Black holes with a massive dilaton

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The modification of dilaton black holes which result when the dilaton acquires a mass are investigated. We derive some general constraints on the number of horizons of the black hole and argue that if the product of the charge Q of the black hole and the dilaton mass m satisfies $Qm \lesssim 1$ then the black hole has only one horizon. We also argue that for $Qm \gtrsim 1$ there may exist solutions with three horizons and we discuss the causal structure of such solutions. We also investigate the possible structures of extremal solutions and the related problem of two-dimensional dilaton gravity with a massive dilaton.

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I. INTRODUCTION

The notion that Einstein's theory of gravity should be modified by the addition of scalar fields has a long history dating back to the pioneering work of Brans and Dicke [1] who were motivated by the desire to more directly incorporate Mach's principle into physical law. In recent times a particular variant of this idea, dilaton gravity, has received attention because of its close connection with low-energy string theory. In this theory neutral black holes are still described by the Schwarzschild metric and the scalar dilaton plays no role. For charged black holes, however, the dilaton plays a crucial role in modifying the causal structure of the solution.

The causal structure of charged black holes described by the Reissner-Nordström metric in Einstein gravity has given rise to a number of puzzles and speculations. The solution has an outer event horizon at r_+, a Cauchy horizon at r_-, and a timelike singularity in place of the spacelike singularity of the Schwarzschild solution. This gives rise to many peculiar features. For example, an observer crossing the Cauchy horizon at r_- would see the whole history of the asymptotically flat region she originated in flash by in finite time at infinite blueshift and then find that her future is no longer determined by her past. These bizarre features suggest that the inner Cauchy horizon is unstable against small perturbations. In contrast, the charged dilaton black hole has a Schwarzschild-type causal structure with only one horizon, and a spacelike singularity, suggesting stability of the solution to perturbations.

In a different direction, there have been recent attempts to unravel the mysteries of Hawking radiation in a class of two-dimensional theories of dilaton gravity [2]. These theories can be viewed as low-energy effective theories of four-dimensional extremal dilaton black holes. In contrast with the extremal Reissner-Nordström black hole, the extremal dilaton black hole has the singularity and horizon merging at "$r=2M$" and the horizon actually becomes an internal null infinity of the spacetime, thus giving the four-dimensional spacetime the causal structure of the two-dimensional linear dilaton vacuum reviewed in [2]. In some attempts to understand resolutions to the puzzles raised by Hawking radiation the infinite "throat" structure of extremal dilaton black holes has played a significant role [3–5]. The idea here is that the infinite volume of the throat can store the arbitrarily large amount of information which may be lost in the standard semiclassical picture of Hawking evaporation of a black hole.

So far the structure of dilaton black holes is understood only in the physically unrealistic limit of vanishing dilaton mass. If string theory and its low-energy limit are relevant to the real world then the dilaton must eventually acquire a mass. We would then like to know how the above features are modified by the presence of a dilaton mass. In particular, we would like to know what causal structures are allowed and whether the feature of an infinite "throat" persists. Unfortunately our current understanding of how the dilaton acquires a mass is rather primitive and is tied to our lack of understanding of supersymmetry breaking. Since we do not have a good model of how the dilaton mass is generated, we perform as much of the analysis as possible for a general choice of dilaton potential and when we need an explicit choice of potential we consider two simple choices of mass term which we hope will reflect the general structure of such solutions.

The outline of the paper is as follows. The second section contains a review of massless dilaton black holes, and serves to establish our notation and conventions. In Sec. III we discuss adding a mass term, derive general constraints on the number of horizons, show that there is only one horizon when $Qm \lesssim 1$, and derive expansions for the behavior of the solutions in various asymptotic regions. In Sec. IV we discuss the structure of the possible extremal limits of massive dilaton black holes. Section V contains a brief discussion of two-dimensional massive dilaton gravity. We end with some brief final comments in Sec. VI.

II. MASSLESS DILATONIC BLACK HOLES

Black holes in dilaton gravity were first analyzed in some generality by Gibbons and Maeda [6]. An elegant form of the solution was given in later work by Garfinkle, Horowitz, and Strominger (GHS) [7] and we will for the most part follow their approach. GHS considered a massless dilaton field coupled to electromagnetism and gravity. Taking the signature of the metric to be $(+,-,-,-)$ the appropriate action is

$$S=\int d^4x\sqrt{-g}\,[-R+2(\nabla\phi)^2-e^{-2\phi}F_{ab}^2]\ , \qquad (2.1)$$

and one wants to find static, spherically symmetric solutions with a nontrivial dilaton field. The metric may be written in the general form

$$ds^2=A^2(r)dt^2-A^{-2}(r)dr^2-C^2(r)(d\theta^2+\sin^2\theta d\phi^2)\ , \qquad (2.2)$$

where $A(r_+)=0$ marks the outermost event horizon, and $C(r_+)^2=\mathcal{A}/4\pi$ is given in terms of the area of the event horizon. The Hawking temperature β^{-1} of the black hole is given by $(A^2)'|_{r_+}=4\pi\beta^{-1}$.

Varying the action (2.1) gives the equations of motion

$$\nabla_a[e^{-2\phi}F^{ab}]=0\ , \qquad (2.3)$$

$$\nabla_a\nabla^a\phi=\tfrac{1}{2}e^{-2\phi}F_{ab}^2\ , \qquad (2.4)$$

and the "Einstein" equations

$$\begin{aligned}
G_0^0&=\frac{1}{C^2}(1-A^2C'^2)-\frac{2AA'C'}{C}-\frac{2A^2C''}{C}=\mathcal{T}_0^0\ ,\\
G_r^r&=\frac{1}{C^2}(1-A^2C'^2)-\frac{2AA'C'}{C}=\mathcal{T}_r^r\ , \qquad (2.5)\\
G_\theta^\theta&=-\frac{1}{2}(A^2)''-\frac{2AA'C'}{C}-\frac{A^2C''}{C}=\mathcal{T}_\theta^\theta\ ,
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{T}_{ab}&=2\nabla_a\phi\nabla_b\phi-2e^{-2\phi}F_{ac}F_b{}^c\\
&\quad-\tfrac{1}{2}g_{ab}[2(\nabla\phi)^2-e^{-2\phi}F_{cd}^2]\ . \qquad (2.6)
\end{aligned}$$

Notice that the field strength of a magnetically charged black hole in Einstein theory, $F=Q\sin\theta\, d\theta\wedge d\phi$, also satisfies the equation of motion (2.3) for dilaton gravity. However, if there is a nonzero electromagnetic field strength then (2.4) demands a nontrivial dilaton field. This is to be contrasted with zero electromagnetic field where a no-hair theorem demands that the dilaton vanish.

Looking for a static monopole solution, and rearranging the equations of motion, yields

$$\begin{aligned}
C''&=-\frac{1}{2}\frac{C}{A^2}(\mathcal{T}_0^0-\mathcal{T}_r^r)=-C\phi'^2\ ,\\
[(A^2)'C^2]'&=-C^2(2\mathcal{T}_\theta^\theta+\mathcal{T}_r^r-\mathcal{T}_0^0)=\frac{2Q^2e^{-2\phi}}{C^2}\\
&=-2(A^2C^2\phi')'\ , \qquad (2.7)\\
(A^2C^2)''&=2-2C^2(\mathcal{T}_\theta^\theta+\mathcal{T}_r^r)=2\ .
\end{aligned}$$

The last two of these equations are readily integrated to yield

$$\begin{aligned}
&(A^2)'C^2+2A^2C^2\phi'=\mathcal{A}/\beta\ ,\\
&A^2C^2=(r-r_+)^2+\frac{\mathcal{A}}{\beta}(r-r_+)\ . \qquad (2.8)
\end{aligned}$$

From this, it is straightforward to show that, choosing $\beta=4\pi r_+=8\pi M$,

$$\begin{aligned}
A^2&=1-\frac{r_+}{r}\ ,\\
C^2&=r\left[r-\frac{2Q^2e^{-2\phi_0}}{r_+}\right]\ , \qquad (2.9)\\
e^{-2\phi}&=e^{-2\phi_0}C^2/r^2\ ,
\end{aligned}$$

where ϕ_0 is the value of the dilaton field at infinity. One obtains an electrically charged solution from the magnetic solution above by performing a modified duality transformation on the electromagnetic field and changing the sign of the dilaton. Starting from the field strength, dilaton, and metric for a magnetic solution, F^M, ϕ^M, and g^M, we obtain the electric solution as

$$\begin{aligned}
F_{\mu\nu}^E&=\tfrac{1}{2}e^{-2\phi^M}\epsilon_{\mu\nu}{}^{\lambda\rho}F_{\lambda\rho}^M\ ,\\
\phi^E&=-\phi^M\ , \qquad (2.10)\\
g_{\mu\nu}^E&=g_{\mu\nu}^M\ .
\end{aligned}$$

The important differences to stress between the dilaton and Einstein magnetic black holes are the horizon and singularity structure and the nature of the extremal limit. The dilaton black hole has only one horizon and a spacelike singularity, giving rise to a Schwarzschild-type Penrose-Carter diagram; on the other hand the typical Einstein magnetic black hole has two horizons and a timelike singularity, giving rise to the familiar "vertically infinite" Penrose-Carter diagram for the Reissner-Nordström solution shown in Fig. 1.

