

$$M_N^c(s; a_1, \dots, a_N; \alpha_1, \dots, \alpha_N) \equiv \sum_{n_1,\dots,n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \cdots + a_N n_N^{\alpha_N} + c^2)^{-s}, \tag{23}$$

for any value of c, such as

$$\begin{aligned} M_N^c(s; a_1, \dots, a_N; \alpha_1, \dots, \alpha_N) = {} & \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \frac{b_{i_r}^{-1/\alpha_{i_r}}}{\alpha_{i_r}} \Gamma\left(\frac{1}{\alpha_{i_r}}\right) \\ & \times \sum_{k_{j_1},\dots,k_{j_{N-p-1}}=0}^{\infty} \Gamma\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right) \\ & \times \prod_{l=1}^{N-p-1} \left[\frac{(-b_{j_l})^{k_{j_l}}}{k_{j_l}!} \zeta(-\alpha_{j_l} k_{j_l})\right] \\ & \times \zeta\left(\alpha_n \left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right)\right) + \Delta_{ER}, \end{aligned} \tag{24}$$

with $b_{i_r} \equiv a_{i_r}/a_N$ [notice the slight erratum in Eq. (3.22) of the second of Refs. [15]], and $1 \le i_1 < \cdots < i_p \le N-1$, $1 \le j_1 < \cdots < j_{N-p-1} \le N-1$, being $i_1, \dots, i_p, j_1, \dots, j_{N-p-1}$, a permutation of $1, 2, \dots, N-1$. The sum on $C_{N-1,p}$ means sum over the $\binom{N-1}{p}$ choices of the indices $i_1, \dots, i_p$ among the $1, 2, \dots, N-1$.

In our case, $a_1 = \cdots = a_p = 1$ and $\alpha_1 = \cdots = \alpha_p = 2$, and the involved general formula above, (24), is simplified considerably. We have, for $c = 0$,

$$E_p(s) = \frac{(-1)^{p-1}}{2^{p-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \Gamma(2s - j) \zeta\left(s - \frac{j}{2}\right) + \Delta_{ER}, \tag{25}$$

and for $c \neq 0$,

$$E_p^c(s) = \frac{(-1)^{p-1}}{2^{p-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \Gamma\left(s - \frac{j}{2}\right) E_1^c\left(s - \frac{j}{2}\right) + \Delta_{ER}. \tag{26}$$

Notice that the poles of this function arise from those of $E_1^c(s - j/2)$, which are obtained for the values of s such that $s - j/2 = 1/2, -1/2, -3/2, \dots$. They are poles of order one at $s = p/2, (p-1)/2, p/2 - 1, \dots$, except for $s = 0, -1, -2, \dots$, in which cases the function is finite [owing to the $\Gamma(s)$ in the denominator]. These poles are removed by ζ-function regularization [16].

Alternatively, a very useful and exact recurrent formula is [15]

$$\begin{aligned}E_N^c(s;a_1,\dots,a_N) &\equiv \sum_{n_1,\dots,n_N=1}^{\infty}(a_1n_1^2+\cdots+a_Nn_N^2+c^2)^{-s}\\ &= -\frac{1}{2}E_{N-1}^c(s;a_2,\dots,a_N)+\frac{1}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_{N-1}^c(s-1/2;a_2,\dots,a_N)\\ &\quad+\frac{\pi^s}{\Gamma(s)}a_1^{-s/2}\sum_{k=0}^{\infty}\frac{a_1^{k/2}}{k!(16\pi)^k}\prod_{j=1}^{k}[(2s-1)^2-(2j-1)^2]\\ &\qquad\times\sum_{n_1,\dots,n_N=1}^{\infty}n_1^{s-k-1}(a_2n_2^2+\cdots+a_Nn_N^2+c^2)^{-(s+k)/2}\\ &\qquad\times\exp\left[-\frac{2\pi}{\sqrt{a_1}}n_1(a_2n_2^2+\cdots+a_Nn_N^2+c^2)^{1/2}\right].\end{aligned} \tag{27}$$

The recurrence starts from expression

$$E_1^c(s;1)=-\frac{c^{-2s}}{2}+\frac{\sqrt{\pi}}{2}\frac{\Gamma(s-1/2)}{\Gamma(s)}c^{-2s+1}+\frac{2\pi^s c^{-s+1/2}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}(2\pi nc). \tag{28}$$

