PHYSICAL REVIEW D VOLUME 47, NUMBER 6 15 MARCH 1993

van Vleck determinants: Geodesic focusing in Lorentzian spacetimes

Matt Visser*
Physics Department, Washington University, St. Louis, Missouri 63130-4899

(Received 13 November 1992)

The van Vleck determinant is a ubiquitous object, arising in many physically interesting situations such as (1) WKB approximations to quantum time evolution operators and Green functions, (2) adiabatic approximations to heat kernels, (3) one-loop approximations to functional integrals, (4) the theory of caustics in geometrical optics and ultrasonics, and (5) the focusing and defocusing of geodesic flows in Riemannian manifolds. While all of these topics are interrelated, the present paper is particularly concerned with the last case and presents extensive theoretical developments that aid in the computation of the van Vleck determinant associated with geodesic flows in Lorentzian spacetimes. *A fortiori* these developments have important implications for the entire array of topics indicated.

PACS number(s): 04.20.Cv, 04.60.+n.

I. INTRODUCTION

The van Vleck determinant is a truly ubiquitous object in mathematical physics. Introduced by van Vleck in 1928 [1], it was first utilized in elucidating the nature of the classical limit of quantum mechanics via WKB techniques. Further developments were due to Morette [2]. A nice discussion of this original line of development can be found in Pauli's lecture notes [3].

In a different vein, DeWitt [4–7] and others have developed extensive and powerful adiabatic expansion methods for approximating quantum Green functions in Lorentzian and Riemannian manifolds. Textbook discussions may be found in Birrell and Davies [8] and Fulling [9].

By extension, these adiabatic techniques lead to powerful point-splitting techniques. These techniques are useful, for example, for estimating the vacuum expectation value of the renormalized stress-energy tensor of a quantum field propagating on a Lorentzian spacetime. As will soon be apparent, in this particular context the van Vleck determinant is essentially a measure of the tidal focusing and/or defocusing of geodesic flows in spacetime.

This paper will develop several formal techniques for getting a handle on the van Vleck determinant. The evolution of the van Vleck determinant as one moves along a geodesic will be explicitly evaluated in terms of the expansion of the geodesic flow. The relationship with the (spacelike, lightlike, or timelike) versions of the Raychaudhuri equation is thus made manifest. Constraints, such as the weak energy condition (WEC), are then used to develop powerful inequalities limiting the behavior of the van Vleck determinant. The evolution of the van Vleck determinant is then reformulated in terms of tidal effects—the evolution of a set of Jacobi fields under the influence of the full Riemann tensor. A formal solution to the resulting integral equation is presented. Using these formal techniques, a weak-field approximation is developed. This weak-field approximation will be seen to depend only on the Ricci tensor of spacetime. In a similar vein, a short distance approximation is developed. Finally, an asymptotic estimate is provided of the behavior of the determinant as one moves out to infinity.

The physical problem that originally stimulated my interest in these investigations was Hawking's chronology protection conjecture [10, 11]. That conjecture led to several different computations of the vacuum expectation value of the renormalized stress-energy tensor for quantum fields propagating in spacetimes on the verge of violating chronology protection [12–15]. The van Vleck determinant is an overall prefactor occurring in all those computations. Another paper, currently in preparation, will discuss the application of these techniques to wormhole spacetimes.

It is nevertheless true that much of the mathematical machinery developed in this paper has a considerably wider arena of applicability. Accordingly, some effort will be made to keep the discussion as general as reasonably possible.

Notation. Adopt units where $c \equiv 1$, but all other quantities retain their usual dimensionalities. In particular $G = \hbar/m_P^2 = \ell_P^2/\hbar$. The metric signature is taken to be $(-,+,\cdots,+)$. General conventions follow Misner, Thorne, and Wheeler [16].

II. THE VAN VLECK DETERMINANT

A. General definition

Consider an arbitrary mechanical system, of n degrees of freedom, governed by a Lagrangian $\mathcal{L}(\dot{q},q)$. Solve the equations of motion to find the (possibly not unique) path γ passing through the points (q_i,t_i) and (q_f,t_f). Then calculate the action of that path

$$S_\gamma(q_i,t_i;q_f,t_f) \equiv \int_\gamma \mathcal{L}(\dot{q},q)dt. \tag{1}$$

The van Vleck determinant is then defined as

*Electronic address: visser@kiwi.wustl.edu

$$\Delta_\gamma(q_i,t_i;q_f,t_f) \equiv (-1)^n \det\left\{\frac{\partial^2 S_\gamma(q_i,t_i;q_f,t_f)}{\partial q_i\,\partial q_f}\right\}. \quad (2)$$

The van Vleck determinant, in this original incarnation, occurs as a prefactor in the WKB approximation to the quantum time evolution operator [1–3]. In the Schrödinger picture

$$\langle q_f|\exp\{-iH(t_f-t_i)/\hbar\}|q_i\rangle \approx$$

$$(2\pi i\hbar)^{-n/2}\sum_\gamma \sqrt{\Delta_\gamma(q_i,t_i;q_f,t_f)} \times \exp\{+iS_\gamma(q_i,t_i;q_f,t_f)/\hbar\}. \quad (3)$$

In the particular case of a geodesic flow on a Lorentzian or Riemannian manifold the appropriate action to be inserted in this definition is the geodetic interval between a pair of points.

