Statistical mechanics of extended black objects

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(Received 4 September 1992)

We extend the considerations of a previous paper on black hole statistical mechanics to the case of black extended objects such as black strings and black membranes in 10-dimensional space-time. We obtain a general expression for the Euclidean action of quantum black p-branes and derive their corresponding degeneracy of states. The statistical mechanics of a gas of black p-branes is then analyzed in the microcanonical ensemble. As in the case of black holes, the equilibrium state is not thermal and the stable configuration is the one for which a single black object carries most of the energy. Again, neutral black p-branes obey the bootstrap condition and it is then possible to argue that their scattering amplitudes satisfy crossing symmetry. Finally, arguments identifying quantum black p-branes with ordinary quantum branes of different dimensionality are presented.

PACS number(s): 04.60.+n, 11.17.+y, 97.60.Lf

I. INTRODUCTION

A complete picture of black hole physics (including the effects of quantum mechanics) is still missing at present. This problem is very much at the core of a larger problem, that of a consistent description of a quantum theory of gravity. However, as is well known, to this day, the only consistent theory of quantum gravity (as well as that of a comprehensive unification of all forces) is string theory. Therefore, it may not be surprising to encounter string physics in black hole physics.

't Hooft, in a series of inspiring articles [1], has led the way for the search for a proper particle Hilbert space in the presence of black holes. The effect of a particle falling into or escaping from the horizon of a Schwarzschild black hole is actually to shift the horizon and thereby produce a so-called gravitational shock wave [2]. The corresponding scattering matrix elements (built on the Hilbert space of the particles' momenta) have been shown to be formally identical with those of a string theory with imaginary string tension [1, 3].

In a recent paper [4], the present authors in collaboration with Cox generalized the above considerations to the case of a dilaton black hole and found that the requirement of real string tension (unitarity) could be achieved for a specific value of the dilaton parameter in the naked singularity domain $r_- > r_+$. This result seems to indicate that quantum black holes can become (or decay into) quantum strings beyond the extreme point $r_- = r_+$, a fact which may find its experimental verification with the discovery of extragalactic cosmic γ ray bursts [5]. However this also indicates that black holes are not quite strings, an assertion supported by the asymptotic behavior of their density of states which grows faster than that of strings as mass increases.

In a more traditional approach, following semiclassical treatments in which the above gravitational back reaction effects were neglected, the area of a black hole event horizon has been interpreted in terms of thermodynamical entropy and the black hole mass related to a canonical temperature called the Bekenstein-Hawking temperature [6, 7]. This picture, however, as we pointed out in previous articles [8–12], leads to difficulties with both thermodynamics and quantum mechanics. This is especially so in the thermally interpreted process of black hole evaporation as pure states are converted into mixed states. Attempts to resolve this problem by taking into account the effects of quantum hair have been made [8–11].

There exists however an alternative interpretation of the semiclassical (WKB) calculation which does not violate the laws of quantum mechanics [12]. Namely, the saddle point approximation to the path integral with Euclidean action can be regarded as a tunneling probability per unit volume of a particle escaping the event horizon. This is basically a quantum mechanical barrier penetration problem and so the tunneling probability is an effective measure of the ratio of a single particle state escaping the black hole to the number of available states $\rho_{\rm BH}$ inside (and including) the horizon. We then arrive at the following approximate semiclassical formula for the black hole degeneracy of states at mass level m,

$$\rho_{\rm BH}(m) \simeq c \exp\left(\frac{S_E(m)}{\hbar}\right) , \tag{1.1}$$

in which the constant c represents general quantum field theoretical corrections and S_E is the Euclidean action (so-called Bekenstein-Hawking entropy) of the classical solutions (instantons) of the Euclidean equations of motion [13, 14]. These instantons are actually periodic instantons as the condition of the vanishing of the conical singularity of the Euclidean space-time in the black hole background requires Euclidean time τ to be a compact space (S^1) with period $\beta_{\rm H}$ (the inverse Hawking temperature). The integral over Euclidean time in Eq. (1.1) is to be evaluated over a single period and, making use of the relation between $\beta_{\rm H}$ and the black hole mass m, the density $\rho_{\rm BH}$ becomes solely a function of mass (and possibly electric charge, angular momentum, etc.). In the case of a four-dimensional Schwarzschild black hole, one obtains ($\hbar = c = G = 1$)

$$\rho_{\rm Schw}(m) \;\sim\; c\, e^{4\pi m^2}\,, \tag{1.2}$$

a result to be compared with the softer behavior for strings:

$$\rho_{\rm string}(m) \;\sim\; c m^a\, e^{bm}\,. \tag{1.3}$$

However, like strings, Schwarzschild black holes have been shown [12] to obey the statistical bootstrap condition [15–17]. Furthermore, arguments were presented [12] which made it very plausible for the scattering amplitudes to obey duality symmetry. That these properties were realized for such a case originated from the fact that extreme Schwarzschild black holes are massless. Quantum black holes may then belong to some class of conformal theories, perhaps p-branes. Electric or magnetic charge (hair) tends to destroy these properties [12].

