Bose-Einstein condensation as symmetry breaking in curved spacetime and in spacetimes with boundaries

David J. Toms
Physics Department, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, United Kingdom
(Received 23 October 1992)

A quantum field theory treatment of Bose-Einstein condensation for a charged, noninteracting scalar field in a static spacetime with a possible spatial boundary is given. An interpretation is presented in terms of symmetry breaking to give a nonconstant scalar field expectation value. The method used is a computation of the effective action in the high-temperature limit. Results are obtained for static spacetimes of general dimension. The critical temperature for Bose-Einstein condensation is obtained in terms of the lowest eigenvalue of the Laplacian with the scalar field subject to the appropriate boundary conditions. A number of applications are provided for flat spacetimes with and without boundaries, and to curved spacetimes. In special cases where the scalar field expectation value is constant, some previously known results are obtained; in other cases, new results are found.

PACS number(s): 03.70.+k, 04.90.+e, 11.15.Ex, 11.30.Qc

I. INTRODUCTION

The theory of Bose-Einstein condensation has been of continuing interest to physicists, theorists and experimentalists alike since its original discovery for nonrelativistic particles [1,2]. Textbook treatments of Bose-Einstein condensation include Refs. [3–5]. Part of the reason for its continuing popularity lies in the fact that it provides at least a partial understanding of the behavior of liquid helium at low temperatures, as suggested originally by London [6,7]. Of course, modeling a physical system such as liquid helium as an ideal gas obeying Bose-Einstein statistics ignores any possible interactions between the particles. The role of interactions in Bose-Einstein condensation has been considered in Refs. [8–10].

Bose-Einstein condensation in relativistic quantum mechanics has also been considered [11–18]. A more complete treatment of Bose-Einstein condensation for relativistic systems has been given using the techniques of modern quantum field theory at finite temperature and density [19–23]. In particular, it was shown that a proper account of antiparticles must be taken, and that unlike the nonrelativistic case, Bose-Einstein condensation can occur at high temperatures. The other important aspect of the quantum field theory approach to Bose-Einstein condensation is that the accumulation of particles in the ground state may be understood as spontaneous symmetry breaking in the sense that the vacuum expectation value of a scalar field becomes nonzero at a critical temperature [20,21].

The work referenced so far considers relativistic or nonrelativistic systems in flat spacetime with no boundaries present. The case of flat spaces with boundaries was originally motivated by a desire to understand the behavior of liquid helium in thin films. (See Refs. [24,25] for example.) Nonrelativistic systems in rectangular cavities with a variety of boundary conditions have been studied extensively by Pathria and his co-workers [26–32]. (This work with further references is reviewed in Ref. [33].) Relativistic systems in rectangular cavities have also been treated [34,35].

The generalization from flat spacetime to curved spacetime has been considered. The spacetime which has received the greatest attention has been the Einstein static universe, for which the spatial section is a three-sphere. The nonrelativistic ideal Bose-Einstein gas was treated by Al'taie [36]. The generalization to relativistic scalar fields in the Einstein universe was given in Ref. [37] for conformal coupling, and in Ref. [38] for minimal coupling. The case of higher-dimensional spheres was considered by Shiraishi [39], whose expression for the critical temperature is only correct for minimally coupled fields. None of these references interpret Bose-Einstein condensation in terms of symmetry breaking.

One basic reason for dealing with cubic enclosures or spheres is that the complete eigenvalue spectrum of the Laplacian is known. In these cases, because the eigenvalues are so simple, it is possible to obtain the partition function exactly, or evaluate it approximately in certain limits, by using a variety of summation techniques. For more general spaces, where the eigenvalues of the Laplacian may not be known explicitly, other methods must be found to study Bose-Einstein condensation. The expansion of the thermodynamic potential in the high-temperature limit in fairly general classes of spacetimes has been the subject of several papers [40–44]. In addition, Kirsten [43] has looked at Bose-Einstein condensation in some cases using this high-temperature expansion, but without the interpretation in terms of symmetry breaking.

The purpose of the present paper is to study Bose-Einstein condensation in the fairly general setting of a static spacetime whose spatial section is an arbitrary manifold with or without boundary. This paper is a detailed version of Ref. [45]. The technique used consists of an evaluation of the effective action using a high-temperature expansion, and is based on that of Actor [46] in flat spacetime. It is similar in spirit, although different in detail to that of Refs. [40–42]. Special attention is given to the interpretation in terms of symmetry breaking. As evident from our earlier work [45], this interpretation leads to results considerably different from previous treatments in flat or curved spacetime.

The outline of our paper is the following. Sec. II contains a short review of the path integral evaluation of the one-loop effective action for a noninteracting scalar field in curved spacetime. The background scalar field is not assumed to be constant. The high-temperature expansion of the effective action is obtained in Sec. III using generalized ζ-function techniques. Sec. IV contains the general treatment of Bose-Einstein condensation, which can be used even if the vacuum expectation value of the scalar field is not constant. Expressions for the critical temperature and the scalar field vacuum expectation value are given. Some applications of the general analysis are given in Sec. V. It is shown how a number of previously known results may be recovered, and a number of new results are derived. The final section contains a short discussion. A number of technical details are given in the Appendix.

II. THE EFFECTIVE ACTION

Consider a $(D+1)$-dimensional globally hyperbolic spacetime $M \simeq R\times\Sigma$, where Σ is a D-dimensional Riemannian manifold with boundary $\partial\Sigma$. (Possibly $\partial\Sigma=\varnothing$.) If the metric signature is $(+-\cdots-)$, we will take

$$S=\int dv_x\{\partial^\mu\Phi^\dagger\partial_\mu\Phi-m^2\Phi^\dagger\Phi-U_0(x)-U_1(x)\Phi^\dagger\Phi\}, \tag{1}$$

as the action functional for a complex scalar field. Here dv_x is the invariant volume element for M, and $U_0(x)$ and $U_1(x)$ are scalar functions defined on M which do not involve the scalar field Φ. This means that the theory is free in the sense that the scalar field only interacts with the classical background spacetime, and not with itself or any other quantized fields. For example, in the case $D=3$, $U_0(x)$ would be quadratic in the curvature of M, and $U_1(x)$ might be chosen to be the familiar ξR nonminimal coupling.

Associated with invariance under a gauge transformation,

$$\Phi(x)\rightarrow\Phi'(x)=\exp[i\theta(x)]\Phi(x) , \tag{2}$$

is the conserved Noether current

$$J_\mu(x)=i(\Phi^\dagger\partial_\mu\Phi-\partial_\mu\Phi^\dagger\Phi) . \tag{3}$$

Since $\nabla^\mu J_\mu=0$, we have a conserved charge,

$$Q=\int_\Sigma d\sigma_x J_0=i\int_\Sigma d\sigma_x(\Phi^\dagger\dot\Phi-\dot\Phi^\dagger\Phi) , \tag{4}$$

if we assume that M is static with line element

$$ds^2=dt^2-g_{ij}(\mathbf{x})dx^i dx^j . \tag{5}$$

$d\sigma_x=\sqrt{\det g_{ij}(\mathbf{x})}d^Dx$ is the volume element on Σ.

Rather than deal with a complex field, it is easier to take two real fields $\varphi_1(x)$ and $\varphi_2(x)$ defined by

$$\Phi(x)=\frac{1}{\sqrt{2}}[\varphi_1(x)+i\varphi_2(x)] . \tag{6}$$

Apart from the factor of $1/\sqrt{2}$, which is just a convenient normalization, $\varphi_1$and φ_2 are just the real and imaginary parts of Φ. [It is possible to use a polar decomposition of $\Phi(x)$, but the calculations are simpler if (6) is used.] In terms of these two new fields, the conserved charge becomes

$$Q=\int_\Sigma d\sigma_x(\varphi_2\dot\varphi_1-\dot\varphi_2\varphi_1) . \tag{7}$$

The momenta canonically conjugate to φ_1 and φ_2 are simply $\pi_1=\dot\varphi_1$ and $\pi_2=\dot\varphi_2$, and the Hamiltonian density $\mathcal{H}$ is

$$\mathcal{H}=\tfrac{1}{2}\pi_1^2+\tfrac{1}{2}\pi_2^2+\tfrac{1}{2}|\nabla\varphi_1|^2+\tfrac{1}{2}|\nabla\varphi_2|^2 +\tfrac{1}{2}[m^2+U_1(x)](\varphi_1^2+\varphi_2^2)+U_0(x) . \tag{8}$$

The charge Q becomes

$$Q=\int_\Sigma d\sigma_x(\varphi_2\pi_1-\varphi_1\pi_2) \tag{9}$$

when expressed in terms of the fields and their conjugate momenta.