B. Geodetic interval

Consider a Lorentzian spacetime of total dimensionality $d+1$. That is, d space dimensions, 1 time dimension. With appropriate changes, these results apply also to a Riemannian manifold of total dimensionality $d+1$. To go from the Lorentzian to the Riemannian case the only significant change is that one should forget all the subtleties associated with lightlike (null) geodesics.

The geodetic interval may be defined by

$$\sigma_\gamma(x,y) \equiv \pm\frac{1}{2}[s_\gamma(x,y)]^2 \quad (4)$$

Here we take the upper (+) sign if the geodesic γ from the point x to the point y is spacelike. We take the lower (−) sign if this geodesic γ is timelike. In either case we define the geodesic distance $s_\gamma(x,y)$ to be positive semidefinite.

Note that, provided the geodesic from x to y is not lightlike,

$$\nabla^x_\mu\sigma_\gamma(x,y) = \pm s_\gamma(x,y)\;\nabla^x_\mu s_\gamma(x,y), \quad (5)$$
$$= +s_\gamma(x,y)\;t_\mu(x;\gamma;x\leftarrow y), \quad (6)$$
$$= +\sqrt{2|\sigma_\gamma(x,y)|}\;\;t_\mu(x\leftarrow y). \quad (7)$$

Here $t^\mu(x;\gamma;x\leftarrow y) \equiv \pm g^{\mu\nu}\nabla^x_\nu s_\gamma(x,y)$ denotes the unit tangent vector at the point x pointing along the geodesic γ away from the point y. When no confusion results we may abbreviate this by $t^\mu(x\leftarrow y)$ or even $t^\mu(x)$.

If the geodesic from x to y is lightlike things are somewhat messier. One easily sees that for lightlike geodesics $\nabla^x_\mu\sigma_\gamma(x,y)$ is a null vector. To proceed further one must introduce a canonical observer, described by a unit timelike vector V^μ at the point x. By parallel transporting this canonical observer along the geodesic one can set up a canonical frame that picks out a particular canonical affine parameter:

$$\nabla^x_\mu\sigma_\gamma(x,y) = +\zeta_\gamma(x,y)\;l_\mu(x;\gamma;x\leftarrow y), \quad (8)$$
$$l_\mu(x;\gamma;x\leftarrow y)\;V^\mu = -1, \quad (9)$$
$$\zeta_\gamma(x,y) = -V^\mu\nabla^x_\mu\;\sigma_\gamma(x,y). \quad (10)$$

Note that this affine parameter ζ can, crudely, be thought of as a distance along the null geodesic as measured by an observer with four-velocity V^μ. By combining the null vector l^μ with the timelike vector V^μ, one can construct a second canonical null vector: $m^\mu \equiv 2V^\mu - l^\mu$. A quick calculation shows

$$\nabla^x_\mu\zeta_\gamma(x,y) = -\frac{1}{2}\;m_\mu(x;\gamma;x\leftarrow y), \quad (11)$$
$$m_\mu(x)\;V^\mu = -1; \quad (12)$$
$$l_\mu(x)\;m_\mu(x) = -2. \quad (13)$$

Unfortunately, while spacelike and timelike geodesics can be treated in a unified formalism as alluded to above, the subtleties involved with lightlike geodesics will require the presentation of several tedious variations on the general analysis.

C. Specific definition: Geodesic flow

Consider geodesic flow in a Lorentzian spacetime. In the present context the (scalarized) van Vleck determinant is defined by

$$\Delta_\gamma(x,y) \equiv (-1)^d\;\frac{\det\{\nabla^x_\mu\nabla^y_\nu\sigma_\gamma(x,y)\}}{\sqrt{g(x)g(y)}}. \quad (14)$$

This definition has a nice interpretation in terms of the Jacobian associated with the change of variables from (x,t) to (x,y). One may specify a geodesic either by (a) specifying a single point x on the geodesic and the tangent vector t at that point, or by (b) specifying two separate points (x,y) on the geodesic. The (scalarized) Jacobian associated with this change of variables is

$$J_\gamma(x,y) \equiv \frac{\det\{\partial(x,t)/\partial(x,y)\}}{\sqrt{g(x)g(y)}} = \frac{\det'\{\partial t/\partial y\}}{\sqrt{g(x)g(y)}}. \quad (15)$$

Here the $\det'$ indicates the fact that we should ignore the known trivial zero eigenvalue(s) in determining this Jacobian.

If the points x and y are spacelike or timelike separated then the one trivial zero arises from the fact that the tangent vector is normalized, $t^\mu t_\mu = \pm 1$, so that $t^\mu(x)\nabla^y_\nu t_\mu(x) = 0$. To see the connection with the van Vleck determinant, observe

$$\nabla^x_\mu\nabla^y_\nu\sigma_\gamma(x,y) = \nabla^y_\nu\left[s_\gamma(x,y)\;t_\mu(x;\gamma;x\leftarrow y)\right], = s_\gamma(x,y)\;\nabla^y_\nu t_\mu(x;\gamma;x\leftarrow y) \pm t_\nu(y)\;t_\mu(x). \quad (16)$$