The extremal limits of these two types of black hole differ both in their charge-mass ratios and in their struc-

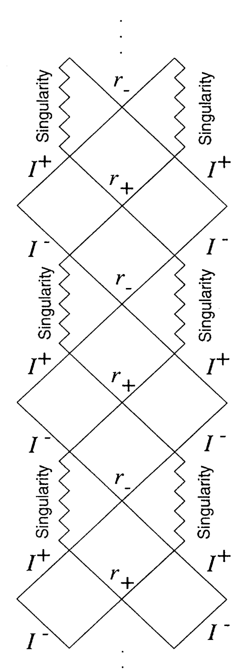

FIG. 1. Penrose-Carter diagram for a Reissner-Nordström black hole.

ture. For the dilaton hole the extremal limit is $Q^2=2M^2e^{2\phi_0}$, as opposed to $Q^2=M^2$ in the Reissner-Nordström case. Further, the event horizon is actually singular in this limit and has zero area, as opposed to the Reissner-Nordström case which has finite area and is nonsingular.

It is also interesting to analyze the solution in terms of the string metric $\hat{g}_{ab}$ defined by

$$\hat{g}_{ab}=e^{2\phi}g_{ab} \ . \tag{2.11}$$

The introduction of $\hat{g}$ is motivated in part by the fact that in fundamental string theory the string world sheet has minimal surface area with respect to the metric $\hat{g}_{ab}$. The line element is then

$$d\hat{s}^2=\frac{1-2Me^{\phi_0}/r}{1-Q^2e^{-\phi_0}/Mr}dt^2 - \frac{dr^2}{(1-2Me^{\phi_0}/r)(1-Q^2e^{-\phi_0}/Mr)}-r^2d\Omega_{\rm II}^2 \tag{2.12}$$

where we have absorbed the factor of $e^{2\phi_0}$ in order that the metric have the canonical asymptotic form. Note that in this metric the singularity at $r_s=Q^2e^{-\phi_0}/M$ corresponds to a two-sphere of area $4\pi Q^4e^{-2\phi_0}/M^2$ rather than to a point as in the Einstein metric.

In the extremal limit $Q^2\rightarrow 2M^2e^{2\phi_0}$ the line element (2.12) becomes

$$d\hat{s}^2=dt^2-\frac{dr^2}{(1-2M/r)^2}-r^2d\Omega_{\rm II}^2 \ . \tag{2.13}$$

In this limit the previous singularity at r_s coincides with the horizon at $r_+=2Me^{\phi_0}$ and both have been pushed off to infinite proper distance. In terms of a new coordinate σ with

$$d\sigma=\frac{dr}{1-2M/r} \ , \tag{2.14}$$

we have, as $r\rightarrow 2M$,

$$d\hat{s}^2\rightarrow[-dt^2+d\sigma^2+(2M)^2d\Omega_{\rm II}^2] \ , \tag{2.15}$$

so that the geometry approaches that of an infinite "tube" of radius $2Me^{\phi_0}$.

However, note that this infinite tube is quite distinct from the infinite tube of the spatial sections of extremal Reissner-Nordström black holes. In this extremal metric, $r=2M=\sqrt{2}Q$ is not only at an infinite spatial distance, but also at an infinite proper distance to any causal observer so that in effect the event horizon provides another internal asymptotic null infinity as can be seen from the Penrose-Carter diagram of Fig. 2. For extremal Reissner-Nordström black holes, however, while $r=M=Q$ is located at the end of an infinite tube in a spatial section of the metric, it is at a finite proper distance for any infalling observer—thus there is no internal asymptotic region, only an event horizon as shown in Fig. 4. It is also worth emphasizing that the infinite throat of the extremal solution in the string metric occurs only for the magnetically charged solution and not for the electrically charged solution.

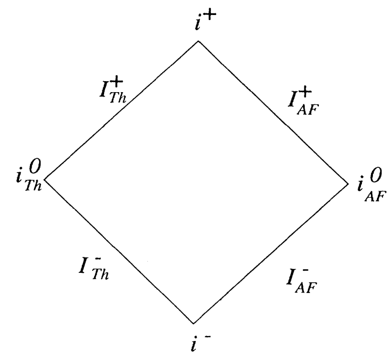

FIG. 2. Penrose-Carter diagram for an extremal dilaton black hole. The subscripts AF and Th refer to the asymptotically flat and "throat" regions of the black hole respectively.

In the following section we will also find useful an alternative parametrization of the string metric in the form (2.2). We can write

$$d\hat{s}^2=\hat{A}^2(\rho)dt^2-\hat{A}^{-2}(\rho)d\rho^2-\hat{C}^2(\rho)d\Omega_{\rm II}^2 \tag{2.16}$$

with

$$\hat{C}^2(\rho)=r^2(\rho) \ , \qquad \hat{A}^2(\rho)=\frac{[1-2M/r(\rho)]e^{2\phi_0}}{1-Q^2e^{-2\phi_0}/Mr(\rho)} \ , \tag{2.17}$$

with $r(\rho)$ defined implicitly by

$$\rho=e^{2\phi_0}(r-r_0)+\frac{Q^2}{M}\ln\left|\frac{r-Q^2e^{-2\phi_0}/M}{r_0-Q^2e^{-2\phi_0}/M}\right| \tag{2.18}$$

and r_0 the (arbitrary) point at which $\rho=0$. Note that the singularity $(r=r_s)$ occurs at $\rho=-\infty$ and that $\rho\rightarrow e^{2\phi_0}r$ as $r\rightarrow\infty$.

In this metric the equations of motion can be seen to admit solutions which are products of two two-dimensional solutions. In particular, they admit a solution of the form $S^2\times\mathcal{M}^2_{\rm BH}$ where S^2 is a two-sphere of constant radius and $\mathcal{M}^2_{\rm BH}$ is a two-dimensional black hole solution. Explicitly this solution is given by

$$\hat{C}^2(\rho)=2Q^2 \ , \qquad \hat{A}^2(\rho)=1-2Me^{-2\lambda\rho} \ , \qquad \phi=-\lambda\rho \ , \tag{2.19}$$

where $\lambda^2=1/(8Q^2)$ and M is the arbitrary mass of the two-dimensional black hole. For a more detailed explanation of the relation between four- and two-dimensional dilaton black holes see [8]. We will see later that the addition of a dilaton potential no longer allows solutions of the above form.

To summarize, in either metric the important features to note are that the equations of motion are readily solved, the horizon structure of a dilaton black hole is different than in Einstein gravity, and the thermodynamical relationships are also different. These latter features will persist when we add a mass term for the dilaton, although the equations will not be so easy to solve.