The reason for adopting a Hamiltonian approach is that this is the easiest way to incorporate finite-temperature effects in quantum field theory. (See [47] for a pedagogical treatment.) First of all perform a Wick rotation $t\rightarrow it$ to obtain a $(D+1)$-dimensional Riemannian spacetime. Since we have restricted M to be static with line element (5), this presents no difficulty. The grand partition function is expressed as

$$Z=\int[d\pi][d\varphi]\exp\left[\int_0^\beta\int_\Sigma d\sigma_x\{i\pi_1\dot\varphi_1+i\pi_2\dot\varphi_2 -\mathcal{H}+\mu q\}\right] , \tag{10}$$

where q is the charge density defined by

$$Q=\int_\Sigma d\sigma_x q , \tag{11}$$

and μ is the chemical potential. The path integral in (10) extends over all fields φ_1 and φ_2 periodic in time with period $\beta=T^{-1}$, adopting units in which Boltzmann's constant is set equal to 1. (The momentum integration in Eq. (10) is unrestricted.) The Riemannian spacetime may be thought of as $M\simeq S^1\times\Sigma$ due to the periodic nature of the boundary conditions on the fields. Because $\mathcal{H}$ is quadratic in the momenta, the integration over the momenta in (10) may be performed simply by completing the square. This leaves a configuration space path integral

$$Z=\int[d\varphi]\exp(-\tilde S[\varphi]) , \tag{12}$$

where

$$\tilde S[\varphi]=\int_0^\beta\int_\Sigma d\sigma_x\{\tfrac{1}{2}(\dot\varphi_1-i\mu\varphi_2)^2+\tfrac{1}{2}(\dot\varphi_2+i\mu\varphi_1)^2+\tfrac{1}{2}|\nabla\varphi_1|^2+\tfrac{1}{2}|\nabla\varphi_2|^2+\tfrac{1}{2}[m^2+U_1(\mathbf{x})](\varphi_1^2+\varphi_2^2)+U_0(\mathbf{x})\} . \tag{13}$$

It is important to note that $\tilde{S}$ differs from the Wick rotation of the classical action functional (1) due to the presence of the chemical potential associated with the conserved charge.

Up to this point the analysis has been essentially the same as that in flat Minkowski spacetime [20,21]. We now wish to make a departure to deal with the more general situation which arises for curved spacetimes or for spacetimes (including flat spacetime) with boundaries. If the boundary conditions are such that the vacuum expectation value of the scalar fields are expected to be constant, then it is possible to compute the thermodynamic potential as in Refs. [20,21]. However, the boundary conditions on the fields may exclude the possibility of a constant solution other than the zero solution. Symmetry breaking, if it occurs, must in such a case involve a scalar field which is not constant. This has been noted before in the case of interacting fields both classically [48] and in quantum field theory [49–53]. At finite temperature and density, it occurs even for a noninteracting scalar field theory [45]. A prototypical example which illustrates the need for the general analysis just mentioned is the following. Suppose that Σ is the region of flat spacetime enclosed by a spherical shell, and that the scalar field vanishes on $\partial\Sigma$. ($\partial\Sigma$ then represents the spherical shell.) It is then obvious, because the scalar field vanishes on $\partial\Sigma$, that the only possible constant solution consistent with the boundary condition is the zero solution. If the ground state of the theory does not correspond to the zero field, then it cannot possibly be constant. We will analyze this example later in the paper.

Rather than ideal with the thermodynamic potential, we will consider the finite-temperature effective action computed using the background-field method [54]. The background scalar field will be chosen to be $\bar{\varphi}_1(x)=\bar{\varphi}(\mathbf{x})$ and $\bar{\varphi}_2(x)=0$. (Because the spacetime is assumed to be static, the scalar field expectation value would not be expected to depend on time.) By expanding about this assumed background in the usual way [54], the effective action turns out to be

$$\Gamma=\tilde{S}[\bar{\varphi}]+\tfrac{1}{2}\ln\det\{\ell^2\tilde{S}_{,ij}[\bar{\varphi}]\}\ , \tag{14}$$

where

$$\tilde{S}_{,ij}[\bar{\varphi}]=\frac{\delta^2\tilde{S}[\bar{\varphi}]}{\delta\bar{\varphi}^i(x)\delta\bar{\varphi}^j(x')}\ . \tag{15}$$

The second term in (14) arises from the Gaussian functional integration, and represents the one-loop quantum correction to the classical action functional. ℓ is a unit of length introduced to keep the argument of the logarithm dimensionless. For $\tilde{S}$ given in (13),

$$\frac{\delta^2\tilde{S}[\bar{\varphi}]}{\delta\bar{\varphi}_1(x)\delta\bar{\varphi}_1(x')}=[-\Box_x+m^2-\mu^2+U_1(\mathbf{x})]\delta(x,x') =\frac{\delta^2\tilde{S}[\bar{\varphi}]}{\delta\bar{\varphi}_2(x)\delta\bar{\varphi}_2(x')}\ , \tag{16}$$

$$\frac{\delta^2\tilde{S}[\bar{\varphi}]}{\delta\bar{\varphi}_1(x)\delta\bar{\varphi}_2(x')}=-\frac{\delta^2\tilde{S}[\bar{\varphi}]}{\delta\bar{\varphi}_2(x)\delta\bar{\varphi}_1(x')}=2i\mu\frac{\partial}{\partial t}\delta(x,x')\ , \tag{17}$$

so that the quantum part of the effective action is independent of the background field. It is then easy to see that

$$\begin{aligned}\Gamma^{(1)}&=\frac{1}{2}\ln\det(\ell^2\tilde{S}_{,ij})\\ &=\frac{1}{2}\ln\det\left[\ell^4(-\Box+m^2-\mu^2+U_1)^2-4\ell^4\mu^2\frac{\partial^2}{\partial t^2}\right]\\ &=\Gamma_++\Gamma_-\ ,\end{aligned} \tag{18}$$

where

$$\Gamma_\pm=\frac{1}{2}\ln\det\left[\ell^2\left[-\Box+m^2-\mu^2+U_1\pm2\mu\frac{\partial}{\partial t}\right]\right]\ . \tag{19}$$

The assumption that the spacetime is static has been used here in order to obtain (18) [otherwise $(\partial/\partial t)U_1\neq0$, and the situation is more complicated]. Using (13) evaluated at the background field, gives

$$\Gamma=\beta\int_\Sigma d\sigma_x\{\tfrac{1}{2}|\nabla\bar{\varphi}|^2+\tfrac{1}{2}[m^2-\mu^2+U_1(\mathbf{x})]\bar{\varphi}^2+U_0(\mathbf{x})\} +\Gamma_++\Gamma_-\ . \tag{20}$$

The high-temperature expansion of (19) will be the object of the next section.

The background scalar field must satisfy the field equation

$$\frac{\delta\Gamma}{\delta\bar{\varphi}(\mathbf{x})}=0\ , \tag{21}$$

which from (20) gives

$$-\nabla^2\bar{\varphi}(\mathbf{x})+[m^2-\mu^2+U_1(\mathbf{x})]\bar{\varphi}(\mathbf{x})=0\ . \tag{22}$$

Because the quantum part of the field does not involve the background field, this is just the classical field equation corresponding to the action $\tilde{S}$. However, it is important to realize that because the chemical potential μ is a derived quantity determined by the overall conserved charge of the system, μ can depend on quantum effects. (This point is made in Ref. [20].) Thus the quantum nature of the system can still influence the background scalar field even though there are no self-interactions included. This will be seen at greater depth in Sec. IV.

III. HIGH-TEMPERATURE EXPANSION OF THE EFFECTIVE ACTION

For a general spacelike hypersurface Σ, it is not possible to compute $\Gamma_\pm$ defined in (19) exactly. In certain special spacetimes, for example $\Sigma=S^D$, it is possible to obtain a closed-form expression for $\Gamma_\pm$. Rather than look at a variety of special cases, we will keep Σ general, and concentrate on computing $\Gamma_\pm$ in the high-temperature limit. The resulting expression can be used to study Bose-Einstein condensation at relativistic temperatures. After obtaining the general results for $\Gamma_\pm$ we will recover previous special cases in some instances; this will be the subject of Sec. IV.

The formal result (19) can be given meaning in a number of ways. We will discuss only one of the possible ways; namely, Hawking's [55] version of ζ-function regularization which involves the eigenvalues of the differential operator in (19). Let

$$\left[-\Box+m^2-\mu^2+U_1(\mathbf{x})\pm 2\mu\frac{\partial}{\partial t}\right]\Psi_n^{\pm}=\lambda_n^{\pm}\Psi_n^{\pm}\ . \qquad (23)$$

Because $M\simeq S^1\times\Sigma$, with the product metric (5) assumed, we have $\Box=\partial^2/\partial t^2+\nabla^2$, where ∇^2 is the scalar Laplacian on Σ. Then (23) may be written as

$$\left[-\left[\frac{\partial}{\partial t}\mp\mu\right]^2-\nabla^2+m^2+U_1(\mathbf{x})\right]\Psi_n^{\pm}=\lambda_n^{\pm}\Psi_n^{\pm}\ . \qquad (24)$$

Let $\{\varphi_N(\mathbf{x})\}$ be a complete set of solutions to

$$[-\nabla^2+U_1(\mathbf{x})]\varphi_N(\mathbf{x})=\sigma_N\varphi_N(\mathbf{x})\ , \qquad (25)$$

subject to the appropriate boundary conditions on Σ. Normalize the solutions by

$$\int_\Sigma d\sigma_x\varphi_N(\mathbf{x})\varphi_{N'}(\mathbf{x})=\delta_{NN'}\ . \qquad (26)$$

Because $\Psi_n^{\pm}$ must be periodic in time with period β, we may take

$$\Psi_n^{\pm}(t,\mathbf{x})=\exp\left[\frac{2\pi i}{\beta}jt\right]\varphi_N(\mathbf{x})\ , \qquad (27)$$

and

$$\lambda_n^{\pm}=\sigma_N+m^2+\left[\frac{2\pi j}{\beta}\pm i\mu\right]^2\ . \qquad (28)$$

Here n stands for the set (j,N) where $j=0,\pm1,\pm2,\dots$, and we assume that the lowest eigenvalue $\lambda_0^{\pm}=\sigma_0+m^2-\mu^2$ is positive.