By adopting suitable coordinates at x, and independently at y, one can make the tangent vectors $t_\mu(x)$ and $t_\mu(y)$ both lie in the t direction (if they are timelike) or both lie in the z direction (if they are spacelike). Then, by inspection,

$$\Delta_\gamma(x,y) = \pm[-s_\gamma(x,y)]^d\;\frac{\det'\{\nabla^y_\nu t_\mu(x;\gamma;x\leftarrow y)\}}{\sqrt{g(x)g(y)}} = \pm[-s_\gamma(x,y)]^d\;J_\gamma(x,y). \quad (17)$$

If the points x and y are lightlike separated then the two trivial zeros arise from the fact that the tangent vector l^μ satisfies two constraints: $l^\mu l_\mu = 0$, and $l^\mu V_\mu = -1$. Thus both $l^\mu(x)\nabla^y_\nu t_\mu(x) = 0$, and $V^\mu(x)\nabla^y_\nu t_\mu(x) = 0$. To precisely determine the connection with the van Vleck determinant requires some subtlety. For a point x close to, but not quite on, a null geodesic emanating from y one may usefully decompose the gradient of the geodetic interval as

$$\nabla^x_\mu \sigma_\gamma(x,y) = +\zeta_\gamma(x,y)\; l_\mu(x;\gamma;x\leftarrow y) \\ +\xi_\gamma(x,y)\; m_\mu(x;\gamma;x\leftarrow y). \tag{18}$$

Here ζ and ξ are to be thought of as curvilinear null coordinates. They are the curved space generalizations of $(t \pm x)/2$. One is now in a position to calculate

$$\nabla^x_\mu \nabla^y_\nu \sigma_\gamma(x,y) = \nabla^y_\nu\left[\zeta(x,y)\; l_\mu(x\leftarrow y) + \xi(x,y)\; m_\mu(x\leftarrow y)\right], \\ = \zeta(x,y)\; \nabla^y_\nu l_\mu(x;\gamma;x\leftarrow y) + \xi(x,y)\; \nabla^y_\nu m_\mu(x;\gamma;x\leftarrow y) - \frac{1}{2}m_\nu(y)\; l_\mu(x) - \frac{1}{2}l_\nu(y)\; m_\mu(x). \tag{19}$$

Now go to the light cone, by setting $\xi = 0$. By adopting suitable coordinates at x one can arrange: $l_\mu(x) = (1,1,0,0)$ and $m_\mu(x) = (1,-1,0,0)$. Independently, one can arrange the same to be true at y. Finally, careful inspection of the above reveals

$$\Delta_\gamma(x,y) = (-1)^d[\zeta_\gamma(x,y)]^{(d-1)}\; \frac{\det'\{\nabla^y_\nu l_\mu(x;\gamma;x\leftarrow y)\}}{\sqrt{g(x)g(y)}} \\ = (-1)^d[\zeta_\gamma(x,y)]^{(d-1)}\; J_\gamma(x,y). \tag{20}$$

D. Elementary results

In view of the equation

$$\nabla^x_\mu \sigma_\gamma(x,y) = s_\gamma(x,y)\; t_\mu(x;\gamma;x\leftarrow y), \tag{21}$$

one has as an exact result (valid also on the light cone)

$$\sigma_\gamma(x,y) = \frac{1}{2}g^{\mu\nu}\; \nabla^x_\mu \sigma_\gamma(x,y)\; \nabla^x_\nu \sigma_\gamma(x,y). \tag{22}$$

Repeated differentiations and contractions result in [4–7]

$$\nabla^\mu_x\left[\Delta_\gamma(x,y)\; \nabla^x_\mu \sigma_\gamma(x,y)\right] = (d+1)\Delta_\gamma(x,y). \tag{23}$$

Assuming that x and y are either timelike separated or spacelike separated, this may be rewritten as

$$\nabla^\mu_x\left[\Delta_\gamma(x,y)\; s_\gamma(x,y)\; t_\mu(x)\right] = (d+1)\Delta_\gamma(x,y). \tag{24}$$

Use of the Leibnitz rule leads to

$$s_\gamma(x,y)\; t_\mu(x)\; \nabla^\mu_x \Delta_\gamma(x,y) \pm t_\mu(x)\; t^\mu(x)\; \Delta_\gamma(x,y) \\ +s_\gamma(x,y)\; \Delta_\gamma(x,y)\; \nabla^\mu_x t_\mu(x) \\ = (d+1)\Delta_\gamma(x,y). \tag{25}$$

One notes that $t^\mu(x)$ defines a normalized spacelike or timelike vector field. Thus

$$t_\mu(x)\; \nabla^\mu_x \Delta_\gamma(x,y) = \left(\frac{d}{s_\gamma(x,y)} - \left[\nabla^\mu_x t_\mu(x)\right]\right)\Delta_\gamma(x,y). \tag{26}$$

Now define $\Delta_\gamma(s)$ to be the van Vleck determinant calculated at a proper distance s along the geodesic γ in the direction away from y and towards x. Note the boundary condition that $\Delta_\gamma(s=0) \equiv 1$; and define

$$t^\mu(x)\nabla^x_\mu f \equiv \frac{df}{ds}. \tag{27}$$

One finally obtains a first-order differential equation governing the evolution of the van Vleck determinant:

$$\frac{d\Delta_\gamma(s)}{ds} = \left(\frac{d}{s} - \theta\right)\Delta_\gamma(s). \tag{28}$$