In this work, we generalize these analyses to cases of quantum black extended objects such as black strings and black membranes in ten-dimensional space-time. Our considerations are based on recent findings by Horowitz and Strominger [18] as well as Gibbons and Maeda [19].

We first obtain an explicit general expression for the degeneracy of states of black $(10-D)$-branes ($4 \leq D \leq 10$) and then proceed to analyze the statistical mechanics of a gas of such objects making use of the microcanonical ensemble. As in the case of black holes, the canonical partition function of black $(10-D)$-branes diverges for all temperatures. Our results closely resemble those for dilaton black holes [12].

Although the canonical partition function is formally divergent, a fact due to the negative microcanonical specific heat, it may be possible to extract information with respect to the nucleation rate (decay rate per unit volume) of a gas of black objects by evaluating this same partition function in the convergence domain of specific parameters appearing in the expression for the density of states. Once the integration is performed, these parameters are then analytically continued back to their original values, an operation which often generates imaginary terms. The imaginary part of the corresponding free energy is then simply related, as has been shown by Langer in condensed matter systems [20], to the decay rate of the metastable black object's gaseous phase.

II. BLACK $(10-D)$-BRANES

In this section, we proceed to derive a general expression for the degeneracy of states of a quantum black $(10-D)$-brane ($4 \leq D \leq 10$) making use of the semiclassical (WKB) approximated expression for the tunneling probability per unit volume. In complete parallel to the black hole case [cf. Eq. (1.1)], we have

$$\rho_{\rm BB}(m) \;\simeq\; c\,\exp\left(\frac{S_E(m)}{\hbar}\right)\,. \tag{2.1}$$

Following Horowitz and Strominger [18], we wish to consider the ten-dimensional action

$$S \;=\; \frac{1}{16\pi}\int d^{10}x\,\sqrt{-g}\,\Big[\, e^{-2\Phi}\left[\, R \;+\; 4\,(\partial\Phi)^2\,\right] \;-\; \frac{2\,e^{2\alpha\Phi}}{(D\,-\,2)!}\,F^2\,\Big]\,, \tag{2.2}$$

in which the field $F_{\mu_1\cdots\mu_{D-2}}$ is a $(D-2)$-form satisfying $dF = 0$ and from which a magnetic charge $Q \propto \int F$ is carried by objects spatially extended in $(10-D)$ dimensions. In the above action, the field Φ represents the dilaton and R is the scalar curvature of the ten-dimensional space-time. Actions similar to this one are found in string theories. Of course, it is also possible to extend trivially the above model to higher dimensions by taking the direct sum of the ten-dimensional metric in (2.2) with that of a flat space of arbitrary dimension D'. The resulting extended object solutions will then have dimensionality $10 + D' - D$. The search for charged black $(10-D)$-brane solutions extremizing the action (2.2) has been considerably simplified by reducing the problem to finding dilaton black hole solutions of an effective D-dimensional action [18]. Such a solution has been given by Gibbons and Maeda [19].

Through field redefinitions, Horowitz and Strominger arrived at the action

$$S \;=\; \frac{L^{10-D}}{16\pi}\int d^Dx\,\sqrt{-\hat g}\,\Big[\;\hat R \;-\; \frac{1}{2}\,(\nabla\phi)^2 \;-\; \frac{2\,e^{-a\phi}}{(D\,-\,2)!}\,F^2\,\Big]\,, \tag{2.3}$$

in which L^{10-D} is the volume of $(10-D)$-space, $\hat g_{\mu\nu}$ is the induced metric of D-dimensional space-time with Riemann curvature $\hat R_{\mu\nu}$, ϕ is the rescaled dilaton field and the dilaton parameter a is given by

$$a \;\equiv\; \sqrt{4\alpha^2 \,+\, 2\alpha(7-D) \,+\, 2\frac{D-1}{D-2}}\,. \tag{2.4}$$