In order to deal with the derivative of the function K above, one can follow two equivalent procedures: either first do the usual analytic continuation, and then take $s=-1/2$ and the derivative, or else first take the derivative of (22), perform the analytic continuation, and put $s=+1/2$ at the end. The result is exactly the same. Either way, other nontrivial series commutations have to be performed (see [17] and references therein). We get, in particular, for $c\neq 0$,

$$E_2^c(s)=-\frac{1}{2}E_1^c(s)+\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_1^c\left(s-\frac{1}{2}\right)+\Delta_{ER}, \tag{29}$$

$$E_3^c(s)=\frac{1}{4}E_1^c(s)-\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_1^c\left(s-\frac{1}{2}\right)+\frac{\pi}{4(s-1)}E_1^c(s-1)+\Delta_{ER},$$

and similar expressions for $c=0$, as it is clear from (25) and (26) (it has been proven in [16] that this case can be obtained from the former by analytically continuing in the parameter c). Remember that Δ_{ER} is the well-known term (here exponentially small), which was found in [17]. As our final interest is numerical approximation (see above), we will not take into account exponentially small terms (let us point out that these expansions are asymptotic and very quickly convergent) [15]. Notice, moreover, that for $c=0$ there is no dependence on λ, so the corresponding term does not contribute to Eq. (22).

From the above expressions, for any value of p there is no difficulty in obtaining the value of λ which solves (22). In particular, for $p=2$ we have

$$\lambda\simeq\left(\frac{3-d}{12\pi}\right)^{2/3}k^{1/3}\rho^2; \tag{30}$$

this is a sensible root for large $k\cdot\rho$ (specifically, for $1/R<<k^{2/3}\rho^2$) and $d\neq 3$. For $p=3$ the result is

$$\lambda\simeq\frac{1}{2^{10}}\left[\frac{3(4-d)}{\pi}\right]^2\frac{k\rho^6}{R^2}, \tag{31}$$

which is a sensible root for large $k\cdot\rho$ (specifically, for $k\rho^4>>1$) and $d\neq 4$. Let us now substitute these values into the expression of V, and look for the derivative $\partial V/\partial R$. We get, for $p=2$, an expression of the form

$$\frac{\partial V_2}{\partial R}=c_1R+\frac{c_{-1}}{R}+\frac{c_{-2}}{R^2}, \tag{32}$$

which always has one real root (at least) R_2. It corresponds to a minimum of V for $d>3$ and reasonable values of the constants involved. Substituting back into $V(R)$ we see that the minimum is attained at

$$V_2(R_2)=kR_2^2\left[3(1+\sigma_0)^{1/2}(1+\sigma_1)-\frac{3}{2}\left(\frac{3-d}{12\pi}\right)^{2/3}(\sigma_0+2\sigma_1)k^{1/3}\rho^2+\frac{(d-3)^2}{48\pi^2}\rho^6\right], \tag{33}$$

i.e., for large k it is obtained at a negative value of V, while for large ρ it is reached at a positive value of V.

The case $p=3$ is quite different. We then get

$$V_3(R)=\alpha_1R^3-\alpha_2R+\frac{\beta}{R}, \tag{34}$$

so that its derivative has two real roots:

$$R_\pm=\pm\left[\frac{\alpha_2+(\alpha_2^2+12\alpha\beta)^{1/2}}{6\alpha_1}\right]^{1/2}, \tag{35}$$

one of which is seen to correspond to a maximum and the other to a minimum of V. Moreover, two additional, complex roots appear. The minimum for V is now attained at a negative value of V when either k or ρ are large enough and, conversely, at a positive value of the potential for small k or ρ. Note that in order to find the critical radius at which the potential becomes complex (so that the static approximation breaks down and tachyons appear) it is necessary to do the analysis with the general expression (15) directly. One can conjecture from our results here that the rigid membrane is tachyon-free (no critical radius exists), as it is also the case with rigid strings (see the second of Refs. [2] and the first of Refs. [3]). At least for the limiting situation discussed above, this is in fact the case.