Here θ is the expansion of the geodesic spray defined by the integral curves of $t^\mu(x)$, that is, $\theta \equiv \nabla_\mu t^\mu$. Direct integration yields

$$\Delta_\gamma(x,y) = s_\gamma(x,y)^d \exp\left(-\int_\gamma \theta ds\right). \tag{29}$$

Here the integration is to be taken along the geodesic γ from the point y to the point x. The interpretation of this result is straightforward: take a geodesic spray of trajectories emanating from the point y. If they were propagating in flat space, then after a proper distance s the (relative) transverse density of trajectories would have fallen to s^{-d}. Since they are not propagating in flat spacetime the actual (relative) transverse density of trajectories is given by the exponential of minus the integrated expansion. The van Vleck determinant is then the ratio between the actual density of trajectories and the anticipated flat space result. Note that by explicit calculation, the van Vleck determinant is completely symmetric in x and y.

An analogous development holds if the points x and y are connected by a null geodesic. The details are, as one has by now grown to expect, somewhat tedious. One returns to Eq. (23) and uses Eq. (18) to derive

$$\nabla^\mu_x\left[\Delta_\gamma(x,y)\{\zeta\; l_\mu(x) + \xi\; m_\mu(x)\}\right] = (d+1)\Delta_\gamma(x,y). \tag{30}$$

Use of the Leibnitz rule, followed by the limit $\xi \to 0$, now leads to

$$\zeta\; l_\mu(x)\; \nabla^\mu_x \Delta_\gamma(x,y) - l_\mu(x)\; m^\mu(x)\; \Delta_\gamma(x,y) \\ +\zeta\; \Delta_\gamma(x,y)\; \nabla^\mu_x l_\mu(x) = (d+1)\Delta_\gamma(x,y). \tag{31}$$

One recalls that, by definition, $l^\mu(x) m_\mu(x) = -2$. Adopting suitable definitions, analogous to the non-null case, one derives

$$\frac{d\Delta_\gamma(\zeta)}{d\zeta} = \left(\frac{d-1}{\zeta} - \hat{\theta}\right)\Delta_\gamma(\zeta). \tag{32}$$

Direct integration yields

$$\Delta_\gamma(x,y) = \zeta_\gamma(x,y)^{d-1}\exp\left(-\int_\gamma \hat{\theta}\, d\zeta\right). \tag{33}$$

In particular, note that while the affine parameter ζ, and the expansion $\hat{\theta} \equiv \nabla_\mu l^\mu$ both depend on the canonical observer V^μ, the overall combination is independent of this choice. The appearance of the exponent $d-1$ for null geodesics, as contrasted with the exponent d for spacelike or timelike geodesics might at first be somewhat surprising. The ultimate reason for this is that in $(d+1)$-dimensional spacetime the set of null geodesics emanating from a point x sweeps out a d-dimensional submanifold. As one moves away from x an affine distance ζ into this submanifold the relative density of null geodesics falls as ζ^{1-d}. In $(3+1)$-dimensional spacetime, this is just the inverse square law for luminosity.

Equation (32) above should be compared with Eq. (22) of Kim and Thorne [12]. The typographical error in that equation (the 3 should be a 2) fortunately does not propagate into the rest of their paper. The virtue of this tedious but elementary analysis is that one now has a unified framework applicable in all generality to cases of timelike, lightlike, or spacelike separated points x and y.

E. Inequalities

Now that one has these general formulas for the van Vleck determinant, powerful inequalities can be derived by applying the Raychaudhuri equation and imposing suitable convergence conditions. Restricting attention to vorticity free (spacelike or timelike) geodesic flows the Raychaudhuri equation particularizes to

$$\frac{d\theta}{ds} = -(R_{\mu\nu}t^\mu t^\nu) - 2\sigma^2 - \frac{\theta^2}{d}. \tag{34}$$

For a concrete textbook reference, this is a special case of Eq. (4.26) of Hawking and Ellis [17], generalized to arbitrary dimensionality. While Hawking and Ellis are discussing timelike geodesics, the choice of notation in this paper guarantees that the discussion can be carried over to spacelike geodesics without alteration. The quantity σ denotes the shear of the geodesic congruence, and σ^2 is guaranteed to be positive semidefinite.

Definition. A Lorentzian spacetime is said to satisfy the (timelike, null, or spacelike) convergence condition if for all (timelike, null, or spacelike) vectors t^μ: $(R_{\mu\nu}t^\mu t^\nu) \geq 0$.

Definition. A Riemannian manifold is said to possess semipositive Ricci curvature if for all vectors t^μ: $(R_{\mu\nu}t^\mu t^\nu) \geq 0$.

If the points x and y are (timelike or spacelike) separated, and the spacetime satisfies the (timelike or spacelike) convergence condition then one has the inequality

$$\frac{d\theta}{ds} \geq -\frac{\theta^2}{d}. \tag{35}$$

This inequality is immediately integrable:

$$\frac{1}{\theta(s)} \geq \frac{1}{\theta(0)} + \frac{s}{d}. \tag{36}$$

For the geodesic spray under consideration, one has $\theta(s) \to d/s$ as $s \to 0$. Therefore $[1/\theta(0)] = 0$. One has thus derived an inequality, valid for geodesic sprays, under the assumed convergence conditions

$$\theta(s) \leq +\frac{d}{s}. \tag{37}$$

This implies that the van Vleck determinant is a monotonic function of arc length, $d\Delta/ds \geq 0$, and immediately integrates to an inequality on the determinant itself $\Delta_\gamma(x,y) \geq 1$.