The equations of motion derived by extremizing the action (2.3) are obtained as follows [18]:

$$\nabla^{\mu_1}[\,e^{-a\phi}F_{\mu_1\cdots\mu_{D-2}}\,] \;=\; 0\,, \tag{2.5}$$

$$\nabla^2\phi \;=\; -\frac{2\alpha}{(D-2)!}\,e^{-a\phi}F^2\,, \tag{2.6}$$

and

$$\hat R_{\mu\nu} \,-\, \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi \;=\; \frac{2}{(D-3)!}\,e^{-a\phi}F_{\mu\lambda_1\cdots\lambda_{D-3}}F_\nu^{\lambda_1\cdots\lambda_{D-3}} - 2\hat g_{\mu\nu}\frac{(D-3)}{(D-2)(D-2)!}\,e^{-a\phi}F^2\,. \tag{2.7}$$

Spherically symmetric solutions describing dilaton black holes in D dimensions have been obtained [18, 19]:

$$d\hat s^2 \;=\; -e^{2\Phi(\hat r)}dt^2 \,+\, e^{2\Lambda(\hat r)}d\hat r^2 \,+\, R^2(\hat r)d\Omega^2_{D-2}\,, \tag{2.8}$$

$$F_{\mu_1 \cdots \mu_{D-2}} = Q\, \epsilon_{\mu_1 \cdots \mu_{D-2}} \,,$$
$$\frac{F^2}{(D-2)!} = \frac{Q^2}{R^{2(D-2)}(\hat{r})} \,, \tag{2.9}$$

and

$$e^{-a\phi(\hat{r})} = \left[1 - \left(\frac{r_-}{r}\right)^{D-3}\right]^{\gamma(D-3)} , \tag{2.10}$$

where

$$e^{2\Phi(\hat{r})} = e^{-2\Lambda(\hat{r})} = \left[1 - \left(\frac{r_+}{r}\right)^{D-3}\right]\left[1 - \left(\frac{r_-}{r}\right)^{D-3}\right]^{1-\gamma(D-3)} , \tag{2.11}$$

$$R^2(\hat{r}) = r^2 \left[1 - \left(\frac{r_-}{r}\right)^{D-3}\right]^{\gamma} , \tag{2.12}$$

$$\gamma = \frac{2a^2(D-2)}{(D-3)[2(D-3) + a^2(D-2)]} \,, \tag{2.13}$$

and where $\hat{r} = \hat{r}(r)$ can be obtained from the relation [18]

$$r^{D-4}dr = R^{D-4}d\hat{r} \,. \tag{2.14}$$

The above solutions are parametrized by two horizons situated at r_+ and r_-, parameters which can be expressed in terms of the charge Q and mass M of the D-dimensional dilaton black hole as

$$Q^2 = \frac{\gamma(D-3)^3(r_+r_-)^{D-3}}{2a^2} \tag{2.15}$$

and

$$M = \frac{(D-2)\pi^{\frac{D-3}{2}}}{8\Gamma(\frac{D-1}{2})} \{ r_+^{D-3} + [1-\gamma(D-3)]r_-^{D-3} \} \,. \tag{2.16}$$

These relations can be inverted to yield

$$r_+^{D-3} = \frac{4\Gamma(\frac{D-1}{2})M}{(D-2)\pi^{\frac{D-3}{2}}} \left[1 + \sqrt{1 - \frac{a^2(D-2)^2Q^2\pi^{D-3}[1-\gamma(D-3)]}{8\gamma(D-3)^3\Gamma^2(\frac{D-1}{2})M^2}}\right] \tag{2.17}$$

and

$$r_-^{D-3} = \frac{a^2Q^2(D-2)\pi^{\frac{D-3}{2}}}{2\gamma(D-3)^3\Gamma(\frac{D-1}{2})M} \left[1 + \sqrt{1 - \frac{a^2(D-2)^2Q^2\pi^{D-3}[1-\gamma(D-3)]}{8\gamma(D-3)^3\Gamma^2(\frac{D-1}{2})M^2}}\right]^{-1} . \tag{2.18}$$

Now in order to compute the semiclassical approximation of the path integral, the above solutions must be analytically continued to Euclidean time τ. In the Euclidean space-time, they become instanton solutions. However, in this Euclidean formulation, a surface term (S_{bd}) [21] must be added to the analytical continuation of the action (2.2), which we denote by S_0. Therefore the full Euclidean action is given by

$$S_E = S_0 + S_{bd} \,. \tag{2.19}$$

In addition, requiring the absence of the conical singularity in the Euclidean space-time yields the so-called inverse Bekenstein-Hawking temperature [6, 7, 10]:

$$\beta_{\mathrm{H}} = \frac{2\pi}{\left[(\partial_{\hat{r}} e^{\Phi(\hat{r})})\, e^{-\Lambda(\hat{r})}\right]_{\hat{r}=\hat{r}(r_+)}} \,. \tag{2.20}$$