III. MASSIVE DILATONIC BLACK HOLES

In this section we consider adding in a potential term for the dilaton field. Instead of the value of the dilaton at infinity, and hence the string coupling constant being arbitrary, it will now be determined by the minimum of the potential. Since we want to generate a mass term for the dilaton, the leading term in the potential should be $V_1(\phi)=2m^2(\phi-\phi_0)^2$ [the factor of 2 is due to the unconventional normalization of the kinetic term in (2.1)]. In string theory the natural variable is e^{ϕ} which plays the role of the dimensionless coupling constant. We thus expect that the true potential for the dilaton will also have higher-order corrections in an expansion in ϕ. Where possible, we will try to make general statements without making detailed assumptions about the form of the potential. When we need to consider specific possibilities we will consider either the potential V_1 or $V_2=2m^2\sinh^2(\phi-\phi_0)$. This latter choice of potential is a simple function of e^{ϕ} which agrees with V_1 to lowest order in ϕ^2 but which is more divergent at the singularity of the black hole than V_1.

Intuitively, we expect that if a field has mass m, then at length scales large compared to m^{-1} the potential will suppress fluctuations in the field while at lengths small compared to m^{-1} it will behave rather like a massless field. Therefore we would expect that at large distances our black hole would look like a Reissner-Nordström black hole, and at small distances like a massless dilaton black hole. We might also expect that the classical treatment here will break down unless the Compton wavelength of the dilaton $1/m$ is small compared to the gravitational radius of the black hole $\sim M/M_P^2$ with M_P the Planck mass. In geometrized units this means $mM \gg 1$.

For large black holes (i.e., those satisfying this criterion) we thus expect the structure to be asymptotically Reissner-Nordström type. If the black hole has large charge $Q\sim M$ then both the outer and inner horizons of the Reissner-Nordström solution occur in a region where we expect the dilaton to play a negligible role. However, as we approach the singularity a new possibility arises. Namely that the dilaton "switches on" and the solution becomes like the massless dilaton black hole which would cause a third horizon and the causal lattice shown in Fig. 3.

As we will see, this final possibility requires a violation of the strong energy condition, which in turn requires the dilaton to become dominant at some length scale, thus validating our intuition. On the other hand, for real physical black holes we should replace the magnetic charge Q by the electric charge Q_e, and we would expect $Q_e \ll 10^{18}\, M$ since otherwise the black hole would neutralize itself by attracting charged particles from the surrounding medium. In this case we expect the dilaton to become important long before we reach the inner horizon of the Reissner-Nordström solution and the exact solution may have one, two, or three horizons.

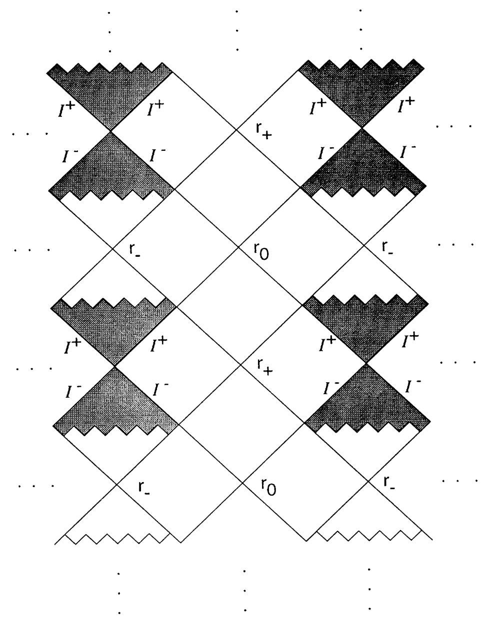

FIG. 3. Penrose-Carter diagram for a massive dilaton black hole with outer horizon r_+, middle horizon r_0, and inner horizon r_-. $I^{\pm}$ indicate future and past null infinity respectively, while the shading indicates regions not included in the spacetime. The figure repeats periodically to tile the plane.

The organization of this section will be as follows. We first set up the equations of motion in the Einstein metric and discuss some general properties of solutions, examining under what circumstances three horizons *may* occur. We then obtain a specific restriction on the number of horizons, given a certain constraint which is satisfied by our test potential V_2, but not V_1. We then examine the equations of motion in the string metric, giving a plausibility argument that for $Qm \lesssim 1$ the solution will have only one horizon. We conclude by commenting on the dual electric solutions. Note that we will be concerned only with the nonextremal black holes in this section.

We begin by stating the general equations of motion. Remaining with the general expression $V(\phi)$, we note that the action becomes

$$S=\int d^4x\sqrt{-g}\,[-R+2(\nabla\phi)^2-V(\phi)-e^{-2\phi}F_{ab}^2]\ . \quad (3.1)$$

Thus the Maxwell equations are unchanged, and the equations of motion and energy momentum are modified to

$$\nabla_a\nabla^a\phi+\frac{1}{4}\frac{\partial V}{\partial\phi}=\frac{Q^2e^{-2\phi}}{C^4}\ , \quad (3.2)$$

$$\mathcal{T}_{ab}=2\nabla_a\phi\nabla_b\phi-2e^{-2\phi}F_{ac}F_b{}^c -\tfrac{1}{2}g_{ab}[2(\nabla\phi)^2-e^{-2\phi}F_{cd}^2-V(\phi)]\ , \quad (3.3)$$

which implies

$$C''=-C\phi'^2\ ,$$
$$[(A^2)'C^2]'=C^2\left[\frac{1}{2}\frac{\partial V}{\partial\phi}-V\right]-2(A^2C^2\phi')'\ , \tag{3.4}$$
$$(A^2C^2)''=2-2C^2V(\phi)\ .$$

Note that the presence of the potential means that we no longer have a first integral to simplify the process of solving these equations. A few general remarks however can be made.

Note that the first of Eqs. (3.4) implies that C' is always decreasing; therefore if we wish to have $C'(r)\to 1$ at infinity, then, as with the massless case, the singularity will occur at positive r, $r_{\rm sg}>0$. (In fact the dominant energy condition guarantees $r_{\rm sg}\geq 0$.) The final equation of (3.4) shows that with a potential, $(A^2C^2)''$ is no longer necessarily positive. Since, roughly speaking, it is the positivity of $(A^2C^2)''$ that guarantees a single horizon, we see that if $C^2V>1$, then multiple horizons are possible. The middle equation of (3.4) actually gives us some more concrete requirements on the stress-energy-momentum tensor, and hence $V(\phi)$, if there are to be three horizons. Note that the existence of three horizons requires (at least) two zeros of $(A^2)'$. Since

$$[(A^2)'C^2]'=-C^2(2\mathcal{T}_\theta^\theta+\mathcal{T}_r^r-\mathcal{T}_0^0)$$

this in turn implies a violation of the strong energy condition (SEC). A violation of the SEC is by no means impossible, for although electromagnetism satisfies the strong energy condition, a massive scalar field does not necessarily; nonetheless, it again emphasizes the point that if the scalar potential is weak, the horizon structure of the massless dilaton black hole will not be altered.

The dilaton equation (3.2) can be used to prove that if the potential $V(\phi)$ is convex, then $\phi'<0$ when $A^2>0$. To show this we first argue that $\phi'<0$ for some range outside an event horizon, and then for the whole range.

Multiplying (3.2) by ϕ' and integrating yields

$$\int\frac{Q^2e^{-2\phi}}{C^4}\phi'=\int\frac{1}{2}\phi'\frac{\partial V}{\partial\phi}-\frac{\phi'}{C^2}(A^2C^2\phi')'$$
$$=\frac{1}{2}[V-A^2\phi'^2]$$
$$-\int\left[\frac{2C'A^2}{C}+\frac{1}{2}(A^2)'\right]\phi'^2\ . \tag{3.5}$$

Evaluating this integral between the horizon and infinity shows

$$\int_{r_+}^{\infty}\frac{Q^2e^{-2\phi}}{C^4}\phi'<0\ , \tag{3.6}$$

i.e., that $\phi'<0$ at some point.