Generalized ζ functions will be defined by

$$\zeta_{\pm}(s)=\sum_{j=-\infty}^{\infty}\sum_N(\lambda_{jN}^{\pm})^{-s}\ . \qquad (29)$$

Since formally we have

$$\ln\det\left[\ell^2\left[-\Box+m^2-\mu^2+U_1\pm 2\mu\frac{\partial}{\partial t}\right]\right]=\sum_N\ln(\ell^2\lambda_N^{\pm})\ , \qquad (30)$$

we may define

$$\Gamma_{\pm}=-\tfrac{1}{2}\zeta_{\pm}'(0)+\tfrac{1}{2}\zeta_{\pm}(0)\ln\ell^2\ , \qquad (31)$$

where $\zeta_{\pm}(0)$ denotes the analytic continuation of (29) from the region of the complex s plane where the summations in (29) converge to $s=0$. Since the summation on j extends from $j=-\infty$ to $j=+\infty$, it is clear that $\zeta_-(s)=\zeta_+(s)$, and hence

$$\Gamma_-=\Gamma_+\ . \qquad (32)$$

This result may also be deduced from the fact that the spacetime line element (5) is invariant under the discrete symmetry $t\to -t$. Because this symmetry interchanges, the differential operators appearing in Γ_+ and Γ_-, (32) must hold. We may therefore concentrate on $\zeta_+(s)$:

$$\zeta_+(s)=\sum_{j=-\infty}^{\infty}\sum_N\left[\left[\frac{2\pi j}{\beta}+i\mu\right]^2+\sigma_N+m^2\right]^{-s}\ . \qquad (33)$$

Because σ_N is not known for general Σ, it is not possible to evaluate (33) explicitly. However, if we are only interested in the high-temperature expansion of the effective action it is possible to obtain the desired results from (33) by generalizing the analysis of Actor [46] from flat spacetime to curved spacetime. (In flat spacetime, the analysis is easier because $\sigma_N\to\mathbf{k}^2$, resulting in considerable simplification.)

We begin by separating off the $j=0$ term in (33) by writing

$$\zeta_+(s)=\sum_N(\sigma_N+m^2-\mu^2)^{-s}+F_+(s)+F_-(s)\ , \qquad (34)$$

where

$$F_{\pm}(s)=\sum_{j=1}^{\infty}\sum_N\left[\left[\frac{2\pi j}{\beta}\pm i\mu\right]^2+\sigma_N+m^2\right]^{-s}\ . \qquad (35)$$

Note that the first summation in (34) is independent of the temperature; however, it should not be confused with the $T=0$ result. [This is clear since $F_{\pm}(s)\neq 0$ when $T=0$.] Using the identity

$$a^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\exp(-at)\ , \qquad (36)$$

which is essentially just the definition of $\Gamma(s)$, we have

$$F_+(s)=\sum_{j=1}^{\infty}\sum_N\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\exp\left\{-t\left[\left[\frac{2\pi j}{\beta}+i\mu\right]^2+\sigma_N+m^2\right]\right\}\ . \qquad (37)$$

The expression for $F_-(s)$ can be obtained from $F_+(s)$ by making the replacement $\mu\to-\mu$. Making the change of variable $t\to\bar\beta^2 t$, where $\bar\beta=\beta/2\pi$, we have

$$F_+(s)=\frac{1}{\Gamma(s)}\bar\beta^{2s}\sum_{j=1}^{\infty}\int_0^\infty dt\ t^{s-1}\Theta(t)\exp\{-t[(j+i\mu\bar\beta)^2+m^2\bar\beta^2]\}\ , \qquad (38)$$

where we have defined

$$\Theta(t)=\sum_N\exp(-t\bar\beta^2\sigma_N)\ . \qquad (39)$$

Up to this point in the calculation no approximations have been made. In order to proceed further we must know something about $\Theta(t)$. Because of our lack of knowledge concerning σ_N, this entails some sort of approximation scheme. $\Theta(t)$ may be recognized as $\operatorname{tr}\exp[-\tau(-\nabla^2+U_1)]$ where $\tau=t\bar{\beta}^2$. The high-temperature limit corresponds to $\bar{\beta}\to 0$, and therefore is associated with the behavior of $\operatorname{tr}\exp[-\tau(-\nabla^2+U_1)]$ for small τ, which possesses a known asymptotic expansion leading to

$$\Theta(t)\simeq(4\pi t\bar{\beta}^2)^{-D/2}\sum_{k=0,1/2,1,\ldots}^{\infty}(t\bar{\beta}^2)^k\theta_k\ . \tag{40}$$

Here θ_k are coefficients which depend on the intrinsic geometry of Σ, the extrinsic geometry of $\partial\Sigma$, and the boundary conditions satisfied by the fields on $\partial\Sigma$. The literature associated with (40) is vast. The expansion of $\operatorname{tr}\exp[-t(-\nabla^2)]$ for ∇^2 the scalar Laplacian on a Riemannian manifold without boundary dates back at least to Ref. [56]. The first use of this type of asymptotic expansion in quantum field theory was by Schwinger [57] and DeWitt [54], and for this reason is sometimes called the Schwinger-DeWitt expansion by physicists. The case of manifolds with boundaries was examined by McKean and Singer [58] and by Greiner [59]. An explicit evaluation of the first few coefficients was performed by numerous authors [60–63].

Using (40) in (38) gives

$$F_+(s)\simeq\frac{\bar{\beta}^{2s-D}}{\Gamma(s)}\sum_{j=1}^{\infty}\sum_{k=0,1/2,\ldots}^{\infty}(4\pi)^{-D/2}\bar{\beta}^{2k}\theta_k\int_0^{\infty}dt\,t^{s-D/2+k-1}\exp\{-t[(j+i\bar{\beta}\mu)^2+\bar{\beta}^2m^2]\}$$

$$=(4\pi)^{-D/2}\bar{\beta}^{2s-D}\sum_{j=1}^{\infty}\sum_{k=0,1/2,\ldots}^{\infty}\frac{\Gamma(s-D/2+k)}{\Gamma(s)}\bar{\beta}^{2k}\theta_k[(j+i\bar{\beta}\mu)^2+\bar{\beta}^2m^2]^{D/2-k-s}\ . \tag{41}$$

[The integration over t has been performed using Eq. (36).] Define

$$S(\lambda)=\Gamma(\lambda)\sum_{j=1}^{\infty}[(j+i\bar{\beta}\mu)^2+\bar{\beta}^2m^2]^{-\lambda}\ , \tag{42}$$

so that

$$F_+(s)\simeq(4\pi)^{-D/2}\bar{\beta}^{2s-D}\frac{1}{\Gamma(s)}\sum_{k=0,1/2,\ldots}^{\infty}\bar{\beta}^{2k}\theta_kS(s+k-D/2)\ . \tag{43}$$

It is at this stage that we make contact with the approach of Actor [46]. The basic idea of the method is to use the binomial expansion in (42) to obtain an expansion of $S(\lambda)$ in powers of $\bar{\beta}$. First write (42) as

$$S(\lambda)=\Gamma(\lambda)\sum_{j=1}^{\infty}j^{-2\lambda}(1+z_j)^{-\lambda}\ , \tag{44}$$

where

$$z_j=\frac{2i\bar{\beta}\mu}{j}+\bar{\beta}^2\frac{m^2-\mu^2}{j^2}\ . \tag{45}$$

[This is why the $j=0$ term was separated off in Eq. (34).] The binomial expansion gives

$$(1+z_j)^{-\lambda}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}z_j^k\ . \tag{46}$$

Substitution of z_j given in (45), followed by a further use of the binomial theorem, leads to

$$(1+z_j)^{-\lambda}=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^k}{l!(k-l)!}\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}\times(2i\mu)^{k-l}\bar{\beta}^{k+l}(m^2-\mu^2)^lj^{-k-l}\ . \tag{47}$$

This result may now be used in (44), and the sum over j performed using the definition of the Riemann ζ function leading to

$$S(\lambda)=\sum_{k=0}^{\infty}\sum_{l=0}^{k}S(k,l)\bar{\beta}^{k+l}\ , \tag{48}$$

where $S(k,l)$ is

$$S(k,l)=\frac{(-1)^k}{l!(k-l)!}\Gamma(\lambda+k)(2i\mu)^{k-l}(m^2-\mu^2)^l\times\zeta_R(2\lambda+k+l)\ . \tag{49}$$

Here $\zeta_R(p)=\sum_{n=1}^{\infty}n^{-p}$ denotes the Riemann ζ function. A simple rearrangement of the summation indices shows that

$$S(\lambda)=\sum_{n=0}^{\infty}x_n(\lambda)\bar{\beta}^{2n}+\sum_{n=0}^{\infty}y_n(\lambda)\bar{\beta}^{2n+1}\ , \tag{50}$$

where

$$x_n(\lambda)=\sum_{k=0}^{n}S(2n-k,k)\ , \tag{51}$$

$$y_n(\lambda)=\sum_{k=0}^{n}S(2n+1-k,k)\ . \tag{52}$$

The first two terms of (51) are given in the Appendix.