We find

$$\beta_{\rm H} = \frac{4\pi r_+}{D-3}\left[1-\left(\frac{r_-}{r_+}\right)^{D-3}\right]^{\frac{\gamma(D-2)}{2}-1} . \tag{2.21}$$

This result can be reexpressed in terms of M and Q with the aid of Eqs. (2.17) and (2.18). We get

$$\begin{aligned}\beta_{\rm H} &= \frac{4\pi}{D-3}\left[\frac{4\Gamma(\frac{D-1}{2})}{(D-2)\pi^{\frac{D-3}{2}}}\right]^{\frac{1}{D-3}} M^{\frac{1}{D-3}}\left(1+\sqrt{1-\lambda[1-\gamma(D-3)]}\right)^{\frac{1}{D-3}} \\ &\quad\times\left[1-\lambda\left(1+\sqrt{1-\lambda[1-\gamma(D-3)]}\right)^{-2}\right]^{\frac{\gamma(D-2)}{2}-1}, \\ \lambda &\equiv \frac{a^2Q^2(D-2)^2\pi^{D-3}}{8\gamma(D-3)^3\Gamma^2(\frac{D-1}{2})M^2} .\end{aligned} \tag{2.22}$$

The calculation of the contribution S_0 to the Euclidean action is straightforward. Contracting Einstein's equation [Eq. (2.7)] with the metric tensor $\hat{g}_{\mu\nu}$ (after rotation to Euclidean time), inserting the result into the action and making use of Eqs. (2.9)–(2.14),we arrive at the expression

$$\begin{aligned}S_0 &= -\frac{L^{10-D}\pi^{\frac{D-3}{2}}Q^2\beta_{\rm H}}{4\Gamma(\frac{D-1}{2})}\left[1+\frac{D-4}{D-2}\right] \\ &\quad\times\int_{r_+}^{\infty} dr\, r^{-(D-2)} ,\end{aligned} \tag{2.23}$$

in which the (D−4)-term originates from the nonvanishing trace of the gauge field energy-momentum tensor for $D>4$. The above action has been evaluated for a single period in the compact Euclidean time. This is the contribution from the periodic instantons. Carrying out the integration finally yields

$$S_0 = -\frac{L^{10-D}\pi^{\frac{D-3}{2}}Q^2\beta_{\rm H}}{2(D-2)\Gamma(\frac{D-1}{2})r_+{}^{D-3}} . \tag{2.24}$$

Now in order to evaluate the surface boundary action, we follow the analyses of Ref. [10]. We have

$$\begin{aligned}S^{(\rm boundary)} &= -\frac{L^{10-D}}{8\pi}(D-3) \\ &\quad\times[\,e^{-\Lambda(\hat{r})}\partial_{\hat{r}}(\text{volume of boundary})\,]_{\hat{r}\to\infty} ,\end{aligned} \tag{2.25}$$

where

$$\text{volume of boundary} = \beta_{\rm H}e^{\Phi(\hat{r})}\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\hat{r}^{D-2} . \tag{2.26}$$

The above action however is divergent in the limit $\hat{r}\to\infty$ and so a subtraction must be performed, namely that of a flat space contribution. Again following the considerations of Ref. [10], the flat space contribution is obtained as

$$\begin{aligned}S^{(\rm boundary)}_{\rm flat} &= -\frac{L^{10-D}}{4\pi}(D-3)\,\beta_{\rm H}e^{\Phi(\hat{r})} \\ &\quad\times\frac{\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\left[\partial_{\hat{r}}\,(\hat{r}^{D-2})\right]_{\hat{r}\to\infty} .\end{aligned} \tag{2.27}$$

Therefore,

$$S_{bd} = S^{(\rm boundary)} - S^{(\rm boundary)}_{\rm flat} . \tag{2.28}$$

Explicit evaluation finally yields

$$\begin{aligned}S_{bd} &= \frac{(D-3)L^{10-D}\pi^{\frac{D-3}{2}}\beta_{\rm H}}{8\Gamma(\frac{D-1}{2})} \\ &\quad\times\{r_+^{D-3}+[1-\gamma(D-3)]r_-^{D-3}\} .\end{aligned} \tag{2.29}$$