Now that we have shown that $\phi'<0$ at some point outside the horizon, we will show that this holds at all points outside the horizon. The proof is by contradiction. If ϕ is not monotonically decreasing outside the horizon then it must have a maximum outside the horizon (possibly at infinity). At this maximum (3.2) then implies

$$\frac{1}{2}\frac{\partial V}{\partial\phi}-\frac{Q^2e^{-2\phi}}{C^4}=A^2\phi''\leq 0 \tag{3.7}$$

(with equality for the case where the maximum is at infinity). As we move from this maximum in towards the horizon, ϕ can either have a minimum or decrease to the horizon. Equation (3.2) then implies that

$$\frac{1}{2}\frac{\partial V}{\partial\phi}-\frac{Q^2e^{-2\phi}}{C^4}=\phi''A^2+(A^2)'\phi'+\frac{2C'A^2\phi'}{C}>0\ . \tag{3.8}$$

The right-hand side (RHS) of (3.8) is positive for either case since at the supposed minimum A^2 and ϕ'' are positive and C is monotonically increasing while in the latter case the RHS of (3.8) is positive at the horizon since $A^2=0$ and $(A^2)'\phi'>0$. So we have established that the left-hand side (LHS) of (3.8) must be positive at or outside the horizon if ϕ' is positive at or outside the horizon. However, for $\phi'>0$ and a convex potential, the LHS of (3.8) is an increasing function of r. Hence it cannot become nonnegative given (3.7). We have thus derived a contradiction and shown that ϕ' remains negative everywhere outside the event horizon. A similar argument shows that we expect $\phi'<0$ in any inner regions of the black hole where $A^2>0$.

Having used the equations to extract some general information, let us now be specific about the form of the solution. We can confirm some of the previous reasoning by solving Eqs. (3.4) in a power series in $1/r$, that is, at large distances from the singularity. To lowest nontrivial order the asymptotically flat solution depends only on the quadratic part of the potential and therefore will essentially be independent of the potential, being given by

$$\phi(r)=\phi_0+\frac{Q^2e^{-2\phi_0}}{m^2r^4}+\cdots+Fe^{-mr}+\cdots\ ,$$
$$A^2(r)=1-\frac{2M}{r}+\frac{Q^2e^{-2\phi_0}}{r^2}-\frac{Q^4e^{-4\phi_0}}{5m^2r^6}+\cdots\ , \tag{3.9}$$
$$C(r)=r-\frac{2Q^4e^{-4\phi_0}}{7m^4r^7}+\cdots\ ,$$

where F is an arbitrary constant. This asymptotic expansion agrees with the Reissner-Nordström solution at large distances, remembering that ϕ_0 shifts the value of the gravitational coupling at infinity, and indeed up to $r^2\sim e^{-\phi_0}Q/m$, which will be past at least the event horizon for large mass black holes. In addition, note that ϕ is monotonically decreasing as claimed.

Examining the equations of motion near the central singularity ($C^2\to 0$), the form of the potential becomes important. For our two test potentials V_1 and V_2 we find the following behavior for the metric and the dilaton:

$$C\sim c_0(r-r_0)^{1/2}+c_1(r-r_0)^{3/2}\ ,$$
$$e^{-2\phi}\sim f_0(r-r_0)-\frac{2c_1}{c_0}f_0(r-r_0)^2\ ,$$
$$A_1^2\sim\frac{1-2f_0Q^2/c_0^2}{c_0c_1}+\frac{2f_0Q^2}{c_0^4}(r-r_0)\ , \tag{3.10}$$
$$A_2^2\sim A_1^2-\frac{m^2(r-r_0)}{2f_0}\ ,$$

where A_i^2 is the g_{00} appropriate for V_i. For V_1, $(A^2)'$ is always positive at the singularity, whereas for V_2 the sign of $(A^2)'$ will depend on how large the dilaton mass is.

Notice that in (3.9) there is only one free parameter, since we are looking for an asymptotically flat solution with a particular charge and mass. As we approach the singularity with Q fixed, there are, however, four residual free parameters. Thus, since our solution space is five dimensional, we do in general expect a solution to exist, however without, for example, a numerical integration, this is not a certainty.[1] However, in certain cases, we can eliminate possibilities for the causal structure of the solutions, and we will therefore concentrate on what we can say analytically about the general properties a potential must have to admit one, two, or three horizons.

We start by proving that the potential $V_2(\phi)$ can have at most two horizons. To do this we integrate the middle equation of (3.4) between the first and second horizons. This gives

$$\int_{r_1}^{r_2} 2C^2 \left[\frac{1}{2}\frac{\partial V}{\partial \phi} - V \right] < 0 . \tag{3.11}$$

At first sight this may not seem at all restrictive, but for the potential $V_2(\phi)$ we have

$$\frac{1}{2}\frac{\partial V_2}{\partial \phi} - V_2 = m^2(1 - e^{-2(\phi-\phi_0)}) . \tag{3.12}$$

Since we have already shown that ϕ is decreasing on the interval $[r_1, r_2]$, (3.11) would require $\phi(r_2) < \phi_0$. But a rearrangement of (3.4),

$$[A^2C^2\phi']' = C^2\left[V + \frac{1}{2}\frac{\partial V}{\partial \phi}\right] + (A^2CC')' - 1 \tag{3.13}$$

integrated in a neighborhood of r_2 gives

$$A^2C^2\phi'|_{r_2+\delta} = \int_{r_2}^{r_2+\delta} m^2C^2(e^{2(\phi-\phi_0)} - 1) - \delta + A^2CC' > 0 \tag{3.14}$$

which implies

$$\int_{r_2}^{r_2+\delta} m^2C^2(e^{2(\phi-\phi_0)} - 1) > 0 \tag{3.15}$$

so that $\phi(r_2) > \phi_0$. So, $V_2(\phi)$ does not admit a black hole with three horizons. V_1 on the other hand has

$$V_1 - \frac{1}{2}\frac{\partial V_1}{\partial \phi} = 2m^2(\phi-\phi_0)(\phi-\phi_0-1) , \tag{3.16}$$

thus as $\phi - \phi_0$ becomes greater than 1, (3.11) can be satisfied. This can be seen to fit in with some of our earlier intuitive arguments. Since V_2 becomes very important for large ϕ compared to V_1, we might expect greater resistance to approaching a GHS massless solution. Therefore for large dilaton mass we might expect the solution to remain much like a Reissner-Nordström black hole except very close to the singularity, by which stage there is no possibility of a third horizon forming.

It is obviously now of interest to determine whether the solution for V_1 can have three horizons, not least because of the bizarre causal structure associated with three horizons. It would also be useful to know whether, and if so when, even two horizons are possible. One of the nice features of the massless dilaton black hole was that it had a (presumably stable) Schwarzschild-like causal structure, with no Cauchy horizons. It would therefore be useful to know if this single horizon structure persists, and if so, for approximately what range of parameters.

In order to get a clearer picture of what is happening, and to simplify some of the arguments, let us transform to the string metric (2.11). The action in the new metric is

$$S = \int d^4x\sqrt{-\hat{g}}\, e^{-2\phi}\{ -\hat{R} - 4\hat{g}^{ab}\partial_a\phi\partial_b\phi - \hat{g}^{ac}\hat{g}^{bd}F_{ab}F_{cd} - e^{-2\phi}V(\phi)\} . \tag{3.17}$$

The equations of motion which follow from this action are

$$8\hat{\nabla}_a\hat{\nabla}^a\phi - 8(\hat{\nabla}\phi)^2 + 4e^{-2\phi}V(\phi) = e^{-2\phi}\frac{\partial V}{\partial \phi} - 2\hat{R} - 2F_{ab}^2 \tag{3.18}$$

$$\hat{R}_{ab} + 2F_{ac}F_b{}^c + 2\hat{\nabla}_a\hat{\nabla}_b\phi = \tfrac{1}{2}\hat{g}_{ab}\{\hat{R} + F_{ab}^2 + e^{-2\phi}V(\phi) + 4\hat{\nabla}_a\hat{\nabla}^a\phi - 4(\hat{\nabla}\phi)^2\} \tag{3.19}$$

where all contractions are taken with the new string metric $\hat{g}_{ab}$. Taking the trace of (3.19) simplifies (3.18),

$$-4\hat{\nabla}_a\hat{\nabla}^a\phi + 8(\hat{\nabla}\phi)^2 = e^{-2\phi}\frac{\partial V}{\partial \phi} - 2F_{ab}^2 , \tag{3.20}$$

and (3.19) then can be written as

$$\hat{G}_{ab} = -2\hat{\nabla}_a\hat{\nabla}_b\phi - 2F_{ac}F_b{}^c + \frac{1}{2}\hat{g}_{ab}\left[3F_{cd}^2 + e^{-2\phi}V(\phi) - e^{-2\phi}\frac{\partial V}{\partial \phi} + 4(\hat{\nabla}\phi)^2\right] . \tag{3.21}$$

The Maxwell field equation is unchanged; hence, $F = Q\sin\theta\, d\theta \wedge d\phi$ is also a solution in this theory.