A particular consequence of (49) and (52) is that $y_n(\lambda)$ is an odd function of μ. [$x_n(\lambda)$ is seen to be an even function of μ.] Because $F_-(s)$ differs from $F_+(s)$ merely by changing the sign of μ, it follows that

$$F_+(s)+F_-(s)\simeq\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\{\theta_k x_n(s-D/2+k)\bar{\beta}^{2k+2n}+\theta_{k+1/2}x_n(s-D/2+k+\tfrac{1}{2})\bar{\beta}^{2k+2n+1}\}\ . \tag{53}$$

Finally we can relabel the summation indices to collect terms which involve the same power of $\bar{\beta}$. This leads to, from (34),

$$\zeta_+(s)\simeq\sum_N(\sigma_N+m^2-\mu^2)^{-s}+\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D}\sum_{r=0}^{\infty}\sum_{k=0}^{r}\{\theta_k x_{r-k}(s-D/2+k)\bar{\beta}^{2r}+\theta_{k+1/2}x_{r-k}(s-D/2+k+\tfrac{1}{2})\bar{\beta}^{2r+1}\}\ . \tag{54}$$

Because $x_{r-k}(\lambda)$ is expressed as a finite sum in (51), this gives us an efficient evaluation of the high-temperature expansion of the generalized ζ function.

It now remains to perform the analytic continuation of (54) to a neighborhood of $s=0$ to obtain $\Gamma_\pm$ using (31). This must be done by examining the cases of D even and odd separately, because of the analytical properties of the Γ and ζ functions which occur. For convenience, define

$$\tilde{\zeta}_+(s)=\sum_N(\sigma_N+m^2-\mu^2)^{-s}\ , \tag{55}$$

$$\zeta_+^{(1)}(s)=\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D}\times\sum_{r=0}^{\infty}\sum_{k=0}^{r}\theta_k x_{r-k}(s-D/2+k)\bar{\beta}^{2r}\ , \tag{56}$$

$$\zeta_+^{(2)}(s)=\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D}\times\sum_{r=0}^{\infty}\sum_{k=0}^{r}\theta_{k+1/2}x_{r-k}(s-D/2+k+\tfrac{1}{2})\bar{\beta}^{2r+1}\ . \tag{57}$$

The expansions of (56) and (57) are found in the Appendix.

We may now use our results to find the dominant terms in Γ_+ when the temperature is high. It is very important at this stage to be consistent in the retention of terms of differing orders in β. First of all, because it is not possible to evaluate $\tilde{\zeta}_+(s)$ for general Σ, we cannot keep terms in Γ_+ which are independent of β. This means that it is not consistent to retain terms in the high-temperature expansion which vanish as $\beta\to 0$. [Of course the evaluation of $\tilde{\zeta}_+(0)$ is easy; it is $\tilde{\zeta}'_+(0)$ which is impossible to obtain in the general case.] We are therefore only interested in terms which behave like β^{-z} for $z>0$, or like $\ln\beta$. This means that $\zeta_+(0)$ may be forgotten, as it is clear from the results of the appendix that it does not contain any terms which are important as $\beta\to 0$. [$\zeta_+(0)$ is interesting for renormalization of the effective action; however, this is not directly relevant to the present paper.] To the order at which we may work, from (31) we may take $\Gamma_+\simeq-\frac{1}{2}\zeta'_+(0)$ and keep only those terms in $\zeta'_+(0)$ which behave like β^{-z} and $\ln\beta$.

For $D=1$, from (A21) and (A25) we find

$$\Gamma_+\simeq-\frac{\pi}{6}T\theta_0-\frac{1}{2}\pi^{-1/2}\theta_{1/2}\ln T+\cdots\ , \tag{58}$$

which is independent of μ. This means, as we will show later, that Bose-Einstein condensation cannot occur for any Σ regardless of the boundary conditions on the fields at high temperature. The reason for this is that the μ-dependent terms vanish as $T\to\infty$.

For $D=2$, from (A15) and (A18),

$$\Gamma_\pm\simeq-\frac{1}{2\pi}\zeta(3)T^2\theta_0-\frac{1}{12}\pi^{1/2}T\theta_{1/2}+\frac{1}{4\pi}[\theta_1-(m^2-\mu^2)\theta_0]\ln T+\cdots\ . \tag{59}$$

For $D=3$, we find, from (A20) and (A24),

$$\Gamma_+\simeq-\frac{\pi^2}{90}T^3\theta_0-\frac{1}{4}\pi^{-3/2}\zeta(3)T^2\theta_{1/2}-\frac{1}{24}T[\theta_1-(m^2-2\mu^2)]+\frac{1}{8}\pi^{-3/2}[(m^2-\mu^2)\theta_{1/2}-\theta_{3/2}]\ln T+\cdots\ . \tag{60}$$

For $D\geq 4$, we find from (A14) and (A17) and (A19) and (A23) that

$$\Gamma_+=-\pi^{-(D+1)/2}\Gamma((D+1)/2)\zeta(D+1)T^D\theta_0-\tfrac{1}{2}\pi^{-(D+1)/2}\Gamma(D/2)\zeta(D)T^{D-1}\theta_{1/2}-\tfrac{1}{4}\pi^{-(D+1)/2}\Gamma((D-1)/2)\zeta(D-1)T^{D-2}\{\theta_1-[m^2-(D-1)\mu^2]\theta_0\}+O(T^{D-3})\ . \tag{61}$$

In this last result, we have not indicated all terms important for high T, but only those important for the discussion of Bose-Einstein condensation.

Again, we wish to emphasize that in special cases, it is possible to obtain better results than the ones shown here. However, in the general case (of general Σ and general boundary conditions) the results found here are the best possible without making further approximations.

IV. BOSE-EINSTEIN CONDENSATION AT HIGH TEMPERATURE

Having obtained the high-temperature expansion of the effective action, we can now use it to study the possibility of symmetry breaking and the connection with Bose-Einstein condensation. The scalar field ground state is a solution to (22) subject to the appropriate boundary conditions. Suppose that we expand

$$\bar{\varphi}(\mathbf{x})=\sum_N C_N\varphi_N(\mathbf{x}) \ , \tag{62}$$

where $\varphi_N(\mathbf{x})$ obeys (25) and (26). The C_N are some expansion coefficients to be determined. Because the $\varphi_N(\mathbf{x})$ are linearly independent, it follows from substituting (62) in (22) and using (25), that

$$C_N(\sigma_N+m^2-\mu^2)=0 \ . \tag{63}$$

If $\mu^2<m^2+\sigma_0$, where σ_0 is the smallest eigenvalue in the set $\{\sigma_N\}$, then the only possible solution to (63) is $C_N=0$. This corresponds to $\bar{\varphi}(\mathbf{x})=0$, which represents the unbroken symmetry phase. If $\mu^2=m^2+\sigma_0$, then C_0 is not determined by (63), but $C_N=0$ for all $N\neq 0$. In this case we have a nontrivial solution

$$\bar{\varphi}(\mathbf{x})=C_0\varphi_0(\mathbf{x}) \ , \tag{64}$$

corresponding to symmetry breaking. The scalar field ground state is then determined by the eigenfunction in (25) which corresponds to the smallest eigenvalue.

In order to see how C_0 may be determined, and to see how this is related to Bose-Einstein condensation, we will examine the charge associated with the ground state. The vacuum expectation value of the charge operator is given in terms of the effective action by

$$Q=-\frac{1}{\beta}\frac{\partial\Gamma}{\partial\mu} \ . \tag{65}$$

Using (20), along with the knowledge that $\Gamma_-=\Gamma_+$, results in the conclusion that

$$Q=Q_0+Q_1 \ , \tag{66}$$

where

$$Q_0=\mu\int_\Sigma d\sigma_x\bar{\varphi}^2(\mathbf{x}) \ , \tag{67}$$

$$Q_1=-\frac{2}{\beta}\frac{\partial\Gamma}{\partial\mu} \ . \tag{68}$$

We may now use the high-temperature expansions established in the last section.

First of all assume that $D\geq 3$. From (60) and (29), it follows that

$$Q_1\simeq\mu V\alpha_D T^{D-1} \tag{69}$$

at high T, where

$$\alpha_D=2\pi^{-(D+1)/2}\Gamma\left[\frac{D+1}{2}\right]\zeta(D-1) \ . \tag{70}$$

We have used the fact that [58–63]

$$\theta_0=V \ , \tag{71}$$

irrespective of the boundary conditions imposed on the fields. From (69) it follows that if T is high enough it is always possible to have $\bar{\varphi}(\mathbf{x})=0$ and to satisfy

$$Q\simeq\mu V\alpha_D T^{D-1} \tag{72}$$

where $\mu^2<\sigma_0+m^2$. This is the unbroken phase of the theory. Q_0 in (66) vanishes in this case, and may be interpreted as the charge associated with particles in the ground state. Q_1 then represents the charge associated with particles in excited states. The unbroken phase corresponds to the total charge distributed among particles in excited states.

As the temperature decreases, it follows from (72) that because the total charge Q is fixed, μ must increase. Eventually, μ reaches a value μ_c which satisfies

$$\mu_c^2=\sigma_0+m^2 \ . \tag{73}$$

The temperature at which (73) holds defines the critical temperature T_c. From the discussion presented earlier in this section, a nonzero value for the scalar field is possible [see Eq. (64)], and this allows an accumulation of charge in the ground state. The critical temperature is easily computed to be

$$T_c=\left[\frac{Q}{\mu_c V\alpha_D}\right]^{1/(D-1)} \tag{74}$$

using (73). This result may be substituted into (69) to give

$$Q_1\simeq Q\left[\frac{T}{T_c}\right]^{D-1} \tag{75}$$

as the charge in excited states when $T\leq T_c$. The ground-state charge follows from (66) as

$$Q_0\simeq Q\left[1-\left[\frac{T}{T_c}\right]^{D-1}\right] \ , \tag{76}$$

for $T\leq T_c$. This result is the same as that which would be obtained in flat Minkowski spacetime. (See Refs. [19,22] for the case $D=3$. Of course the value for T_c may differ from that in Minkowski spacetime.) It is important to emphasize the universality of this result: it holds irrespective of the spacetime; it holds irrespective of the boundary conditions on the fields.