For null geodesic flows the Raychaudhuri equation becomes

$$\frac{d\hat{\theta}}{d\zeta} = -(R_{\mu\nu}l^\mu l^\nu) - 2\sigma^2 - \frac{\hat{\theta}^2}{d-1}. \tag{38}$$

See Eq. (4.35) of Hawking and Ellis [17]. If the spacetime now satisfies the null convergence condition one can deduce the inequality

$$\frac{d\hat{\theta}}{d\zeta} \geq -\frac{\hat{\theta}^2}{d-1}. \tag{39}$$

Upon integration, and use of the relevant boundary condition, one derives in a straightforward manner the inequality

$$\hat{\theta}(\zeta) \leq +\frac{d-1}{\zeta} \tag{40}$$

which implies that the van Vleck determinant is a monotonic function of the null affine parameter, $d\Delta/d\zeta \geq 0$, and immediately integrates to an inequality on the determinant itself $\Delta_\gamma(x,y) \geq 1$. One is now in a position to enunciate the following results.

Theorem. In any Lorentzian spacetime, if the points x and y are (timelike, null, or spacelike) separated, and the spacetime satisfies the (timelike, null, or spacelike) convergence condition, then the van Vleck determinant is a monotonic function of affine parameter and is bounded from below: $\Delta_\gamma(x,y) \geq 1$.

Theorem. In any Riemannian manifold of semipositive Ricci curvature, the van Vleck determinant is a monotonic function of an affine parameter and is bounded from below: $\Delta_\gamma(x,y) \geq 1$.

To see the physical import of the (timelike, null, or spacelike) convergence conditions, consider a type-I stress-energy tensor. The cosmological constant, if present, is taken to be subsumed into the definition of the stress energy. Then $T_{\mu\nu} \sim \text{diag}[\rho; p_i]_{\mu\nu}$, where the p_i are the d principal pressures. From the Einstein equations, $R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$, one deduces $R_{\mu\nu} \sim 4\pi G\, \text{diag}[\rho + \sum_j p_j; \rho + 2p_i - \sum_j p_j]_{\mu\nu}$.

The timelike convergence condition implies: (1) $\rho + \sum_j p_j \geq 0$; (2) $\forall i, \rho + p_i \geq 0$. This is equivalent to the strong energy condition.

The null convergence condition implies: $\forall i, \rho + p_i \geq 0$. This is a somewhat weaker restriction than the WEC. (The WEC would require the additional constraint $\rho \geq 0$.)

The spacelike convergence condition implies: (1) $\forall i, \rho + 2p_i - \sum_j p_j \geq 0$; (2) $\forall i, \rho + p_i \geq 0$. This is a rather strong constraint that is not equivalent to any of the standard energy conditions. Particularize to 3+1 dimensions, then the constraints (1) can be made more explicit as $\rho + p_1 - p_2 - p_3 \geq 0$, $\rho + p_2 - p_3 - p_1 \geq 0$, $\rho + p_3 - p_1 - p_2 \geq 0$. By adding the first two of these constraints one may deduce $\rho - p_3 \geq 0$. Similarly $\forall i, \rho - p_i \geq 0$. Thus the spacelike convergence condition implies the dominant energy condition, which in turn implies the WEC. The implication does not hold in the reverse direction. In particular, for an electromagnetic field of zero Poynting flux $T_{\mu\nu} \sim \rho \text{ diag}[+1; -1; +1; +1]_{\mu\nu}$. This stress-energy tensor satisfies the dominant energy condition but not the spacelike convergence condition. A canonical example of a stress-energy tensor that *does* satisfy the spacelike convergence condition is a perfect fluid that satisfies the dominant energy condition: $\rho + p \geq 0$; $\rho - p \geq 0$.

Note that the timelike and spacelike convergence conditions separately imply the null convergence condition.

The physical reason behind the inequality satisfied by the van Vleck determinant is now manifest: "Ordinary" matter produces an attractive gravitational field. An attractive gravitational field focuses geodesics, so that they do not spread out as much as they would in flat space. Then the van Vleck determinant is bounded from below, and in fact continues to grow as one moves along any geodesic.

One should note, however, that several interesting classes of spacetimes violate one or more of these convergence conditions. This makes the inequalities derived here somewhat less useful than they might otherwise be, and suggests that it would be useful to explore the possibility of deriving more general inequalities based on weaker energy conditions. For example, spacetimes containing the interesting class of objects known as traversable wormholes violate all three of these convergence conditions [18, 19]. Indeed in that particular class of spacetimes the required presence of "exotic" matter leads to van Vleck determinants that are arbitrarily close to zero [12].

F. Reformulation: Tidal focusing

Direct computations using the preceding results are unfortunately rather difficult. To get a better handle on the van Vleck determinant we shall first apparently make the problem more complicated by deriving a second-order differential equation for the van Vleck determinant. By going to the second-order formalism it is possible to relate the van Vleck determinant directly to the tidal forces induced by the full Riemann tensor — more specifically to the focusing and defocusing effects induced by tidal forces.