Looking for a static spherically symmetric solution as before with the metric

$$d\hat{s}^2 = \hat{A}^2 d\tau^2 - \hat{A}^{-2}d\rho^2 - \hat{C}^2\{d\theta^2 + \sin^2\theta\, d\phi^2\} \tag{3.22}$$

gives, for the dilaton,

$$[\hat{A}^2\hat{C}^2(e^{-2\phi})']' = \frac{2Q^2e^{-2\phi}}{\hat{C}^2} - \frac{1}{2}\hat{C}^2e^{-4\phi}\frac{\partial V}{\partial \phi} , \tag{3.23}$$

and the equations of motion for the metric variables can be boiled down to

[1] This issue is currently being addressed by Horne and Horowitz, and we thank them for discussions on this point.

$$\hat{C}''=\hat{C}\phi'' ,$$

$$[(\hat{A}^2)'\hat{C}^2 e^{-2\phi}]' = -\hat{C}^2 e^{-4\phi}\left[V-\frac{1}{2}\frac{\partial V}{\partial \phi}\right] , \tag{3.24}$$

$$[\hat{A}^2(\hat{C}^2)'e^{-2\phi}]' = e^{-2\phi}\left[2-\frac{4Q^2}{\hat{C}^2}-\hat{C}^2 e^{-2\phi}\left[V-\frac{1}{2}\frac{\partial V}{\partial \phi}\right]\right] .$$

Note immediately that the last two "gravity" equations imply

$$\{[\hat{A}^2(\hat{C}^2)'-(\hat{A}^2)'\hat{C}^2]e^{-2\phi}\}' = \left[2-\frac{4Q^2}{\hat{C}^2}\right]e^{-2\phi} . \tag{3.25}$$

Thus if $\hat{C}^2>2Q^2$ for all ρ, then $[\hat{A}^2(\hat{C}^2)'-(\hat{A}^2)'\hat{C}^2]e^{-2\phi}$ must be monotonically increasing; this is not compatible with a third inner horizon. In fact, (3.25) shows that if there are to be three horizons not only is $\hat{C}<2Q$ required, but also a turning point in $\hat{C}$ before the final inner horizon is reached. Note that although $C(r)$ in the original metric is strictly increasing, this does not imply a similar result for $\hat{C}(\rho)$, since

$$\begin{aligned}\frac{d}{d\hat{s}}\hat{\mathcal{A}}(\hat{s}) &= 8\pi\hat{C}|\hat{A}^2|^{1/2}\hat{C}'(\rho)\\ &= e^{-\phi}\frac{d}{ds}(e^{2\phi}\mathcal{A})\\ &= 8\pi C|A^2|^{1/2}e^{\phi}[C'(r)+C\phi'(r)] ,\end{aligned} \tag{3.26}$$

hence if $\phi'(r)<-C'/C$, $\hat{C}'(\rho)$ can be negative. Thus in the string metric, the event horizon actually masks undulations in the $t=$const surfaces, areas of the two-spheres actually increase before the innermost horizon.

Now let us examine the equations of motion for a weak potential (by which we mean $\sup_{\phi>0}2Q^2e^{-2\phi}V<1$) at a putative inner horizon. We first note that since $C(r)$ is an increasing function of r, $e^{-2\phi}\hat{C}^2$ is an increasing function of ρ. Thus

$$[\hat{A}^2(e^{-2\phi}\hat{C}^2)']' = \left[2-\frac{2Q^2}{\hat{C}^2}-\hat{C}^2e^{-2\phi}V(\phi)\right]e^{-2\phi} \tag{3.27}$$

implies that the term in square brackets on the RHS of (3.27) is negative at the inner horizon. We would like to use this to establish an upper bound on $\hat{C}^2$. First notice that (3.25) implies that $\hat{C}^2<2Q^2$ for some range between the outer horizon and the putative inner horizon. If we assume that $\hat{C}^2$ is strictly increasing between the inner and outer horizon then we also know that the previous inequality is satisfied at the inner horizon. Thus solving (3.27) as a quadratic for $2Q^2/\hat{C}^2$ gives

$$\hat{C}^2 < \frac{2Q^2}{1+\sqrt{1-2Q^2e^{-2\phi}V}} . \tag{3.28}$$

The equation of motion for the dilaton (3.23) implies

$$\frac{2Q^2}{\hat{C}^2} < \frac{1}{2}\hat{C}^2 e^{-2\phi}\frac{\partial V}{\partial \phi} . \tag{3.29}$$

Equations (3.28) and (3.29) thus give us

$$\frac{2Q^2}{\sqrt{Q^2e^{-2\phi}\partial V/\partial\phi}} < \hat{C}^2 < \frac{2Q^2}{1+\sqrt{1-2Q^2e^{-2\phi}V}} \tag{3.30}$$

which in turn requires

$$1 < \frac{Q^2e^{-2\phi}(V+\frac{1}{2}\partial V/\partial\phi)^2}{\partial V/\partial\phi} . \tag{3.31}$$

If we are looking for a lower bound on Qm, then we maximize the RHS of (3.31) with respect to ϕ. The maximum occurs when

$$\left[2\frac{\partial V}{\partial \phi}+\frac{\partial^2 V}{\partial\phi^2}\right]\left[\frac{1}{4}\left[\frac{\partial V}{\partial\phi}\right]^2-V^2\right]=0 . \tag{3.32}$$

If the potential is convex, $\partial V/\partial\phi$ will be positive for positive ϕ; hence the maximum will occur when

$$V(\phi_{\rm max}) = \frac{1}{2}\frac{\partial V}{\partial\phi}\bigg|_{\phi_{\rm max}} \Longrightarrow 1 < 2V(\phi_{\rm max})Q^2e^{-2\phi_{\rm max}} . \tag{3.33}$$

Obviously this is in contradiction with our initial supposition that the potential was weak. So, since generically this RHS will have an order of magnitude of (Q^2m^2), hence, for $Qm \lesssim 1$, there can be only one horizon. For our test potentials, (3.33) gives for V_1, $\phi_{\rm max}-\phi_0=1$, $Qm>e^{1+\phi_0}/2$ for an inner horizon, and for V_2, $\phi_{\rm max}-\phi_0$"$=$"∞ giving $Qm>e^{\phi_0}$. Although we have found several arguments that suggest the validity of this reasoning, which is further supported by the analysis of extremal solutions in the following section, we do not have a watertight proof that $\hat{C}'>0$.

We end this section by mentioning that the transformation (2.10) can be used to trivially construct electrically charged solutions from the magnetic solutions discussed above as long as the potential $V(\phi)$ is an even function of ϕ.

IV. EXTREMAL SOLUTIONS

In thinking about the extremal limit(s) of a black hole with a massive dilaton, the situation is more diverse than either Reissner-Nordström or massless dilaton black holes. In these cases, there is a unique extremal limit: For Reissner-Nordström black holes, $Q=M$ and the inner and outer horizons merge, on the verge of disappearing and leaving a naked singularity. The resulting Penrose-Carter diagram is shown in Fig. 4.

For massless dilaton black holes the singularity and horizon merge, the singularity again on the verge of becoming naked, although this time by moving "outside" the event horizon with a Penrose-Carter diagram as shown in Fig. 2. For the case of massive dilatons however, we have several options, depending on the number of horizons. For example, if we have only one horizon, we might expect an extremal limit similar to the massless case, but for $V_1=2m^2(\phi-\phi_0)^2$ there is also another possibility, namely that A^2 develops first a stationary, then two turning points, and finally an additional double horizon. In other words, black holes with massive dilatons can exhibit both kinds of extremality. We will first look

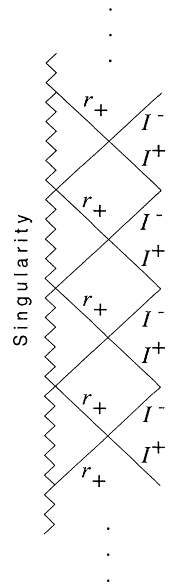

FIG. 4. Penrose-Carter diagram for an extremal Reissner-Nordström black hole.

at the conditions for Reissner-Nordström extremality, before analyzing the massless dilaton type of extremality.