Using (64) in (67), along with the fact that $\varphi_0(\mathbf{x})$ is normalized [see Eq. (26)], leads to

$$C_0\simeq V^{1/2}\alpha_D^{1/2}(T_c^{D-1}-T^{D-1})^{1/2} \tag{77}$$

if $T\leq T_c$. This determines the value of the ground state when $T\leq T_c$ to be

$$\bar{\varphi}(\mathbf{x})=V^{1/2}\alpha_D^{1/2}(T_c^{D-1}-T^{D-1})^{1/2}\varphi_0(\mathbf{x}) \ . \tag{78}$$

A significant difference with the result in flat Minkowski spacetime is that the charge density in the ground state will not be constant in general, but has a spatial dependence determined by $\varphi_0^2(\mathbf{x})$. [See Eq. (67).]

Next suppose that $D=2$. From (59) we find

$$Q_1\simeq\frac{\mu}{\pi}VT\ln T+\cdots \ , \tag{79}$$

at high T. Unfortunately, it is not possible to study Bose-Einstein condensation using this expression. The reason for this is that the terms not calculated may be significant here, as is already apparent from the known flat spacetime result [19]. Based upon the result of Ref. [19] it is conjectured that

$$Q_1 \simeq -\frac{\mu}{2\pi} VT \ln \left[\frac{\sigma_0 + m^2 - \mu^2}{T^2} \right] + \cdots , \quad (80)$$

and following this reference it would be concluded that Bose-Einstein condensation does not occur at relativistic temperatures. However, as we are unable to verify (80) using the methods of the present paper, this conclusion cannot be reliably supported. A more detailed look at $D=2$ is warranted.

For $D=1$, using (58) we have $\partial\Gamma_+/\partial\mu = O(T^{-1})$. This results in $Q_1 \sim 1$ at large T. The argument presented earlier, that at sufficiently high temperatures all of the charge can exist in excited states, no longer follows. There does not exist a critical temperature at which symmetry breaking occurs at high temperature.

V. SOME APPLICATIONS

In order to illustrate the general method presented in the previous sections, we wish to apply it to a number of specific examples. Some of the examples will reproduce known results, whereas a number of others will be new. Some of the examples discussed would be extremely difficult to obtain by attempting to calculate an exact result for the partition function.

A. Flat Minkowski spacetime

First of all, we will show how our results reproduce those of Haber and Weldon [19,20] and Kapusta [21]. Because the theory considered here is in flat spacetime, we will take $U_1(\mathbf{x})=0$. σ_N are the eigenvalues of $-\nabla^2$. Because there is nothing to prevent a constant eigenfunction, the lowest eigenvalue is $\sigma_0=0$, corresponding to φ_0 constant. If we introduce box normalization, then $\varphi_0 = V^{-1/2}$, where V is the box volume which is taken to infinity at the end. According to (73), $\mu_c = \pm m$, and from (74) it follows that

$$T_c = \left[\frac{q}{m\alpha_D} \right]^{1/(D-1)} \quad (81)$$

where $q = Q/V$ is the charge density. The vacuum solution for $T \le T_c$ is, from (78) with $\varphi_0 = V^{-1/2}$,

$$\bar{\varphi} = \alpha_D^{1/2} (T_c^{D-1} - T^{D-1})^{1/2} . \quad (82)$$

If we consider the special case of four-dimensional spacetime, corresponding to $D=3$, using (70) we have $\alpha_3 = \frac{1}{3}$, giving $T_c \simeq (3q/m)^{1/2}$, and $\bar{\varphi} = (q/m)^{1/2}[1-(T/T_c)^2]^{1/2}$. These results agree with Refs. [19–21].

B. The torus

Suppose that $\Sigma = T^D = S^1 \times \cdots \times S^1$ (D times) is the D-dimensional torus. If the boundary conditions on the scalar field are chosen to be periodic in all D directions, then a constant field solution is permitted. The analysis is identical to that for Minkowski spacetime, except of course that V represents the volume of the torus here.

If instead of periodic boundary conditions, antiperiodic boundary conditions are imposed in some of the toroidal directions, then it is clear that the only allowed constant field is one which vanishes identically. This is the simplest example where the analysis presented in the present paper is needed. Let $L_1, \ldots, L_D$ represent the circumferences of the circles comprising the D torus. Take $U_1=0$ again. If φ_1 is antiperiodic in D_A directions, and periodic in D_P directions, where $D_A + D_P = D$, then

$$\varphi_N(\mathbf{x}) = \prod_{i=1}^{D} \left[\frac{2}{L_i} \right]^{1/2} \cos \left[\frac{2\pi}{L_i}(n_i+\delta_i)(x_i - a_i) \right] \quad (83)$$

are the normalized eigenfunctions of $-\nabla^2$. In this expression, $\delta_i=0$ for $i=1,\ldots,D_P$ and $\delta_{i=\frac{1}{2}}$ for $i=D_P+1,\ldots,D$, a_i are arbitrary constants, $n_i=0,1,2,\ldots$, and N stands for the D-tuple $(n_1,\ldots,n_D)$. The eigenvalues of $-\nabla^2$ corresponding to (83) are

$$\sigma_N = \sum_{i=1}^{D} \left[\left[\frac{2\pi}{L_i} \right] (n_i+\delta_i) \right]^2 . \quad (84)$$

Because σ_N is a sum of non-negative terms, it is easy to see that the lowest eigenvalue corresponds to $n_i=0$ for all $i=1,\ldots,D$. Thus

$$\sigma_0 = \pi^2 \sum_{i=1}^{D_A} L_i^{-2} . \quad (85)$$

From (73) μ_c satisfies

$$\mu_c^2 = m^2 + \pi^2 \sum_{i=1}^{D_A} L_i^{-2} , \quad (86)$$

and

$$T_c = \left[\frac{q}{m\alpha_D} \right]^{1/(D-1)} \left[1 + \pi^2 \sum_{i=1}^{D_A} (mL_i)^{-2} \right]^{-1/[2(D-1)]} . \quad (87)$$

The presence of the second factor demonstrates the dependence of the critical temperature on the geometry. It is observed from this result that the effect of imposing antiperiodic boundary conditions on the fields lowers the critical temperature relative to the case of periodic boundary conditions. This is another manifestation of how the global properties of the fields (i.e., the boundary conditions) can alter the behavior of the theory from that in Minkowski spacetime.

It is also interesting to observe that if the original scalar field is massless, then

$$T_c = \left[\frac{q}{\pi \left[\sum_{i=1}^{D_A} L_i^{-2} \right]^{1/2} \alpha_D} \right]^{1/(D-1)} . \quad (88)$$

This gives a relation between the critical temperature and the length scales associated with the spatial geometry for which Bose-Einstein condensation is possible at high temperature. This possibility does not exist in either Minkowski spacetime, or for the torus with periodic boundary conditions.

C. The rectangular cavity

Let Σ represent the region enclosed by a D-dimensional rectangular box in D-dimensional Euclidean space. Suppose first of all that the scalar field satisfies Dirichlet boundary conditions; that is, the scalar field vanishes on the box walls. (If either periodic or antiperiodic boundary conditions are imposed, then this example reduces to the preceding one.) Let $L_1, \ldots, L_D$ be the dimensions of the sides of the box. Then

$$\varphi_N(\mathbf{x}) = \prod_{i=1}^{D} \left[\frac{2}{L_i} \right]^{1/2} \sin\frac{\pi n_i x_i}{L_i} , \tag{89}$$

gives the normalized eigenfunctions of $-\nabla^2$ which vanish at $x_i=0$ and $x_i=L_i$. $n_i=1,2,\ldots$, with $n_i=0$ excluded since it corresponds to an eigenfunction which vanishes, and is therefore not normalizable. The eigenvalues of $-\nabla^2$ corresponding to (89) are

$$\sigma_N = \pi^2 \sum_{i=1}^{D} \left[\frac{n_i}{L_i} \right]^2 , \tag{90}$$

and hence $\sigma_0=\pi^2\sum_{i=1}^{D} L_i^{-2}$ is the smallest eigenvalue. Apart from the replacement of D_A with D, the critical temperature is again given by (87). The spatial variation in the ground-state charge density is $\prod_{i=1}^{D}\sin^2(\pi x_i/L_i)$, which is largest when $x_i=L_i/2$ corresponding to the midpoint of the box. The charge density drops to zero near the walls.

If Neumann boundary conditions are imposed on the field, that is, the normal derivative of the field vanishes on the walls of the box, then the sine functions in (89) are replaced by cosine functions. This time $n_i=0$ is allowed, and the lowest possible eigenvalue becomes $\sigma_0=0$. The analysis is similar to that for the torus with periodic boundary conditions.