Making use of Eqs. (2.19), (2.24), and (2.29), the full Euclidean action for the periodic instanton solutions is finally expressed as

$$S_E = \frac{L^{10-D}\pi^{\frac{D-1}{2}}}{2\Gamma(\frac{D-1}{2})}R^{D-2}(\hat{r}) \equiv \frac{\mathcal{A}}{4} , \tag{2.30}$$

where $\mathcal{A}$ is the horizon area of the black (10−D)-brane. In terms of $\mathcal{M}$ and Q, we get

$$\begin{aligned}S_E(\mathcal{M},Q) &= \frac{L^{10-D}\pi^{\frac{D-1}{2}}}{2\Gamma(\frac{D-1}{2})}\left[\frac{4\Gamma(\frac{D-1}{2})}{(D-2)\pi^{\frac{D-3}{2}}}\right]^{\frac{D-2}{D-3}} M^{\frac{D-2}{D-3}}\left(1+\sqrt{1-\lambda[1-\gamma(D-3)]}\right)^{\frac{D-2}{D-3}} \\ &\quad\times\left[1-\lambda\left(1+\sqrt{1-\lambda[1-\gamma(D-3)]}\right)^{-2}\right]^{\frac{\gamma(D-2)}{2}} ,\end{aligned} \tag{2.31}$$

where λ has been defined in Eq. (2.22). This is the Bekenstein-Hawking entropy. It is easy to check that

$$\frac{dS_E}{d\mathcal{M}} = \frac{1}{L^{10-D}}\frac{dS_E}{dM} = \beta_{\rm H} , \tag{2.32}$$

where $\mathcal{M} \equiv ML^{10-D}$ is the total mass of the black (10−D)-brane. Actually, the overall proportionality constant in the expression (2.16) for the mass density M was chosen in such a way that the relation (2.32) held.

According to Horowitz and Strominger [18], dual black branes with electric charges can be obtained from the substitution

$$\alpha \to -\alpha, \quad D \to 14 - D\,. \tag{2.33}$$

We close this section by providing the expression for the mass density of the extreme black $(10-D)$-branes. It is derived from the condition $r_+ = r_-$. We find

$$M_0 = \frac{\sqrt{D-2}\,\pi^{\frac{D-3}{2}} Q}{2\Gamma(\frac{D-1}{2})\sqrt{a^2(D-2)+2(D-3)}}\,. \tag{2.34}$$

III. STATISTICAL MECHANICS

For small charge, according to Eqs. (2.1) and (2.31), black $(10-D)$-branes are characterized by the following degeneracy of states,

$$\rho_{\rm BB}(\mathcal{M},Q) \simeq c\exp\left(\sigma(D)\,M^{\frac{D-2}{D-3}}\left[1+O(Q^2/M^2)\right]\right), \tag{3.1}$$

in which we defined

$$\sigma(D) \equiv \frac{L^{10-D}\pi^{\frac{D-1}{2}}}{2\Gamma(\frac{D-1}{2})}\left[\frac{8\Gamma(\frac{D-1}{2})}{(D-2)\pi^{\frac{D-3}{2}}}\right]^{\frac{D-2}{D-3}}, \tag{3.2}$$

and the constant c represents the quantum field theoretical effects.

The microcanonical density of states $\Omega(\mathcal{E},Q,V)$ of a gas of identically charged black objects of charge Q, degeneracy $\rho_{\rm BB}(\mathcal{M},Q)$, total energy $\mathcal{E}$ and enclosed in a nine-dimensional volume V is expressed as

$$\Omega(\mathcal{E},Q,V) = \sum_{n=1}^{\infty}\Omega_n(\mathcal{E},Q,V)\,, \tag{3.3}$$

in which the contribution from n black objects is given as

$$\Omega_n(\mathcal{E},Q,V) = \left[\frac{V}{(2\pi)^9}\right]^n\frac{1}{\Gamma(n+1)}\prod_{i=1}^{n}\left[\int_{\mathcal{M}_0}^{\infty}d\mathcal{M}_i\,\rho_{\rm BB}(\mathcal{M}_i,Q)\int_{-\infty}^{\infty}d^9p_i\right]\delta\left(\mathcal{E}-\sum_{i=1}^{n}\mathcal{E}_i\right)\delta^9\left(\sum_{i=1}^{n}\vec{p}_i\right), \tag{3.4}$$

where $\mathcal{M}_0$ is the mass of the lightest (extremal) objects in the gas. Now for small charge, the degeneracy of states (3.1) actually belongs to the class of degeneracies $\rho_p(\mathcal{M})$ defined as

$$\rho_p(\mathcal{M}) \equiv f(\mathcal{M})\exp\left(b\mathcal{M}^p\right), \tag{3.5}$$

where $f(\mathcal{M})$ is a polynomial in $\mathcal{M}$ and $p = \frac{D-2}{D-3}$ (> 1). Recalling the classic results obtained by Frautschi [16], the dominant configuration for such a case is the one for which a single (say the nth) black object carries most of the energy while the $n-1$ others carry energies $\mathcal{E}_i = \mathcal{M}_0$ $(i = 1,\ldots,n-1)$. So $\mathcal{E}_n = \mathcal{E}-(n-1)\mathcal{M}_0$. At high energy $\mathcal{E}$, Eq. (3.4) therefore becomes

$$\Omega_n \simeq \left[\frac{V}{(2\pi)^9}\right]^n\frac{1}{\Gamma(n+1)}\rho_{\rm BB}(\mathcal{E}-(n-1)\mathcal{M}_0)\times\rho_{\rm BB}(\mathcal{M}_0)^{n-1}\,. \tag{3.6}$$