The most direct route to this end is to pick up a standard reference such as Hawking and Ellis [17]. Equation (4.20) on page 83 implies that

$$\int \theta \, ds = \ln \det[A^\mu{}_\nu(s)]. \tag{41}$$

Here $A^\mu{}_\nu(s)$ is the $d \times d$ matrix describing the evolution of the separation of infinitesimally nearby geodesics. This formalism may also be reformulated in terms of the Jacobi fields associated with the geodesic γ. Note that the $d \times d$ matrix $A^\mu{}_\nu(s)$ may equivalently be thought of as a $(d+1) \times (d+1)$ matrix that is trivial in the extra entries: $A_{(d+1)} = A_d \oplus s$. In terms of A,

$$\Delta_\gamma(x,y) = s_\gamma(x,y)^d \, \det(A_d^{-1}) = \det\left(s_\gamma(x,y) A^{-1}\right), \tag{42}$$

where the last determinant can be taken in the sense of either a $d \times d$ or a $(d+1) \times (d+1)$ matrix.

An alternative, more explicit, but also more tedious route to establishing the preceding equation is to define the object

$$\tilde{A}(s) \equiv P \exp\left\{ \int_\gamma (\nabla \otimes \mathbf{t}\,) ds \right\}. \tag{43}$$

Here the symbol P denotes the path ordering process. By definition of path ordering,

$$\frac{d\tilde{A}}{ds} = (\nabla \otimes \mathbf{t}\,) P \exp\left\{ - \int_\gamma (\nabla \otimes \mathbf{t}\,) ds \right\}. \tag{44}$$

On the other hand, $\tilde{A}(0) \equiv I$. Comparison with Eq. (4.10) of Hawking and Ellis now shows that $\tilde{A}(s) \equiv A(s)^\mu{}_\nu \, \partial_\mu \otimes dx^\nu$. Finally note that

$$\det \tilde{A} = \exp\left\{ \int_\gamma \text{tr}(\nabla \otimes \mathbf{t}\,) ds \right\} = \exp\left\{ \int_\gamma \theta \, ds \right\}. \tag{45}$$

This reformulates the van Vleck determinant in terms of a path ordered exponential

$$\Delta_\gamma(x,y) = s_\gamma(x,y)^d \, \det\left[P \exp\left\{ - \int_\gamma (\nabla \otimes \mathbf{t}\,) ds \right\} \right]. \tag{46}$$

This observation serves to illustrate formal similarities (and differences) between the van Vleck determinant and the Wilson loop variables of gauge theories.

By taking a double derivative with respect to arc length the matrix $A(s)$ may be shown to satisfy the second-order differential equation

$$\frac{d^2}{ds^2} A^\mu{}_\nu(s) = -\left(R^\mu{}_{\alpha\sigma\beta} t^\alpha t^\beta \right) A^\sigma{}_\nu. \tag{47}$$

See Hawking and Ellis [17], Eq. (4.21) on page 83, particularized to a geodesic flow. The boundary condition on $A(s)$ is that $A^\mu{}_\nu(s) \to s\delta^\mu{}_\nu$ as $s \to 0$, this being the flat space result for a geodesic spray. This is the promised tidal formulation for the van Vleck determinant.

A particularly pleasant feature of the tidal reformulation is that the case of null geodesics can be handled without too much special case fiddling. The analogue of the tidal equation is

$$\frac{d^2}{d\zeta^2}\hat{A}^\mu{}_\nu(s) = -\left(R^\mu{}_{\alpha\sigma\beta}\, l^\alpha l^\beta\right)\hat{A}^\sigma{}_\nu. \tag{48}$$

See Hawking and Ellis [17], Eq. (4.33) on page 88. Note that two of the eigenvectors of $\hat{A}$ suffer trivial evolution. This is a consequence of the two normalization conditions on the null tangent vector. Thus $\hat{A}$ can be thought of as either a $(d+1)\times(d+1)$ matrix with two trivial entries, or as a reduced $(d-1)\times(d-1)$ matrix. The formulations are related by $\hat{A}_{(d+1)} = \hat{A}_{(d-1)} \oplus \zeta I_2$. Other results can be simply transcribed as needed.

G. Reformulation: Formal solution

To solve the above differential equation (in a formal sense), introduce the one-dimensional retarded Green function

$$G_R(s_f, s_i) = \{s_f - s_i\}\Theta(s_f - s_i). \tag{49}$$

Here $\Theta(s)$ denotes the Heaviside step function. For notational convenience define

$$Q^\mu{}_\nu(s) = -\left(R^\mu{}_{\alpha\nu\beta}t^\alpha t^\beta\right). \tag{50}$$

Integration of the second-order "tidal" equation for Δ_γ, with attention to the imposed boundary condition, leads to the integral equation

$$A^\mu{}_\nu(s) = s\delta^\mu{}_\nu + \int_0^s G_R(s,s')Q^\mu{}_\sigma(s')A^\sigma{}_\nu(s')ds'. \tag{51}$$

More formally, one may suppress the explicit integration by regarding $G_R(s,s')$, multiplication by $Q(s)$, and multiplication by s, as functional operators. Then

$$(I - G_R Q)A = \{sI\}. \tag{52}$$

This has the formal solution

$$A = (I - G_R Q)^{-1}\{sI\} = \left(I + [G_R Q] + [G_R Q]^2 + \cdots\right)\{sI\}. \tag{53}$$