D. The spherical cavity

Suppose that Σ is the region of D-dimensional Euclidean spacetime contained within and bounded by a spherical shell of radius a. On symmetry grounds, the eigenfunction corresponding to the lowest eigenvalue would be expected to be spherically symmetric about the center of the sphere. It is easily seen that

$$\varphi_0 = N r^{1-D/2} J_{(D-2)/2}(kr) , \tag{91}$$

where N is a normalization constant, and J_ν denotes the Bessel function of order ν. The eigenvalue of $-\nabla^2$ corresponding to (91) is

$$\sigma_0 = k^2 . \tag{92}$$

If we apply the boundary condition $\varphi_0=0$ at $r=a$, then it is seen that

$$\sigma_0 = a^{-2} z_{(D-2)/2,1} \tag{93}$$

where $z_{\nu,n}$ denotes the nth positive zero of $J_\nu(z)$.

The result in (93) again illustrates the dependence of the critical temperature on the geometry of the spatial region. In the case where D is odd, say $D=2\delta+1$, $J_{\delta-1/2}(z)$ can be evaluated in terms of elementary functions. (See Ref. [64].) For example, in the physically interesting case $D=3$,

$$J_{1/2}(z) = (2/\pi z)^{1/2}\sin z .$$

In this case, $\sigma_0=\pi a^{-2}$ and we find

$$T_c = \left[\frac{3q}{m} \right]^{1/2} \left[1 + \frac{\pi}{m^2 a^2} \right]^{-1/4} . \tag{94}$$

The result for Neumann boundary conditions was given in Ref. [45]. When $\varphi=0$ is not the ground state, the vacuum solution is given in terms of (91) as discussed in the previous section.

This example illustrates the utility of the general method presented in this paper. Any attempt to perform the calculation exactly would encounter summations over terms which involve the zeros of the Bessel function. Since it is not possible to obtain explicit expressions for these zeros, except numerically or in special cases such as $D=3$, it is difficult to proceed without making further assumptions equivalent to those in the present paper. As far as we are aware, Bose-Einstein condensation in a spherical cavity has not been studied.

E. The Einstein static universe

The curved spacetime which has received the most attention with regard to Bose-Einstein condensation is the static Einstein universe. This usually implies that the spacetime is four dimensional, and that $\Sigma=S^3$. We will assume this to be the case initially, and discuss some generalizations later. We will assume that $U_1=\xi R$. Bose-Einstein condensation has been studied in detail in the cases $\xi=0$ [38] and $\xi=\frac{1}{6}$ [37]. The eigenvalues of $-\nabla^2$ on S^3 are $-n(n+2)a^{-2}$, where a is the radius of S^3 and $n=0,1,2,\ldots$. (A reference for spheres of arbitrary dimension is Ref. [65]. An early paper in the physics literature with the result is Ref. [66].) Since the scalar curvature is constant, $R=6a^{-2}$, we have

$$\sigma_N = n(n+2)a^{-2} + 6\xi a^{-2} . \tag{95}$$

Here N stands for the complete set of integers labeling the eigenfunctions of $-\nabla^2+U_1$. The corresponding eigenfunctions may be found in Ref. [67]. The lowest eigenvalue is $\sigma_0=6\xi a^{-2}$, and we assume that $\xi\geq 0$ here. The critical temperature is

$$T_c = \left[\frac{3q}{m} \right]^{1/2} \left[1 + \frac{6\xi}{m^2 a^2} \right]^{-1/4} . \tag{96}$$

For $\xi=0$ this is seen to be identical to the critical temperature in Minkowski spacetime [38]. The case $\xi=\frac{1}{6}$ agrees with the high-temperature limit found in Ref. [37]. The existence of spacetime curvature is seen to lower the critical temperature from that for flat spacetime if $\xi \neq 0$. As $a \to \infty$, corresponding to $R \to 0$, the Minkowski spacetime result is recovered regardless of the value chosen for ξ. The eigenfunction corresponding to $\sigma_0 = 6\xi a^{-2}$ is just

$$\varphi_0 = V^{-1/2} = (2\pi^2 a^3)^{-1/2}$$

since constant eigenfunctions are not forbidden.

A simple modification of the Einstein static universe which forbids constant nonzero eigenfunctions is obtained by identifying antipodal points on S^3. (An early discussion of the spacetime which results from this was given by Eddington [68].) If the field takes the same value at antipodal points, then the analysis is identical to the standard Einstein universe discussed above. However, we could also impose the boundary condition that the field takes opposite values at antipodal points. As in the case of antiperiodic boundary conditions on the torus, constant eigenfunctions, other than zero, are not allowed. In order to analyze what happens, it is convenient to regard S^3 as the surface $x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2$ embedded in four-dimensional Euclidean space. The antipodal identification, under which the value of the scalar field changes sign, is

$$(x_1, x_2, x_3, x_4) \sim (-x_1, -x_2, -x_3, -x_4) .$$

Standard polar angles (χ, θ, ϕ) may be introduced by

$$x_1 = a \sin\theta \cos\phi \sin\chi ,$$

$$x_2 = a \sin\theta \sin\phi \sin\chi ,$$

$$x_3 = a \cos\theta \sin\chi ,$$

$$x_4 = a \cos\chi .$$

The antipodal identification is

$$(\chi, \theta, \phi) \sim (\pi - \chi, \pi - \theta, \pi + \phi)$$

in these coordinates. The volume of the space is $\pi^2 a^3$ which is just half that of the standard three-sphere due to the point identification.

The eigenvalues of $-\nabla^2 + \xi R$ are still given by (95). The corresponding eigenfunctions are polynomials in x_1, x_2, x_3, x_4 which are harmonic, and homogeneous of degree n. Because of the boundary condition requiring the field to change sign at antipodal points, n must be restricted to be an odd integer. The lowest eigenvalue is seen to be $\sigma_0 = (6\xi + 3)a^{-2}$, which is just obtained from $n = 1$ in (95). There are four possible eigenfunctions corresponding to this eigenvalue: namely, x_1, x_2, x_3, x_4. They are all related by the action of SO(4), which is the isometry group of S^3. We are free to choose, without loss of generality, any one of these solutions. (This was noted by Unwin [52].) We can choose

$$\varphi_0 = 2\pi^{-1} a^{-3/2} \cos\chi , \tag{97}$$

which when used in (78) gives the ground state for $T \le T_c$. The expression for T_c is

$$T_c = \left[\frac{3q}{m} \right]^{1/2} \left[1 + \frac{6\xi + 3}{m^2 a^2} \right]^{-1/4} . \tag{98}$$

Unlike the standard Einstein static universe, even if $\xi = 0$, the critical temperature is changed from the flat spacetime value due to the spatial curvature.

It is also possible to study Bose-Einstein condensation in a higher-dimensional generalization of the Einstein static universe in which the spatial hypersurface is taken to be a D-dimensional sphere, or a D-dimensional sphere with an antipodal identification of points. The generalization of (95) to either of these cases is

$$\sigma_N = n(n + D - 1)a^{-2} + D(D-1)\xi a^{-2} . \tag{99}$$

The eigenvalues of $-\nabla^2$ on S^D can be found in Ref. [65]. The scalar curvature is $R = D(D-1)a^{-2}$. In the case $\Sigma = S^D$, the lowest eigenvalue is seen to be $\sigma_0 = D(D-1)\xi a^{-2}$, corresponding to the eigenfunction $\varphi_0 = V^{-1/2}$. The critical temperature is given by (96) with the replacement $6\xi \to D(D-1)\xi$.

In the case where we identify antipodal points on the sphere, and impose antiperiodic boundary conditions, then the lowest eigenvalue σ_0 is obtained by taking $n = 1$ in (99) for the same reason as discussed above for the case $D = 3$. It follows that

$$\sigma_0 = [1 + (D-1)\xi] D a^{-2} , \tag{100}$$

and therefore the critical temperature is

$$T_c = \left[\frac{3q}{m} \right]^{1/2} \left[1 + \frac{D[1 + (D-1)\xi]}{m^2 a^2} \right]^{-1/4} . \tag{101}$$

As for the case $D = 3$, even if the scalar field is minimally coupled ($\xi = 0$), the critical temperature is altered from the flat spacetime value due to the spatial curvature. For temperatures below T_c the ground state is not constant, and is given by a simple generalization of (97).

VI. DISCUSSION AND CONCLUSIONS

In the sections above, a general method for studying Bose-Einstein condensation has been presented. This method is useful in cases where the scalar field vacuum expectation value is not constant. It generalizes previous work which used the effective potential and was therefore restricted to constant scalar field expectation values. The method of the present paper is suitable for dealing with curved spacetimes whether or not the scalar field is constant.

So far only noninteracting scalar fields have been considered. This restriction is obviously unrealistic for any physical system, and a generalization which includes interactions is necessary. In flat Minkowski spacetime this has been done by Haber and Weldon [20] for the large N limit of the O(N) model, and in Ref. [23] for $\lambda\phi^4$ theory. The inclusion of interactions is complicated since it is well known that perturbation theory breaks down for temperatures close to the critical temperature, and in particular that the one-loop approximation is not valid. (See, for example, Refs. [69–71].) To obtain a reliable approximation it is necessary to sum an infinite class of diagrams. This has been done for Bose-Einstein condensation in $\lambda\phi^4$ theory in Ref. [23] in an evaluation of the effective potential. Some work has also been done obtaining reliable approximations in curved spacetime [72]. It is of interest to examine the effects of interactions using the approach of the present paper. This is not quite a straightforward generalization of previous work on the effective potential if it is assumed that the background scalar field is not constant. In this case one needs a reliable approximation to the effective action which includes derivatives of the fields.