VI. DISCUSSION OF THE GENERAL CASE

Having done the calculation for this special case, corresponding to the limit of large spacetime dimensionality, and armed with the full equation (24), we can now be more ambitious and consider the one-loop effective potential, (8)–(9), without further limit or approximation. For the sake of conciseness, we shall restrict ourselves to $p=2$, but it is obvious that we could consider as well any other value of p. We rely on Eqs. (27) and (28), which specialized to $p=2$ yield, after some work,

$$\sum_{n_1,n_2=1}^{\infty}\sqrt{\left(\frac{n_1}{a_1}\right)^2+\left(\frac{n_2}{a_2}\right)^2}=\frac{1}{24}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)-\frac{\zeta(3)}{8\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right)-\frac{\pi^{3/2}}{2\sqrt{a_1a_2}}\left[\exp\left(-2\pi\frac{a_1}{a_2}\right)[1+O(10^{-3})]\right], \qquad (36)$$

and (this one after additional regularization, see above)

$$\sum_{n_1,n_2=1}^{\infty}\sqrt{\left(\frac{n_1}{a_1}\right)^2+\left(\frac{n_2}{a_2}\right)^2+c^2}=\frac{c}{4}-\frac{\pi}{6}a_1a_2c^3+\left(\frac{1}{4\pi}\sqrt{\frac{c}{a_2}}-\frac{ca_1}{4\pi a_2}\right)\left\{\exp\left(-2\pi c a_2\right)\left[1+O(10^{-3})\right]\right\}. \qquad (37)$$

In both cases we have assumed (this is, of course, no restriction) that $a_2 \le a_1$.

These expressions are really valuable. They are asymptotic, the last term (already of exponential kind) being of order 10^{-3} with respect to the two first ones, and the not explicitly written contributions being of order 10^{-6}. To our knowledge, the second expression, which can be termed as of inhomogeneous Epstein type, has never been discussed in the literature.

For fixed-end boundary conditions, and not taking into account exponentially small terms, we obtain

$$V_{\text{fe}} \simeq ka_1a_2+\frac{(d-3)\pi}{24}\left[\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)-\frac{3\zeta(3)}{2\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right)+\frac{3}{\pi}\sqrt{k}\rho-\frac{2}{\pi^2}k^{3/2}\rho^3a_1a_2\right]. \qquad (38)$$

It is now straightforward to perform the analysis of extrema of V. For brevity, we shall only discuss here some particular cases. First will be the one which is obtained from the two Lagrange equations for the extrema of V as a function of a_1 and a_2 only, for $a_1=a_2\equiv a$, which is reached for

$$a=\left(\frac{\frac{3-d}{8}[3\zeta(3)-\pi^2]}{12\pi k+(3-d)k^{3/2}\rho^3}\right)^{1/3}. \qquad (39)$$

It can be seen that for $12\pi k+(3-d)k^{3/2}\rho^3<0$ this point is a minimum. On the contrary, it is a maximum for $12\pi k+(3-d)k^{3/2}\rho^3>0$. Consistency with the range of validity of the series expansion above is obtained for

$$2\pi\sqrt{k}\rho>\frac{6}{\rho^2}>>1. \qquad (40)$$

Typical values for which this is valid are $\rho\simeq 2/3$, $k\simeq 4$, and $2\pi c\simeq 8$.

Now keeping a_1 and a_2 fixed (but arbitrary), we see that (for $d>3$) in terms of ρ, V is unbounded from below, being always negative for large enough ρ. Considering V as a function of k, the situation is similar. Finally, in a sense the analysis of Ref. [8] is still valid here when we fix the values of k, ρ and of the area $A=a_1a_2$; the minima of the potential are obtained for elongated (stringy) membranes (small a_1/a_2). Notice, however, that even for the particular case considered in [8], our asymptotic expansion provides a more universal expression, because it is valid for any value of $a_2 \le a_1$ (this is again not restrictive, in the end). It also goes without saying that, from our general formula (38) for the potential $V=V(a_1,a_2,k,\rho,d)$, one can perform a simultaneous analysis on *all* the different parameters at the same time—e.g., in order to look for local minima of the potential hypersurface—the explicit dependences on k and ρ being also basic outcomes of the present work.

In the case of toroidal boundary conditions, again neglecting exponentially small contributions, we get (for a very detailed discussion of the relations between the different boundary conditions see the last of Refs. [15])

$$V_{\rm tor} \simeq ka_1a_2 + \frac{(d-3)\pi}{2}\left[-\frac{\zeta(3)}{\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right)\frac{1}{\pi}\sqrt{k}\rho - \frac{1}{6\pi^2}k^{3/2}\rho^3a_1a_2\right]. \qquad (41)$$

The particular extremum for $a_1 = a_2 \equiv a$ is a minimum of V provided that $\sqrt{k}\rho^3 > 12\pi$ (it is a maximum for $\sqrt{k}\rho^3 < 12\pi$). Consistency with the series expansion now implies

$$\sqrt{k}\rho > \frac{12\pi}{\rho^2} >> 1, \qquad (42)$$

which can be met typically for values of $\rho \simeq 3$, $k \simeq 9$, and $2\pi c \simeq 9$, but, of course, as in the former case, the range of allowed values is much wider.