The most probable configuration $\Omega_N(\mathcal{E},Q,V)$ is the one satisfying the condition

$$\left[\frac{d\Omega_n(\mathcal{E},Q,V)}{dn}\right]_{n=N(\mathcal{E},Q,V)} = 0\,. \tag{3.7}$$

In complete analogy with the case of the four-dimensional dilaton black hole [12], we find the following solution for N:

$$\Psi(N+1) = \ln\left[\frac{cV}{(2\pi)^9}\right] + S_E(\mathcal{M}_0,Q) - \mathcal{M}_0\beta_{\rm H}(\mathcal{E}-(N-1)\mathcal{M}_0,Q)\,, \tag{3.8}$$

where $\Psi(z)$ is the psi function and in which use has been made of the relation (2.32). At high energy $[\mathcal{E} >> (N-1)\mathcal{M}_0]$, we get the approximate relation

$$\Psi(N+1) \sim \ln\left[\frac{cV}{(2\pi)^9}\right] - \frac{4\pi}{D-3}\left[\frac{8\Gamma(\frac{D-1}{2})}{(D-2)\pi^{\frac{D-1}{2}}}\right]^{\frac{1}{D-3}}\times\mathcal{M}_0\mathcal{E}^{\frac{1}{D-3}} + O(Q^2)\,. \tag{3.9}$$

It is easy to see that $\frac{\partial N}{\partial \mathcal{E}} < 0$ and so the most probable configuration at high energy is again the one for which N is as small as possible, reaching $N = 1$ at a high energy "ionization point" $\mathcal{E}_c$. So, in complete analogy with the case of the four-dimensional dilaton black hole, the most probable equilibrium configuration of a gas of black $(10-D)$-branes in nine-dimensional space is described as

$$\begin{aligned}(N-1)\mathcal{M}_0 << \mathcal{E} << \mathcal{E}_c \quad (N >> 1)\,,\\ \mathcal{E} = \mathcal{E}_c \quad (N = 1)\,.\end{aligned} \tag{3.10}$$

The critical energy $\mathcal{E}_c$ is determined by the formula

$$\mathcal{M}_0\beta_{\rm H}(\mathcal{E}_c) = S_E(\mathcal{M}_0,Q) + \ln\left[\frac{cV}{(2\pi)^9}\right] - \Psi(2)\,. \tag{3.11}$$

An approximate solution for small charge is

$$\mathcal{E}_c \simeq \left(\frac{D-3}{4\pi\mathcal{M}_0}\right)^{D-3}\frac{(D-2)\pi^{\frac{D-3}{2}}}{8\Gamma(\frac{D-1}{2})}\left[\ln\left(\frac{cV}{(2\pi)^9}\right) - \Psi(2)\right]^{D-3}. \tag{3.12}$$

Now since $\Omega(\mathcal{E},Q,V) \sim \Omega_N(\mathcal{E},Q,V)$, Eq. (3.6) shows that the statistical bootstrap condition is trivially satisfied at $\mathcal{E} = \mathcal{E}_c$ ($N = 1$) since there is a single object in the gas. We remark that for $\mathcal{E} > \mathcal{E}_c$, there is no equilibrium configuration.

The total entropy of the gas is written as

$$S(\mathcal{E},Q,V) \simeq \ln\Omega_N(\mathcal{E},Q,V) = N\ln\left[\frac{cV}{(2\pi)^9}\right] - \ln\Gamma(N+1) + S_E(\mathcal{E}-(N-1)\mathcal{M}_0,Q,V) + (N-1)S_E(\mathcal{M}_0,Q) \,. \tag{3.13}$$

The microcanonical temperature is given by

$$\beta(\mathcal{E},Q,V) \equiv \frac{dS(\mathcal{E},Q,V)}{d\mathcal{E}} = \frac{\partial S_E(\mathcal{E}-(N-1)\mathcal{M}_0,Q,V)}{\partial\mathcal{E}} = \beta_{\rm H}(\mathcal{E}-(N-1)\mathcal{M}_0,Q,V) \,, \tag{3.14}$$

with $N(\mathcal{E},Q,V)$ given by Eq. (3.8) and where Eq. (3.7) has been used.