One natural place to apply possible consequences of Bose-Einstein condensation at high temperatures is in the early Universe. Some authors have considered this already. Two early papers on the cosmological role of Bose-Einstein condensation are Refs. [73,74]. These papers use the standard nonrelativistic formula. The flat spacetime field theory analysis has been used in Refs. [22,23]. Parker and Zhang [38] base a discussion around their results for the static Einstein universe. More recently, Madsen [75] has claimed that a Bose condensate can have interesting consequences for galaxy formation and the dark matter problem. In view of this, it is of interest to extend the results of the present paper to a Robertson-Walker spacetime which describes the early Universe. A study of the self-consistent solutions to the coupled Einstein and Klein-Gordon equations should be possible. At the very least, this would provide a reliability check on the use of flat spacetime or nonrelativistic results for Bose-Einstein condensation in a cosmological setting.

It is also possible to present a detailed study of Bose-Einstein condensation in general cavities in flat spacetime using the results of the present paper. The critical temperature was shown to depend on the smallest eigenvalue of the Laplacian for the field subject to the appropriate boundary conditions. In many cases it is possible to approximate the required eigenvalue in terms of the geometry of the cavity even in cases where it is not possible to solve the eigenvalue problem exactly. This would allow a much wider class of systems to be studied than have been considered so far, because a detailed knowledge of the exact eigenvalue spectrum is not required.

APPENDIX: HIGH-TEMPERATURE EXPANSION

In this appendix we will present the high-temperature expansion of the generalized ζ function for both even- and odd-dimensional spacetimes. The results are used to calculate the high-temperature expansion of the effective action as discussed in Sec. III.

We found it convenient to split the ζ function into three pieces defined in (55)–(57). The first piece $\tilde{\zeta}_+(s)$ obviously has no explicit temperature dependence, but does depend on the spatial hypersurface Σ and the boundary conditions on the fields. It may be ignored for the purposes of this appendix. If we write out the first few terms of (56) and (57) we have

$$\zeta_+^{(1)}(s)=\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D}\{\theta_0 x_0(s-D/2)+\bar{\beta}^2[\theta_0 x_1(s-D/2)+\theta_1 x_0(s-D/2+1)]+O(\bar{\beta}^4)\}\ , \tag{A1}$$

$$\zeta_+^{(2)}(s)=\frac{2}{\Gamma(s)}(4\pi)^{-D/2}\bar{\beta}^{2s-D+1}\{\theta_{1/2}x_0(s-D/2+\tfrac{1}{2})+\bar{\beta}^2[\theta_{1/2}x_1(s-D/2+\tfrac{1}{2})+\theta_{3/2}x_0(s-D/2+\tfrac{3}{2})]+O(\bar{\beta}^4)\}\ . \tag{A2}$$

The expressions for x_0 and x_1 may be found from (49) and (51) to be

$$x_0(\lambda)=\Gamma(\lambda)\zeta(2\lambda)\ , \tag{A3}$$

$$x_1(\lambda)=-[m^2+(2\lambda+1)\mu^2]\Gamma(\lambda+1)\zeta(2\lambda+2)\ . \tag{A4}$$

The first dependence on μ occurs in $x_1(\lambda)$. This is therefore the most important term for the study of Bose-Einstein condensation. The analytic property of the terms in (A1) and (A2) as a function of s depend on whether D is even or odd. We will examine these two cases separately.

1. $D=2\delta$ is even

With $D=2\delta$ we have, from (A3),

$$\frac{x_0(s-D/2)}{\Gamma(s)}=\frac{\Gamma(s-\delta)\zeta(2s-2\delta)}{\Gamma(s)}\ . \tag{A5}$$

Using properties of $\Gamma(z)$ and $\zeta(z)$ (see Ref. [64] for example), it may be shown that

$$\frac{x_0(s-D/2)}{\Gamma(s)}=\pi^{2s-2\delta-1/2}\Gamma(\delta+\tfrac{1}{2}-s)\frac{\zeta(2\delta+1-2s)}{\Gamma(s)}\ . \tag{A6}$$

$\zeta(z)$ is an analytic function of z except at $z=1$ where it has a simple pole with residue 1:

$$\zeta(z)=\frac{1}{z-1}+\gamma+O(z-1)\ . \tag{A7}$$

We want to evaluate (A6) near $s=0$, so (A7) tells us that if $\delta\geq 1$, $\zeta(2\delta+1-2s)$ is analytic as $s=0$. Since, near $s=0$,

$$\frac{1}{\Gamma(s)}=s+\gamma s^2+O(s^3)\ , \tag{A8}$$

it follows that

$$\frac{x_0(s-\delta)}{\Gamma(s)}=s\pi^{-2\delta-1/2}\Gamma(\delta+\tfrac{1}{2})\zeta(2\delta+1)+O(s^2)\quad\text{if } \delta\geq 1 \tag{A9}$$

near $s=0$. If $\delta=0$, then from (A5) we have

$$\frac{x_0(s)}{\Gamma(s)}=\zeta(2s)=-\frac{1}{2}-s\ln(2\pi)+O(s^2) \quad (A10)$$

near $s=0$.

In a similar way, it may be shown from (A4), that

$$\frac{x_1(s-\delta)}{\Gamma(s)}=-s[m^2-(2\delta-1)\mu^2]\pi^{3/2-2\delta}\times\Gamma(\delta-\tfrac{1}{2})\zeta(2\delta-1)+O(s^2) \quad (A11)$$

if $\delta\geq 2$, and

$$\frac{x_1(s-1)}{\Gamma(s)}=\frac{1}{2}(m^2-\mu^2)+s[\mu^2+(m^2-\mu^2)\ln(2\pi)]+O(s^2)\ . \quad (A12)$$

These results are sufficient to show (recall that $\bar{\beta}=\beta/2\pi$)

$$\zeta_+^{(1)}(0)=\begin{cases}0+\cdots & \text{if } \delta\geq 2\ (D\geq 4),\\ \dfrac{1}{4\pi}[(m^2-\mu^2)\theta_0-\theta_1] & \text{if } \delta=1\ (D=2)\ ,\end{cases} \quad (A13)$$

and

$$\zeta_+^{(1)\prime}(0)=2\pi^{-\delta-1/2}\beta^{-2\delta}\Gamma(\delta-\tfrac{1}{2})\{(\delta-\tfrac{1}{2})\zeta(2\delta+1)\theta_0+\tfrac{1}{4}\beta^2\zeta(2\delta-1)[\theta_1-(m^2-(2\delta-1)\mu^2)\theta_0]+\cdots\} \quad (A14)$$

if $\delta\geq 2$;

$$\zeta_+^{(1)\prime}(0)=\frac{1}{\pi}\beta^{-2}\zeta(3)\theta_0+\frac{1}{2\pi}[\mu^2\theta_0+(m^2-\mu^2)\ln\beta\theta_0-\theta_1\ln\beta]+\cdots \quad (A15)$$

if $\delta=1$. The result in (A13) for $\delta=1$ is exact and β independent, whereas the results of (A12) for $\delta\geq 2$ and in (A14) and (A15) are expansions valid only for small β.

The calculation of $\zeta_+^{(2)}(s)$ proceeds in a similar way. It may be shown that

$$\zeta_+^{(2)}(0)=\begin{cases}0+\cdots & \text{if } \delta\geq 2\\ \tfrac{1}{8}\pi^{-3/2}\beta(\theta_{3/2}-m^2\theta_{1/2}) & \text{if } \delta=1\ ,\end{cases} \quad (A16)$$

and

$$\zeta_+^{(2)\prime}(0)=\pi^{-1/2-\delta}\beta^{1-2\delta}\Gamma(\delta)\zeta(2\delta)\theta_{1/2}+\tfrac{1}{4}\pi^{-1/2-\delta}\beta^{3-2\delta}\Gamma(\delta-1)\zeta(2\delta-2)[\theta_{3/2}-(m^2+2(1-\delta)\mu^2)\theta_{1/2}]+\cdots\ , \quad (A17)$$

if $\delta\geq 2$, and

$$\zeta_+^{(2)\prime}(0)=\tfrac{1}{6}\pi^{1/2}\beta^{-1}\theta_{1/2}+\tfrac{1}{4}\pi^{-3/2}\beta[(\gamma+\ln(\beta/4\pi))\theta_{3/2}-(\mu^2+\gamma m^2+m^2\ln(\beta/4\pi))\theta_{1/2}]+\cdots\ , \quad (A18)$$

if $\delta=1$. (It is possible to evaluate $\Gamma(\delta)\zeta(2\delta)$ in terms of π and the Bernoulli numbers if desired [64].)

2. $D=2\delta+1$ is odd

The analysis of odd-dimensional Σ is performed in essentially the same way as that described for even-dimensional Σ. The only ingredients are the properties of $\Gamma(z)$ and $\zeta(z)$. In the even-dimensional case, $D=2$ had to be treated separately from $D=4,6,8,\ldots$ due to the properties of $\Gamma(z)$ and $\zeta(z)$. For odd dimensions, $D=1,3$ must be treated separately from $D=5,7,\ldots$.