ACKNOWLEDGMENTS

We would like to thank Professor Domènec Espriu, Professor Ulf Lindström, Professor Kelly Stelle, and Professor Robin Tucker for fruitful discussions on connected problems. S.D.O. thanks the members of the Department E.C.M. of Barcelona University for their kind hospitality, and E.E. the Alexander von Humboldt Foundation for continued attentions. This work has been supported by Dirección General de Investigación Científica y Técnica (DGICYT), research projects PB90-0022 and SAB92-0072.

[1] A.M. Polyakov, Nucl. Phys. **B268**, 406 (1986); T.L. Curtright, G.I. Ghandhour, and C.K. Zachos, Phys. Rev. Lett. **57**, 799 (1986); Phys. Rev. D **34**, 3811 (1986); H. Kleinert, Phys. Lett. B **174**, 335 (1986).

[2] F. Alonso and D. Espriu, Nucl. Phys. **B283**, 393 (1987); E. Braaten, R.D. Pisarski, and Sze-man Tze, Phys. Rev. Lett. **58**, 93 (1987); H. Kleinert, *ibid.* **58**, 1915 (1987); P. Olesen and S.-K. Yang, Nucl. Phys. **B283**, 74 (1987).

[3] R. Pisarski, Phys. Rev. Lett. **58**, 1300 (1987); V.V. Nesterenko and N.R. Shvetz, Z. Phys. C **55**, 265 (1992).

[4] E. Bergshoeff, E. Sezgin, and P.K. Townsend, Ann. Phys. (N.Y.) **185**, 330 (1988); M.J. Duff, Class. Quantum Grav. **6**, 1577 (1989); A. Achúcarro, J.M. Evans, P.K. Townsend, and D.L. Wiltshire, Phys. Lett. B **198**, 441 (1987); A.A. Bytsenko and S.D. Odintsov, Fortschr. Phys. (to be published).

[5] M.J. Duff, T. Inami, C.N. Pope, E. Sezgin, and K.S. Stelle, Nucl. Phys. **B297**, 515 (1988).

[6] K. Fujikawa and J. Kubo, Phys. Lett. B **199**, 75 (1987).

[7] K. Fujikawa, Phys. Lett. B **206**, 18 (1988); Uji Research Center report, 1991 (unpublished).

[8] E.G. Floratos, Phys. Lett. B **220**, 61 (1989); E.G. Floratos and G.K. Leontaris, *ibid.* **223**, 37 (1989).

[9] S.D. Odintsov and D.L. Wiltshire, Class. Quantum Grav. **7**, 1499 (1990); A.A. Bytsenko and S.D. Odintsov, *ibid.* **9**, 391 (1992).

[10] D.H. Hartley, M. Önder, and R.W. Tucker, Class. Quantum Grav. **6**, 30 (1989).

[11] M.J. Duff, P.S. Howe, T. Inami, and K. Stelle, Phys. Lett. B **191**, 70 (1987); A. Achúcarro, P. Kapusta, and K.S. Stelle, *ibid.* **232**, 302 (1989); U. Lindström, Stockholm report, 1990 (unpublished).

[12] S.D. Odintsov, Phys. Lett. B **247**, 21 (1990).

[13] Y. Nambu, in *Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970); T. Goto, Prog. Theor. Phys. **46**, 1560 (1971); T. Eguchi, Phys. Rev. Lett. **44**, 126 (1980).

[14] O. Alvarez, Phys. Rev. D **24**, 440 (1981).

[15] E. Elizalde, J. Math. Phys. **31**, 170 (1990); J. Phys. A **22**, 931 (1989); Nuovo Cimento B **104**, 685 (1989).

[16] E. Elizalde and A. Romeo, Int. J. Mod. Phys. A **7**, 7365 (1992); K. Kirsten, J. Math. Phys. **32**, 3008 (1991).

[17] E. Elizalde and A. Romeo, Phys. Rev. D **40**, 436 (1989); A. Actor, Nucl. Phys. **B265** [FS15], 689 (1986); H. A.