Therefore, as in black hole statistical mechanics in four dimensions, the microcanonical temperature of a gas of black $(10-D)$-branes in ten dimensions is the same as the Bekenstein-Hawking temperature of the most massive black object in the gas.

The microcanonical specific heat is now given as

$$C_V = -\beta^2\frac{d\mathcal{E}}{d\beta} \,. \tag{3.15}$$

Explicit evaluation yields

$$C_V(\mathcal{E},Q,V) = C_V^{(\rm Hawking)}(\mathcal{E}-(N-1)\mathcal{M}_0,Q) \times\left[1 - \mathcal{M}_0\frac{\partial N}{\partial\mathcal{E}}\right]^{-1} \,, \tag{3.16}$$

where

$$C_V^{(\rm Hawking)} \equiv -\beta_{\rm H}^2\frac{\partial\mathcal{E}}{\partial\beta_{\rm H}} \,. \tag{3.17}$$

At high energy, as argued previously, $\frac{\partial N}{\partial\mathcal{E}} < 0$ and so the sign of the microcanonical specific heat is determined by that of the Hawking specific heat.

The case of neutral ($Q = 0$) black branes is somewhat different as the extreme limit is massless ($\mathcal{M}_0 = 0$). The degeneracy of states for such objects reads

$$\rho_{\rm BB}(\mathcal{M}) \simeq c\exp\left[\sigma(D)\mathcal{M}^{\frac{D-2}{D-3}}\right] \,. \tag{3.18}$$

At high energy, one finds the following expression for the corresponding microcanonical density of states:

$$\Omega(\mathcal{E},V) \simeq \left[\frac{V}{(2\pi)^9}\right]^N \frac{c^{N-1}}{\Gamma(N+1)}\rho_{\rm BB}(\mathcal{E}) \,, \tag{3.19}$$

which corresponds to a gas consisting of a single supermassive black object and $(N-1)$ massless others. Therefore we have $\mathcal{E} = \mathcal{M}$. Now the most probable configuration is again determined from the condition

$$\frac{\partial\Omega}{\partial N} = 0 \,. \tag{3.20}$$

The solution to this equation is given by Eq. (3.8) with $\mathcal{M}_0 = 0$:

$$\Psi(N+1) = \ln\left[\frac{cV}{(2\pi)^9}\right] \,. \tag{3.21}$$

Unlike the charged case, we find that the most probable number of objects in the neutral gas becomes effectively independent of energy in the high energy domain.

The total entropy of this system is given as

$$S(\mathcal{E},V) \simeq N\ln\left[\frac{cV}{(2\pi)^9}\right] - \ln\Gamma(N+1) + S_E(\mathcal{E},Q=0) \,. \tag{3.22}$$

The corresponding microcanonical temperature is obtained as

$$\beta(\mathcal{E},V) = \beta_{\rm H}(\mathcal{E},Q=0) \,. \tag{3.23}$$

The microcanonical specific heat is negative,

$$C_V(\mathcal{E},V) = -4\pi\left[\frac{8\Gamma(\frac{D-1}{2})}{(D-2)\pi^{\frac{D-3}{2}}}\right]^{\frac{1}{D-3}}\mathcal{E}^{\frac{D-2}{D-3}} \,, \tag{3.24}$$

a situation analogous to Schwarzschild black holes in four dimensions [12].

As is clear from Eq. (3.19), unlike the case of a gas of charged black objects, the statistical bootstrap condition can be met for the neutral gas provided

$$\left[\frac{cV}{(2\pi)^9}\right]^N \frac{1}{\Gamma(N+1)} = c \,. \tag{3.25}$$

Again, this is due to the fact that extreme neutral black $(10-D)$-branes are massless.

Finally, for all cases treated in this section, the microcanonical equation of state of a gas of black $(10-D)$-branes in ten dimensions is found to be identical to that of an ideal gas: namely,

$$\beta P = \frac{N}{V} \,. \tag{3.26}$$

This is consistent with the fact that we neglected collision processes.

IV. DISCUSSION

In this work, we presented a generalization of previous considerations on black hole statistical mechanics to the case of black (10−D)-brane solutions recently discovered by Horowitz and Strominger [18].

The results found here are somewhat similar to those found in the case of four-dimensional black holes [12], except that, to leading order in charge expansion, the Euclidean action (Bekenstein-Hawking entropy) behaves like $M^{\frac{D-2}{D-3}}$ where M is the mass density per unit (10−D)-volume.