It is found, after a short calculation, that

$$\zeta_+^{(1)}(0)=\begin{cases}0+\cdots, & \delta\geq 1\ (\text{i.e. } D\geq 3)\\ \tfrac{1}{4}\pi^{-1/2}\beta[\theta_1-m^2\theta_0], & \delta=0\ (\text{i.e., } D=1)\end{cases} \quad (A19)$$

with

$$\zeta_+^{(1)\prime}(0)=2\pi^{-\delta-1}\beta^{-2\delta-1}\Gamma(\delta)\{\delta\zeta(2\delta+2)\theta_0+\tfrac{1}{4}\beta^2\zeta(2\delta)[\theta_1-(m^2-2\delta\mu^2)\theta_0]+\cdots\}\quad \text{if } \delta\geq 1\ , \quad (A20)$$

$$\zeta_+^{(1)\prime}(0)=\frac{\pi}{3}\beta^{-1}\theta_0+\frac{1}{2}\pi^{-1/2}\beta[(\gamma+\ln(\beta/4\pi))\theta_1-(\mu^2+\gamma m^2+m^2\ln(\beta/4\pi))\theta_0]+\cdots\quad \text{if } \delta=0\ . \quad (A21)$$

The second term in the ζ function results in

$$\zeta_+^{(2)}(0)=\begin{cases}0+\cdots & \text{if } \delta\geq 2\\ \tfrac{1}{8}\pi^{-3/2}[(m^2-\mu^2)\theta_{1/2}-\theta_{3/2}] & \text{if } \delta=1\ (\text{i.e., } D=3)\\ -\tfrac{1}{2}\pi^{-1/2}\theta_{1/2} & \text{if } \delta=0\ (\text{i.e., } D=1)\ ,\end{cases} \quad (A22)$$

and

$$\zeta_+^{(2)\prime}(0)=\pi^{-\delta-1}\beta^{-2\delta}\Gamma(\delta+\tfrac{1}{2})\zeta(2\delta+1)\theta_{1/2} + \tfrac{1}{4}\pi^{-\delta-1}\beta^{2-2\delta}\Gamma(\delta-\tfrac{1}{2})\zeta(2\delta-1)[\theta_{3/2}-\theta_{1/2}(m^2-(2\delta-1)\mu^2)]+\cdots \quad \text{if } \delta\geq 2 \tag{A23}$$

$$\zeta_+^{(2)\prime}(0)=\tfrac{1}{2}\pi^{-3/2}\zeta(3)\beta^{-2}\theta_{1/2}+\tfrac{1}{4}\pi^{-3/2}[\theta_{1/2}(\mu^2+(m^2-\mu^2)\ln\beta)-\theta_{3/2}\ln\beta]+\cdots \quad \text{if } \delta=1 \text{ (i.e., } D=3) , \tag{A24}$$

$$\zeta_+^{(2)\prime}(0)=-\pi^{-1/2}\theta_{1/2}\ln\beta+\tfrac{1}{24}\pi^{-1/2}\beta^2[\theta_{3/2}-(m^2+\mu^2)\theta_{1/2}]+\cdots \quad \text{if } \delta=0 \text{ (i.e., } D=1) . \tag{A25}$$

This presents all of the results required to study Bose-Einstein condensation at high temperature. Extending the expansions to higher order in β is not difficult, and can be used to obtain the high-temperature expansion of the effective action in curved spacetime at finite density. The results agree with those of Kirsten [43,44] whenever we have compared them.

[1] S. N. Bose, Z. Phys. **26**, 178 (1924).
[2] A. Einstein, Sitzungsber. Preus. Akad. Wiss. **22**, 261 (1924).
[3] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, London, 1969).
[4] R. P. Feynman, *Statistical Mechanics* (Addison-Wesley, London, 1972).
[5] R. K. Pathria, *Statistical Mechanics* (Pergamon, London, 1972).
[6] F. London, Phys. Rev. **54**, 947 (1938).
[7] F. London, *Superfluids* (Wiley, New York, 1954), Vol. II.
[8] N. N. Bogolyubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).
[9] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1959).
[10] A. E. Glassgold, A. N. Kaufman, and K. M. Watson, Phys. Rev. **120**, 660 (1960).
[11] D. S. Kothari and B. N. Singh, Proc. R. Soc. London **A178**, 135 (1940).
[12] P. T. Landsberg and J. Dunning-Davies, Phys. Rev. **138**, A1049 (1965).
[13] R. Beckmann, F. Karsch, and D. E. Miller, Phys. Rev. Lett. **43**, 1277 (1979).
[14] F. Karsch and D. E. Miller, Phys. Rev. A **22**, 1210 (1980).
[15] R. Beckmann, F. Karsch, and D. E. Miller, Phys. Rev. A **25**, 561 (1982).
[16] C. Aragao de Carvalho and Jr. S. Goulart Rosa, J. Phys. A **13**, 3233 (1980).
[17] J. Dunning-Davies, J. Phys. A **14**, 3005 (1981).
[18] H. O. da Frota and Jr. S. Goulart Rosa, J. Phys. A **15**, 2221 (1982).
[19] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981).
[20] H. E. Haber and H. A. Weldon, Phys. Rev. D **25**, 502 (1982).
[21] J. I. Kapusta, Phys. Rev. D **24**, 426 (1981).
[22] J. Bernstein and S. Dodelson, Phys. Rev. Lett. **66**, 683 (1991).
[23] K. Benson, J. Bernstein, and S. Dodelson, Phys. Rev. D **44**, 2480 (1991).
[24] M. F. M. Osborne, Phys. Rev. **76**, 396 (1949).
[25] J. M. Ziman, Philos. Mag. **44**, 548 (1953).
[26] R. K. Pathria, Phys. Rev. A **5**, 1451 (1972).
[27] S. Greenspoon and R. K. Pathria, Phys. Rev. A **8**, 2657 (1973).
[28] S. Greenspoon and R. K. Pathria, Phys. Rev. A **9**, 2103 (1974).
[29] S. Greenspoon and R. K. Pathria, Phys. Rev. A **11**, 1080 (1975).
[30] C. S. Zasada and R. K. Pathria, Phys. Rev. A **14**, 1269 (1976).
[31] C. S. Zasada and R. K. Pathria, Phys. Rev. A **15**, 2439 (1977).
[32] H. R. Pajkowski and R. K. Pathria, J. Phys. A **10**, 561 (1977).
[33] R. K. Pathria, Can. J. Phys. **63**, 228 (1983).
[34] S. Singh and R. K. Pathria, Phys. Rev. A **30**, 442 (1984).
[35] S. Singh and P. Pandita, Phys. Rev. A **28**, 1752 (1983).
[36] M. B. Al'taie, J. Phys. A **11**, 1603 (1978).
[37] S. Singh and R. K. Pathria, J. Phys. A **17**, 2983 (1984).
[38] L. Parker and Y. Zhang, Phys. Rev. D **44**, 2421 (1991).
[39] K. Shiraishi, Prog. Theor. Phys. **77**, 975 (1987).
[40] J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).
[41] J. S. Dowker and J. P. Schofield, Phys. Rev. D **38**, 3327 (1988).
[42] J. S. Dowker and J. P. Schofield, Nucl. Phys. **B327**, 267 (1989).
[43] K. Kirsten, J. Phys. A **24**, 3281 (1991).
[44] K. Kirsten, Class. Quantum Grav. **8**, 2239 (1991).
[45] D. J. Toms, Phys. Rev. Lett. **69**, 1152 (1992).
[46] A. Actor, J. Phys. A **20**, 5351 (1987).
[47] J. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[48] S. Avis and C. J. Isham, Proc. R. Soc. London **A363**, 581 (1978).
[49] L. H. Ford, Phys. Rev. D **22**, 3003 (1980).
[50] R. Banach, J. Phys. A **14**, 901 (1981).
[51] D. J. Toms, Phys. Rev. D **25**, 2536 (1982).
[52] S. D. Unwin, J. Phys. A **15**, 841 (1982).
[53] R. Laura and D. J. Toms, J. Phys. A **15**, 3725 (1982).
[54] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[55] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
[56] S. Minakshisundaram and Å. Pleijel, Can. J. Math. **1**, 242 (1949).
[57] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[58] H. P. McKean and I. M. Singer, J. Diff. Geometry **1**, 43 (1967).
[59] P. Greiner, Arch. Rational Mech. Anal. **41**, 163 (1971).
[60] G. Kennedy, J. Phys. A **11**, L173 (1978).
[61] I. G. Moss, Class. Quantum Grav. **6**, 759 (1989).
[62] I. G. Moss and J. S. Dowker, Phys. Lett. B **229**, 261 (1989).
[63] T. P. Branson and P. B. Gilkey, Commun. Part. Diff. Eqns. **15**, 245 (1990).
[64] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals*

Series and Products (Academic, London, 1965).
[65] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. II.
[66] E. Schrödinger, Commentat. Pontif. Acad. Sci. **2**, 321 (1938).
[67] L. H. Ford, Phys. Rev. D **11**, 3370 (1975).
[68] A. S. Eddington, *The Mathematical Theory of Relativitiy* (Cambridge University Press, Cambridge, England, 1922).
[69] S. Weinberg, Phys. Rev. D **9**, 3357 (1974).
[70] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3220 (1974).
[71] D. A. Kirzhnits and A. D. Linde, Ann. Phys. (N.Y.) **101**, 195 (1976).
[72] A. Stylianopoulos and B. L. Hu, Phys. Rev. D **39**, 3647 (1989).
[73] J. D. Becker and L. Castell, Acta Phys. Austriaca Suppl. **18**, 885 (1977).
[74] V. A. Kuzmin and M. E. Shaposhnikov, Phys. Lett. **69A**, 462 (1979).
[75] J. Madsen, Phys. Rev. Lett. **69**, 571 (1992).