In looking for a Reissner-Nordström-type extremal solution, the defining feature is the repeated horizon. This corresponds to a repeated zero of A^2, or indeed $\hat{A}^2$. In other words $A^2=(A^2)'=0$ at such a point. Examining the equations of motion in the string metric yields the following equalities that must then be satisfied:

$$2-\frac{2Q^2}{\hat{C}^2}-\hat{C}^2e^{-2\phi}V=0\ ,$$
$$\frac{2Q^2}{\hat{C}^2}-\frac{1}{2}\hat{C}^2e^{-2\phi}\frac{\partial V}{\partial\phi}=0\ . \tag{4.1}$$

These equalities can be solved for $\hat{C}^2$ and give

$$\frac{2Q^2}{1\pm\sqrt{1-2Q^2e^{-2\phi}V}}=\frac{2Q^2}{\sqrt{Q^2e^{-2\phi}\partial V/\partial\phi}} \tag{4.2}$$

which can then be solved for ϕ, giving

$$\frac{\partial V}{\partial\phi}=Q^2e^{-2\phi}\left[V+\frac{1}{2}\frac{\partial V}{\partial\phi}\right]^2 . \tag{4.3}$$

We will examine each test potential in turn.

For V_1, these boil down to a modified cubic for ϕ,

$$(\phi-\phi_0)(\phi-\phi_0+1)^2e^{-2\phi}=1/m^2Q^2\ , \tag{4.4}$$

and, for V_2,

$$e^{-2\phi_0}\tanh(\phi-\phi_0)=1/m^2Q^2\ . \tag{4.5}$$

The former relation generically has two solutions for ϕ, whereas the latter relation has only one positive-ϕ solution. This is of course because there can only be one possible extremal type for the $\sinh^2\phi$ potential, namely a positive-A^2 Reissner-Nordström type, since it does not admit three horizons. It is therefore appropriate that in the limiting value for $Qm=e^{\phi_0}$, it also corresponds to $\phi-\phi_0=\infty$, i.e., the singularity. For V_1, $(\phi-\phi_0)(\phi-\phi_0+1)^2e^{-2\phi}$ has a maximum of $4e^{-2\phi_0}/e^2$ at $\phi-\phi_0=1$; therefore, at the limiting value $Qm=e^{\phi_0}e/2$, the two roots of (4.4) coincide at $\phi-\phi_0=1$. At such a point $(A^2)''$ also vanishes, and there is a triple horizon.

In Einstein gravity the throat region of the extreme Reissner-Nordström metric is described by an exact solution of the Einstein equations with constant radius two-spheres sometimes known as the Bertotti-Robinson electromagnetic universe [9]. A similar situation prevails here. Looking for a solution of (3.23) and (3.24) with $\hat{C}$ and ϕ constant we find a solution with ϕ given by (4.4) or (4.5) and with

$$\hat{C}^2=\frac{2Qe^{\phi}}{\sqrt{\partial V/\partial\phi}}\ ,$$
$$(\hat{A}^2)''=-e^{-2\phi}\left[V-\frac{1}{2}\frac{\partial V}{\partial\phi}\right]\ , \tag{4.6}$$

where the RHS of (4.6) is to be evaluated at the solution of (4.4) or (4.5) depending on the choice of potential.

For the other type of extremal solution, the defining feature is that the singularity and horizon coincide, in other words, that $A^2=0$ at the singularity. Searching for an expansion of the solution near the singularity reveals an interesting difference between the massless and massive cases, which is reflected in the two-dimensional theory, as we will see in the next section. In the massless case, the solution in the neighborhood of the singularity was a linear dilaton vacuum (LDV), i.e., $\phi=-\alpha\rho$, with (vanishing) corrections of the form $e^{\alpha\rho}$. As we will see, the form of these corrections may alter, although the LDV will still persist. Since this "throat" structure of the original massless dilaton black holes was so attractive for hiding information, it is important to demonstrate that this structure remains.

We are looking for a solution of the form

$$\hat{A}^2=a_1+a_2(\rho)e^{\alpha\rho}+\cdots\ ,$$
$$\hat{C}=c_0+c_1(\rho)e^{\alpha\rho}+\cdots\ , \tag{4.7}$$
$$e^{-2\phi}=f_0e^{\alpha\rho}+f_1(\rho)e^{2\alpha\rho}+\cdots$$

as $\rho\to\infty$. Using $\hat{C}''=\hat{C}\phi''$ readily shows that $f_1=-2f_0c_1(\rho)/c_0$, independent of whether or not there is a potential; however, the other two equations in (3.24) rapidly show that while $c_0^2=2Q^2$ as with the zero-mass case, the situation for the other variables is quite different. The corrections c_1 and a_1 must now satisfy nontrivial differential equations:

$$[(a_2e^{\alpha\rho})'e^{\alpha\rho}]'=-f_0e^{2\alpha\rho}\left[V-\frac{1}{2}\frac{\partial V}{\partial\phi}\right]\ ,$$
$$[(c_1e^{\alpha\rho})'e^{\alpha\rho}]'=\frac{1}{2Q^2a_1}e^{2\alpha\rho}\left[4c_1-2\sqrt{2}Q^3f_0\left[V-\frac{1}{2}\frac{\partial V}{\partial\phi}\right]\right]\ , \tag{4.8}$$

and

$$2Q^2\alpha^2 a_1 = 1 - Q^2\left[e^{-2\phi}\frac{\partial V}{\partial \phi}\right]_{\phi=-(1/2)(\alpha\rho+\ln f_0)} . \quad (4.9)$$

These equations can be solved for one's chosen potential. For example, for $V_1 = 2m^2(\phi-\phi_0)^2$, $a_1 = e^{\phi_0}/2Q^2\alpha^2$, as in the zero-mass case, but the corrections take the form

$$a_2 = -\frac{f_0 m^2}{16\alpha^2}[2\rho^2\alpha^2 - 2\rho\alpha(1-2\ln f_0) + 1 - 2\ln f_0 + 2\ln^2 f_0] ,$$
$$c_1 = -\frac{m^2 Q^3 e^{-3\phi_0} f_0}{162\sqrt{2}}[27\alpha^3\rho^3 + 27\alpha^2\rho^2(2+\ln f_0) - (3\alpha\rho - 1)(4 - 16\ln f_0 - 9\ln^2 f_0)] + K/3\alpha , \quad (4.10)$$

where K is an integration constant.

For V_2 the result is much simpler: $a_1 = (e^{2\phi_0} - Q^2 m^2)/2Q^2\alpha^2$, $a_2 = m^2 f_0/2\alpha^2$, and

$$c_1 = -f_0 Q e^{-\phi_0}/\sqrt{2} + K e^{\kappa\rho} , \quad (4.11)$$

where κ is a root of

$$k^2 + 3\alpha k - \frac{2\alpha^2 m^2 Q^2}{e^{2\phi_0} - m^2 Q^2} = 0 .$$

The main thing to note about this solution is that if $Qm > e^{\phi_0}$, then $a_1 < 0$. In other words, down the throat of the black hole, space and time would actually reverse roles. Clearly by continuity, this can only happen if there is an event horizon ($\hat{A}^2 < 0$ near the singularity implies $\hat{A}^2$ has a zero *before* the singularity), which clearly means that this is not a GHS extremal solution, where the singularity is on the verge of becoming naked. However, it can be an extremal solution in the sense of a transition between two and one horizons, where the singularity and the inner horizon merge in the interior of the black hole. In this case, therefore, the "throat" of this inner extremal solution is an inverted LDV—space and time have swapped roles. A possible Penrose-Carter diagram for such a solution is shown in Fig. 5.