This formal solution may also be derived directly from Eq. (51) by continued iteration. To understand what these symbols mean, note that

$$[G_R Q]\{sI\} \equiv \int_0^s G_R(s,s')[Q^\mu{}_\nu(s')]s'ds' = \int_0^s (s-s')[Q^\mu{}_\nu(s')]s'ds'. \tag{54}$$

Similarly

$$[G_R Q]^2\{sI\} = \int_0^s ds' \int_o^{s'} ds''(s-s') \times [Q^\mu{}_\alpha(s')](s'-s'') \times [Q^\alpha{}_\nu(s'')]s''. \tag{55}$$

The formal solution in terms of continued iteration is particularly advantageous in the case where matter is concentrated on thin shells [20–22]. In that case the matrix $[Q^\mu{}_\nu(s)]$ is described by a series of δ functions. Because of the presence of the Heaviside step function, the formal expansion terminates in a finite number of steps

$$A = \left(I + [G_R Q] + [G_R Q]^2 + \cdots + [G_R Q]^N\right)\{sI\}. \tag{56}$$

Here N denotes the total number of shell crossings. This type of calculation will be explored in considerable detail in a subsequent publication [?]. For the time being, one may observe from the above, that when the Riemann curvature is concentrated on thin shells, the matrix A is a piecewise linear continuous matrix function of arc length. Consequently the reciprocal of the van Vleck determinant $\Delta_\gamma(s)^{-1} = \det\{A/s\}$ is piecewise a Laurent polynomial in arc length.

H. Weak-field approximation

These formal manipulations permit the derivation of a very nice weak-field approximation to the van Vleck determinant. Observe

$$\begin{aligned}\Delta_\gamma(s)^{-1} &= \det(s^{-1}A),\\ &= \exp[\mathrm{tr}\ln\{s^{-1}(I - G_R Q)^{-1}s\}],\\ &= \exp[\mathrm{tr}\ln\{I + s^{-1}[G_R Q]s + s^{-1}[G_R Q]^2 s + \cdots\}],\\ &= \exp\left(\mathrm{tr}\{s^{-1}[G_R Q]s + O(Q^2)\}\right).\end{aligned} \tag{57}$$

Now, recall that the determinant and trace in these formulas are to be taken in the sense of $(d+1)\times(d+1)$ matrices, so, in terms of the Ricci tensor,

$$\mathrm{tr}[Q] = -\left(R_{\alpha\beta}t^\alpha t^\beta\right). \tag{58}$$

The $\mathrm{tr}[O(Q^2)]$ terms involve messy contractions depending quadratically on the Riemann tensor. In a weak-field approximation one may neglect these higher-order terms and write

$$\Delta_\gamma(s) = \exp\{-s^{-1}(G_R \mathrm{tr}[Q])s + O(Q^2)\}. \tag{59}$$

More explicitly

$$\Delta_\gamma(x,y) = \exp\left(\frac{1}{s}\int_0^s (s-s')(R_{\alpha\beta}t^\alpha t^\beta)s'ds' + O([\text{Riemann}]^2)\right). \tag{60}$$

This approximation, though valid only for weak fields, has a very nice physical interpretation. The implications of constraints such as the (timelike, null, or spacelike) convergence conditions can now be read off by inspection.

Another nice feature of the weak-field approximation is that the results for null geodesics can also be read off by inspection:

$$\Delta_\gamma(x,y) = \exp\left(\frac{1}{\zeta}\int_0^\zeta (\zeta-\zeta')(R_{\alpha\beta}l^\alpha l^\beta)\zeta'd\zeta' + O([\text{Riemann}]^2)\right). \tag{61}$$

Consider, for example, the effect of a thin perfect fluid in an almost flat background geometry. The Ricci tensor is related to the density and pressure via the Einstein equations, with the result that, for a null geodesic,

$$\Delta_\gamma(x,y) \approx \exp\left(8\pi G \int_0^\zeta [\rho + p] \frac{(\zeta-\zeta')\zeta'}{\zeta} d\zeta'\right). \tag{62}$$

Unsurprisingly, one sees that energy density and/or pressure located at the halfway point of the null geodesic is most effective in terms of focusing.

I. Short distance approximation

Another result readily derivable from the formal solution (53) is a short distance approximation. Assume that x and y are close to each other, so that $s_\gamma(x,y)$ is small. Assume that the Riemann tensor does not fluctuate wildly along the geodesic γ. Then one may approximate the Ricci tensor by a constant, and explicitly perform the integration over arc length, keeping only the lowest order term in $s_\gamma(x,y)$:

$$\Delta_\gamma(x,y) = 1 + \frac{1}{6}\left(R_{\alpha\beta}t^\alpha t^\beta\right) s_\gamma(x,y)^2 + O(s_\gamma(x,y)^3). \tag{63}$$

This result can also be derived via a tedious combination of index manipulations and point-splitting techniques [4], see Eq. (1.76) on page 233.

By adopting Gaussian normal coordinates at x one may write $s_\gamma(x,y)\ t_\mu(x;\gamma;x \leftarrow y) = (x-y)^\mu + O[s^2]$ to yield

$$\Delta_\gamma(x,y) = 1 + \frac{1}{6}[R_{\alpha\beta}(x-y)^\alpha(x-y)^\beta] + O(s_\gamma(x,y)^3). \tag{64}$$

In this form the result is blatantly applicable to null geodesics without further difficulty.