As in the case of the four-dimensional Schwarzschild black holes, neutral black branes also satisfy the statistical bootstrap condition, a fact related to the massless nature of the extreme limit. Also in parallel with Schwarzschild black holes, should we consider quantum black brane scattering processes, the total number of open channels, as a simple consideration would show, actually grows precisely in parallel with the degeneracy of states as the center of mass energy is increased. Therefore,

$$N(m) \;=\; \sum_{n=1}^{\infty} N_n(m) \;\sim\; \rho_{\rm BB}(m) \;\; (m \to \infty)\;. \quad (4.1)$$

Again, it is then plausible to argue that neutral black brane scattering amplitudes satisfy the duality (crossing) symmetry characteristic of string theories. Like strings, they perhaps belong to a class of conformal field theories, e.g., N-branes.

Calculations of the degeneracy of states of higher dimensional structures such as quantum N-branes have been presented almost two decades ago by a few authors [22–24] and happily rediscovered more recently by the authors of Ref. [25]. According to these calculations, the asymptotic behavior of the degeneracy of states of N-branes at large energy (mass) is given as

$$\rho_N(\mathcal{E}) \;\propto\; \exp\,[b\,\mathcal{E}^{\frac{2N}{N+1}}]\,, \quad (4.2)$$

where N is the dimensionality of the extended objects. The above asymptotic behavior seems to be valid in any space-time dimension although the parameter b may be dependent upon the space-time dimensionality. It is interesting to compare this result with our Eq. (3.18) describing the behavior of the degeneracy of states of a quantum neutral black (10−D)-brane in ten space-time dimensions (or analogously a neutral black hole in D dimensions). One finds the relation

$$N \;=\; \frac{D-2}{D-4}\,. \quad (4.3)$$

Only three solutions for integer N exist in the allowed range $4 \leq D \leq 10$, namely $N = 2\,(D = 6)$, $N = 3\,(D = 5)$ and $N = \infty\,(D = 4)$. We then arrive at the conclusion that black 4-branes in ten dimensions are 2-branes, black 5-branes are 3-branes and black 6-branes are ∞-branes, or analogously six-dimensional Schwarzschild black holes are 2-branes, the five-dimensional ones are 3-branes and finally four-dimensional Schwarzschild black holes are ∞-branes. This last result, already pointed out in Ref. [25], may come somewhat as a surprise in view of the usual membrane (2-brane) viewpoint on four-dimensional black holes [26]. Although our comparison of the density of states leads to the identification of quantum black p-branes with ordinary q-branes, a deeper physical understanding of why this is so is still lacking and certainly warrants further investigation.

The considerations presented here with regard to the interpretation of the semiclassical approximation of the Euclidean path integral are completely quantum mechanical. Periodicity of the instanton solutions certainly does not constrain one to a thermal interpretation. Actually, such an interpretation would lead to the following paradoxical situation, namely, at least for neutral black objects (including black holes), should the semiclassical approximation of the path integral be interpreted as the canonical partition function of a gas of black objects at inverse temperature $\beta_{\rm H}$, then its corresponding (statistical mechanical) density of states would be the same as the one obtained from a quantum system with degeneracy of states $\rho_{\rm BB}$ obeying the bootstrap condition. However, we know that the thermal partition function for a gas of objects with such a degeneracy of states does not exist for any finite temperature.

Clearly, the resolution of this paradox lies in the fact that the gas does not achieve thermal equilibrium (the microcanonical specific heat is negative) and consequently the microcanonical and canonical ensembles are not equivalent. It is well known that in this situation the saddle point approximation fails when passing from one ensemble to the other (recall that the canonical partition function is the Laplace transform of the density of states). For such systems of course, one should trust the microcanonical ensemble because it is more fundamental in ensemble theory.

Although formally infinite for all temperatures, it is however possible to extract information from the canonical partition function by evaluating it in the convergence domain of certain parameters. Once the integration (over mass) has been performed these parameters are then analytically continued back to their original values. This procedure usually produces a finite but complex partition function. The nucleation rate (decay rate per unit volume) of the black objects gaseous phase is then simply related to the imaginary part of the corresponding free energy [20], a calculation not unlike the determination of the decay rate of the false vacuum when dealing with a complex effective potential in ordinary quantum field theory [27,28]. Analogous situations also occur in string theories [29–31].

ACKNOWLEDGMENTS

We are grateful to G. Venturi for making us aware of the work of Refs. [22–24] as well as P. Townsend for guidance in the literature on p-brane theories. This research was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-84ER40141.

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