It is interesting that these more massive dilatons cannot exhibit a GHS extremal solution, which can only occur for $Qm \le e^{\phi_0}$, the case of equality being the combined Reissner-Nordström–massless-dilaton extremal type already discussed above. Thus the type of extremal solutions for this potential can be quite neatly cataloged: For $Qm < e^{\phi_0}$ there can only be a massless-dilaton-type extremal solution, the singularity and horizon merge. For $Qm > e^{\phi_0}$ there can only be a Reissner-Nordström-type extremal solution where the singularity is about to become naked, with also the possibility of an inverted GHS extremal solution in the interior. For $Qm = e^{\phi_0}$ there is a situation analogous to the triple horizon of V_1, namely that the singularity merges with a repeated horizon.

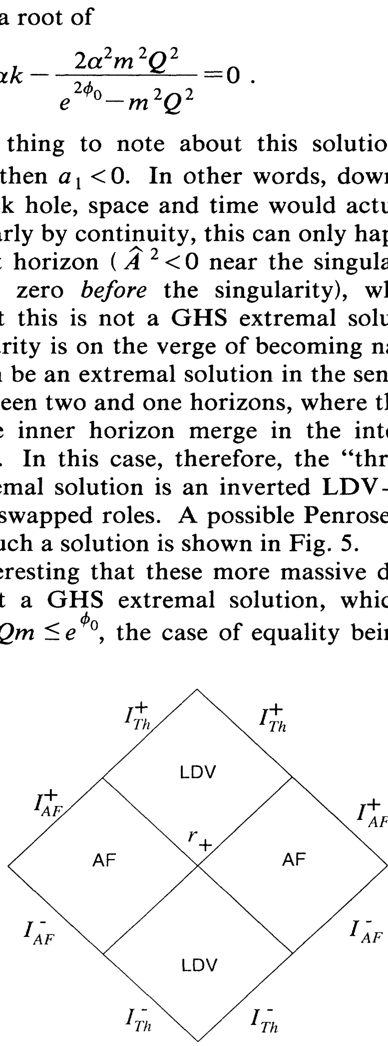

FIG. 5. Penrose-Carter diagram for an extremal massive dilaton black hole where one starts from an asymptotically flat region (AF) and approaches the linear dilaton vacuum (LDV) *after* passing through an event horizon.

Of course we also expect some relation between Q, M, and m to be satisfied in order to obtain these various types of extremal solution; however, unlike the massless case, we do not have an analytic closed-form solution from which we can extract these formulas. However, it is possible to obtain approximate information as to the extremal mass-charge relationships.

For $mQ \gg e^{\phi_0}$, searching for a Reissner-Nordström repeated horizon $[(A^2)'' > 0]$, both (4.4) and (4.5) give the same result, namely

$$(\phi - \phi_0) = e^{2\phi_0}/m^2Q^2 . \quad (4.12)$$

Using (3.4) and (3.2), this implies

$$C^2 = Q^2 e^{-2\phi_0}\left[1 - \frac{1}{m^2Q^2}\right] ; \quad (4.13)$$

therefore

$$[(A^2)'C^2]_{r_H}^{\infty} = 2M = \int_{r_H}^{\infty}\left[\frac{2Q^2e^{-2\phi}}{C^2} - C^2 V\right] < \frac{2Q^2 e^{-2\phi_0}}{C_H} . \quad (4.14)$$

Knowing that $mM \gg e^{\phi_0}$, we can use (3.9) to find the exact form of the corrections which turns out to be

$$M = Qe^{-\phi_0}\left[1 - \frac{1}{10m^2Q^2e^{-2\phi_0}}\right] . \quad (4.15)$$

Thus, perhaps not surprisingly, the effect of the dilaton is to increase the charge carried by an extremal solution. As mQ decreases, this discrepancy obviously increases, so we will estimate it at presumably what is its maximum—the triple extremal solution.

For V_1, this occurs at $(\phi-\phi_0) = 1$, $C^2 = Q^2 e^{-\phi_0}/em = 1/2m^2$, $Qm = e^{1+\phi_0}/2$. We then use

$$[(A^2)'C^2]' < \frac{2Q^2 e^{-2\phi}}{C^2} < \frac{2Q^2 C' e^{-2\phi_0}}{C^2} \tag{4.16}$$

integrated between the horizon and infinity to conclude

$$M < \frac{e}{\sqrt{2}} Q e^{-\phi_0} = \frac{e^2}{2\sqrt{2}m} . \tag{4.17}$$

Obviously, this is not a concrete result; nonetheless, it will give the correct order of magnitude for M. It is interesting to note that while this looks similar to the massless black hole charge-mass ratio when expressed in terms of Q, when expressed in terms of the dilaton mass the inequality is more eloquent. It shows that unless the dilaton mass is very small, only mini black holes are capable of attaining this special extremal limit. For example, if the dilaton acquires a mass of around 1 TeV, then the extremal black hole would have to be *no heavier* than about 10^{10} g while if the dilaton mass is 10^{18} GeV then the black hole mass would have to be less than the Planck mass. Since the dilaton mass could lie anywhere in this range, such solutions would be relevant only for primordial black holes.

For GHS-type extremal solutions, by integrating the final two equations of (3.4), we find

$$M = \frac{2}{3} \int_{r_{sg}}^{\infty} \frac{Q^2 e^{-2\phi}}{C^2} . \tag{4.18}$$

Estimating this integral as $\lim_{r \to r_{sg}} Q^2 e^{-\phi_0}/C$ gives, for V_1, $M \sim \sqrt{2}Q/3$. It must be stressed that this is only an estimate; therefore we should not compare its numerical value to that of the massless extremal limit, or indeed the Reissner-Nordström limit; however, it does show that the extremal mass-charge relationship is in the same ballpark as these other two cases. Without a numerical solution and integration, however, we cannot be more specific. For V_2 this estimate gives $M \sim \sqrt{2}Qe^{\phi_0}/3$, again, roughly the same order of magnitude.

V. TWO DIMENSIONS

Addition of a dilaton potential no longer allows exact solutions of the form $S^2 \times \mathcal{M}^2_{\rm BH}$ with constant radius S^2 and a two-dimensional black hole spacetime $\mathcal{M}^2_{\rm BH}$. Nonetheless it is of some interest to study black holes in two-dimensional massive dilaton gravity even if there is no longer a direct connection with four dimensions. Solutions to two-dimensional dilaton gravity with a dilaton potential have also been studied in [10].

Motivated by the four-dimensional Lagrangian (3.17) we take as our starting action

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g}\, e^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2 - e^{-2\phi}V(\phi)] . \tag{5.1}$$

Note that in this section we change our conventions to comply with those commonly used in two-dimensional dilaton gravity [metric signature $(-,+)$ etc.]. The equations of motion following from (5.1) are

$$2e^{-2\phi}\{\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}[(\nabla\phi)^2 - \nabla^2\phi - \lambda^2 + \tfrac{1}{4}e^{-2\phi}V(\phi)]\} = 0 , \tag{5.2}$$

$$e^{-2\phi}\left[R + 4\lambda^2 + 4\nabla^2\phi - 4(\nabla\phi)^2 + e^{-2\phi}\left[\frac{1}{2}\frac{\partial V}{\partial\phi} - 2V(\phi)\right]\right] = 0 , \tag{5.3}$$

with (5.2) resulting from variation of the metric and (5.3) from variation of the dilaton.

Looking for static solutions in a "Schwarzschild" gauge with

$$ds^2 = -A^2(\sigma)d\tau^2 + A^{-2}(\sigma)d\sigma^2 \tag{5.4}$$

the dilaton equation becomes

$$-(A^2)'' + 4\lambda^2 + 4(A^2\phi')' - 4A^2(\phi')^2 + e^{-2\phi}\left[\frac{1}{2}\frac{\partial V}{\partial\phi} - 2V\right] = 0 \tag{5.5}$$

while the metric equation and constraints may be written as

$$\begin{aligned} &\phi'' = 0 , \\ &A^2(\phi')^2 - AA'\phi' - \lambda^2 + \tfrac{1}{4}e^{-2\phi}V(\phi) = 0 . \end{aligned} \tag{5.6}$$

Now $\phi''=0$ implies $\phi = p_0 + p_1\sigma$ and if $p_1 \neq 0$ we can choose $p_0 = 0$ by shifting σ. There are thus two cases to consider: $\phi = p_1\sigma$ or $\phi = p_0$.