J. Asymptotics

To proceed beyond the weak-field approximation, consider an arbitrarily strong gravitational source in an asymptotically flat spacetime. Let the point y lie anywhere in the spacetime, possibly deep within the strongly gravitating region. Consider an otherwise arbitrary spacelike or timelike geodesic that reaches and remains in the asymptotically flat region.

Far from the source the metric is approximately flat and the Riemann tensor is of order $O[M/r^3]$. Using the tidal evolution equation, a double integration with respect to arc length provides the estimate

$$A^\mu{}_\nu(s) = (A_0)^\mu{}_\nu + s(B_0)^\mu{}_\nu + O[M/s], \tag{65}$$

this estimate being valid for large values of s, where one is in the asymptotically flat region. Here (A_0) and (B_0) are constants that effectively summarize gross features of the otherwise messy strongly interacting region. For the van Vleck determinant $\Delta_\gamma(s)^{-1} = \det\{A/s\} = \det\{B_0 + (A_0/s) + O[1/s^2]\}$. Thus the van Vleck determinant approaches a finite limit at spatial or temporal infinity, with $\Delta_\gamma(\infty) = \det\{B_0\}^{-1}$. Finally, define $J = (B_0)^{-1}(A_0)$, to obtain

$$\Delta_\gamma(s) = \frac{\Delta_\gamma(\infty)}{\det\{I + (J/s) + O[1/s^2]\}}, \tag{66}$$

which is our desired asymptotic estimate of the van Vleck determinant.

III. DISCUSSION

This paper has presented a number of formal developments with regard to evaluating the van Vleck determinant. The evolution of the van Vleck determinant as one moves along a geodesic has been studied in some detail. By utilizing the Raychaudhuri equation stringent limits have been placed on this evolution. There is some hope that it might be possible to formulate more general theorems requiring weaker convergence hypotheses than those discussed in this paper. The evolution of the van Vleck determinant has also been reformulated in terms of tidal effects. Such a presentation has several technical advantages. Among other things, the presentation manifests the influence of the full Riemann tensor. A formal solution to the resulting integral equation was presented. This formal solution in terms of an iterated integral equation has the very nice property of terminating in a finite number of steps if the curvature is confined to thin shells. Using these formal techniques, a weak-field approximation was also developed. This weak-field approximation depends only on the Ricci tensor. In a similar vein, a short distance approximation was developed. Finally, the asymptotic behavior at infinity was investigated. The mathematical machinery developed in this paper has a wide arena of applicability. Accordingly, some effort has been made to keep the discussion as general as reasonably possible.

Stimulated by Hawking's chronology protection conjecture [10, 11], a subsequent paper will present applications of this machinery to spacetimes containing traversable wormholes.

ACKNOWLEDGMENTS

I wish to thank Carl Bender for useful discussions. I also wish to thank the Aspen Center for Physics for its hospitality. This research was supported by the U.S. Department of Energy.

[1] J. H. van Vleck, Proc. Natl. Acad. Sci. U.S.A. **14**, 176 (1928).
[2] C. Morette, Phys. Rev. **81**, 848 (1951).
[3] W. Pauli, *Pauli Lectures on Physics: Volume 6, Selected Topics in Field Quantization*, edited by C. P. Enz (MIT, Cambridge, 1973) {see especially pp. 161–174}.
[4] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960) {see especially pp. 226–235}.
[5] B. S. DeWitt, in *Relativity, Groups, and Topology*, Proceedings of the Les Houches Summer School, Les Houches, France, 1963, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964) {see especially pp. 734–745}.
[6] B. S. DeWitt, Phys. Rep. **19**, 295 (1975) {see especially pp. 342–345}.
[7] B. S. DeWitt, in *Relativity, Groups, and Topology II*, Proceeding of the Les Houches Summer School, Les Houches, France, 1983, edited by B. S. DeWitt and R. Stora, Les Houches Summer School Proceedings Vol. 41 (North-Holland, Amsterdam, 1984) {see especially pp. 531–536}.
[8] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Spacetime* (Cambridge University Press, Cambridge, England, 1982).
[9] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-time* (Cambridge University Press, Cambridge, England, 1989).
[10] S. W. Hawking, in *Sixth Marcel Grossmann Meeting on General Relativity*, Proceedings, Kyoto, Japan, 1991, edited by H. Sato (World Scientific, Singapore, 1992).
[11] S. W. Hawking, Phys. Rev. D **46**, 603 (1992).
[12] S. W. Kim and K. S. Thorne, Phys. Rev. D **43**, 3929 (1991).
[13] G. Klinkhammer, Phys. Rev. D **46**, 3388 (1992).
[14] M. Visser, Phys. Rev. D **47**, 554 (1993).
[15] M. Visser, "Hawking's Chronology Protection Conjecture: Singularity Structure of the Quantum Stress-energy Tensor," Washington University Report No. WASH-U-HEP-92/75 (unpublished).
[16] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[17] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).
[18] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[19] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[20] C. Lanczos, Phys. Z. **23**, 539 (1922); Ann. Phys. (Leipzig) **74**, 518 (1924).
[21] N. Sen, Ann. Phys. (Leipzig) **73**, 365 (1924).
[22] W. Israel, Nuovo Cimento **44B**, 1 (1966); **49B**, 463 (1967).