We first consider solutions with ϕ constant. As is clear from the second equation of (5.6), there are no such solutions when $V=0$. With ϕ constant and choosing the potential to be $V(\phi) = m^2\phi^2$ the equations reduce to

$$e^{-2p_0}p_0^2 = 4\lambda^2/m^2 \tag{5.7}$$

and an equation that says that the curvature is constant:

$$R \equiv -(A^2)'' = 4\lambda^2(1 - 1/p_0) . \tag{5.8}$$

The function $p_0^2 e^{-2p_0}$ has a minimum at $p_0 = 0$, a maximum at $p_0 = 1$ where it equals e^{-2}, and approaches $+\infty$ as $p_0 \to -\infty$ and 0 as $p_0 \to +\infty$. Thus (5.7) has one solution if $4\lambda^2/m^2 > e^{-2}$ with $R > 0$, two solutions if $4\lambda^2/m^2 = e^{-2}$ with $R > 0$ at the solution with $p_0 < 0$ and $R = 0$ at the solution with $p_0 = 1$, and three solutions if $4\lambda^2/m^2 < e^{-2}$ with $R > 0$ for $p_0 < 0$, $R < 0$ for $0 < p_0 < 1$, and $R > 0$ for $p_0 > 1$. These solutions are the two-dimensional analogue of the throat solutions (4.6) discussed in the previous section.

We next look for "linear dilaton" solutions with $\phi = p_1\sigma$. Equations (5.5) and (5.6) then reduce to

$$\begin{aligned} &-(A^2)'' + 4\lambda^2 + 4p_1(A^2)' - 4A^2p_1^2 + e^{-2\phi}\left[\frac{1}{2}\frac{\partial V}{\partial\phi} - 2V\right]\bigg|_{\phi=p_1\sigma} = 0 , \\ &4A^2p_1^2 - 2(A^2)'p_1 - 4\lambda^2 + e^{-2\phi}V(\phi)|_{\phi=p_1\sigma} = 0 . \end{aligned} \tag{5.9}$$

When $V=0$ these equations have a two-dimensional black hole solution [11] given by

$$\phi=-\lambda\sigma\ ,\qquad A^2=1-2Me^{-2\lambda\sigma}\ , \tag{5.10}$$

with M the (arbitrary) mass of the black hole.

With $V\neq 0$ adding the two equations in (5.9) gives

$$(A^2)''-2p_1(A^2)'=e^{-2\phi}\left[\frac{1}{2}\frac{\partial V}{\partial\phi}-V\right]\Bigg|_{\phi=p_1\sigma}=m^2e^{-2p_1\sigma}(p_1\sigma-p_1^2\sigma^2) \tag{5.11}$$

for the potential $V(\phi)=m^2\phi^2$. This is easily integrated to give [using (5.9) as well]

$$\phi=\mp\lambda\sigma\ ,\qquad A^2=1-2Me^{\mp 2\lambda\sigma}-\frac{m^2}{64\lambda^2}e^{\pm 2\lambda\sigma}(8\lambda^2\sigma^2\mp 4\lambda\sigma+1)\ , \tag{5.12}$$

with M arbitrary. If we want to obtain a solution which is asymptotically flat at one end of our one-dimensional world we must take $M=0$. With the usual convention that the singularity occurs at $\sigma\to-\infty$ we then have as a solution (5.12) with the lower choice of sign and $M=0$. In contrast with the usual two-dimensional black hole of (5.10) which has a singularity at strong string coupling $g_s\equiv e^{\phi}=e^{-\lambda\sigma}\to+\infty$, this solution has a singularity at weak string coupling with $g_s=e^{\lambda\sigma}\to 0$, with the potential playing a crucial role.

The causal structure of this solution depends as before on the ratio λ^2/m^2. In particular, the function $f(x)=e^{-2x}(8x^2+4x+1)$ appearing in (5.12) has a maximum at $x=(1+\sqrt{3})/4$ where it takes the value $f_M=e^{-(1+\sqrt{3})/2}(4+2\sqrt{3})$ and the solution has one, two, or three horizons depending on whether the ratio $64\lambda^2/m^2$ is larger than f_M, equal to f_M, or less than f_M, respectively.

Of course it is not clear that this choice of potential plays any particular role in two dimensions, and one might argue that it is not physically sensible to add a term which dominates at weak coupling. Another choice of potential of some interest is $V(\phi)=e^{2\phi}m^2\phi^2$. Repeating the previous analyses with this potential we find two types of de Sitter (constant curvature) solutions. If $\phi=p_0$ is constant we now find a solution with

$$\phi^2=p_0^2=4\lambda^2/m^2,\quad R=-(A^2)''=-m^2p_0\ , \tag{5.13}$$

and if $\phi=p_1\sigma$ we find a general solution

$$A^2=1-\frac{m^2}{4p_1}\sigma-\frac{m^2}{4}\sigma^2-2Me^{2p_1\sigma} \tag{5.14}$$

with p_1 given by

$$p_1^2=\lambda^2\left[1-\frac{m^2}{8\lambda^2}\right]. \tag{5.15}$$

This represents a two-dimensional black hole with a singularity at $\sigma\to-\infty$ (for $p_1<0$) and which asymptotically approaches de Sitter space with constant curvature $R=m^2/2$ as $\sigma\to+\infty$.

VI. CONCLUSION

We have seen that an addition of a dilaton potential allows for a richer variety of charged black hole solutions than is present either in the case of Einstein gravity or massless dilaton gravity. Depending on the values of the black hole mass and charge and the dilaton mass and potential it is possible to have solutions with either one, two, or three horizons, the single horizon having a Schwarzschild structure, the double a Reissner-Nordström causal structure, and the triple horizon the causal lattice of Fig. 3. We were able to establish that for our second test potential,

$$V_2(\phi)=2m^2\sinh^2(\phi-\phi_0)^2\ ,$$

only one or two horizons were possible. Our first test potential, however, $V_1=2m^2(\phi-\phi_0)^2$, could possibly have three horizons, provided mQ, the product of the black hole charge and the dilaton mass, was sufficiently large.

We examined the various types of extremal solutions, which again were more varied than either Einstein or massless dilaton gravity. We found that there could be Reissner-Nordström-type extrema, with two horizons coinciding, and also GHS type extrema, with the singularity and event horizon coinciding. In this case the causal structure in the string metric would contain an infinite throat as in the massless dilaton case. However, whereas for V_1, the singularity always coincided with an $(A^2)'>0$ horizon, for V_2, the singularity could only be of GHS type if $Qm<e^{\phi_0}$. For $Qm>e^{\phi_0}$, the event horizon and singularity meet in the interior of the black hole and the throat has an inverted LDV structure leading to the Penrose-Carter diagram of Fig. 5. Unfortunately, this type of extremal solution is not very useful as a "cornucopion" since one is always doomed to traveling down the throat, never to return to the outside world to pass on all the information one has found; in any case, it does not even qualify as a remnant—the Hawking temperature $(A^2)'_{r_+}/4\pi$ is most definitely not zero. It might be possible for these solutions to radiate until the outer event horizon merges with the inner horizon/singularity, provided sufficient charge is lost; however, it is also possible that they would turn into a Reissner-Nordström extremal solution. This would depend on how preferential it was for the black hole to discharge.

In addition to the above extremal solutions, there are also special triple extremal solutions, where g_{00} has a stationary point of inflection. These correspond to the three horizons meeting for the potential V_1, and the two horizons and singularity meeting for V_2. For V_1 this solution has an absolute upper bound on its mass, independent of the charge on the black hole.

Finally, we should remark that, just as in the case of massless dilaton gravity, for every magnetic black hole solution, there is a corresponding electric black hole solution, given by the special duality transformation (2.10).

In conclusion, while the addition of a potential destroys the simplicity of the solution, which cannot apparently be written in closed form, it greatly increases the wealth of the possible spacetimes. It seems that massive dilatons allow for many more black hole causal structures than either their massless cousins of Einstein gravity. It would be interesting to further investigate these extremal solutions since we expect some of them to share the stability of both extremal Reissner-Nordström and massless dilaton black holes, while the presence of a dilaton potential would seem to forbid embedding them in a theory with unbroken supersymmetry.

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