Self-consistent improvement of the finite-temperature effective potential

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We present a self-consistent calculation of the finite-temperature effective potential for $\lambda\phi^4$ theory, using the composite operator effective potential in which an infinite series of the leading diagrams is summed up. Our calculation establishes the proper form of the leading correction to the perturbative one-loop effective potential.

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I. INTRODUCTION

Temperature-induced symmetry-changing phase transitions in quantum field theory are important ingredients in cosmological scenarios. The existence of the high-temperature phase transitions was suggested by Kirzhnits and Linde [1] and was shown quantitatively by Dolan and Jackiw [2], Weinberg [3], and Kirzhnits and Linde [4]. The approximate critical temperature of a given phase transition can be obtained by calculating the one-loop finite-temperature effective potential. However, cosmological scenarios often rely on the detailed nature of the phase transition, namely, whether it is of first or second order. A more precise determination of the critical temperature and the nature of the phase transition requires an analysis of higher-loop contributions even when the coupling constants in the theory are very small. Weinberg has argued, by using power counting, that the leading contributions at very high temperatures come from all those loops with a superficial degree of divergence $D > 1$. This implies that in order to obtain more accurate information one must study infinite series of certain classes of multiloop diagrams in perturbation theory. For example, in $\lambda\Phi^4$ scalar theory the leading contributions come from the multiloop graphs shown in Fig. 1, which are called *daisy* and *superdaisy*. For this reason, Dolan and Jackiw, in their early paper, studied the effect of these graphs on the temperature-dependent effective mass.

Recently, there has been much interest in the nature of the electroweak phase transition due to the idea that the baryon asymmetry may be generated at the electroweak scale if the transition is of first order [5]. Several authors have calculated the high-temperature effective potential in the standard model and in the $\lambda\Phi^4$ theory, taking into account leading (and subleading) contributions from multiloop diagrams, in order to obtain a correct form of the high-temperature effective potential [6–14]. Some of the authors have calculated an "improved" one-loop effective potential in which the tree-level propagators are replaced by temperature-dependent effective propagators, which were obtained by summing the dominant high-temperature contributions from infinite series of certain classes of self-energy graphs in perturbation theory. When one considers only the leading corrections to the effective propagators all results are in agreement with each other. However, there have been various disagreements when the subleading corrections to the effective propagators are included. On the other hand, the subleading contributions could be important in determining the nature of the phase transition.

We find that in the improved one-loop calculations the difficulties arise because of the fact that the naive substitution of improved propagators in the one-loop effective potential is an *ad hoc* approximation. One needs a self-consistent loop expansion of the effective potential in terms of the full propagator. Such a technique was developed some time ago by Cornwall, Jackiw, and Tomboulis (CJT) in their effective action formalism for composite operators [15]. One considers a generalization $\Gamma(\phi,G)$ of the usual effective action $\Gamma(\phi)$, which depends not only on $\phi(x)$, a possible expectation value of the quantum field $\Phi(x)$, but also on $G(x,y)$, a possible expectation value of the time-ordered product $T\Phi(x)\Phi(y)$. The physical solutions satisfy stationary requirements:

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)}=0 , \tag{1.1}$$

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)}=0 . \tag{1.2}$$

The conventional effective action $\Gamma(\phi)$ is given by $\Gamma(\phi,G)$ at the solution $G_0(\phi)$ of (1.2):

$$\Gamma(\phi)=\Gamma(\phi,G_0(\phi)) . \tag{1.3}$$

In this formalism it is possible to sum a large class of ordinary perturbation-series diagrams that contribute to the effective action $\Gamma(\phi)$, and the gap equation which determines the form of the propagator is obtained by a variational technique, as in (1.2).

For translationally invariant solutions, we set

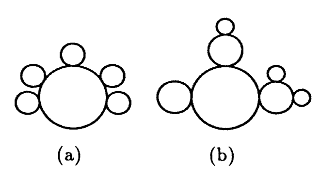

FIG. 1. Examples of (a) daisy and (b) superdaisy graphs.

ϕ=const, take $G(x,y)$ to be a function of only $(x-y)$, and obtain the effective potential for the full propagator:

$$V(\phi,G) \equiv -\Gamma(\phi,G)^{\text{trans inv}} \Big/ \int d^4x \ . \tag{1.4}$$

The purpose of this paper is to understand the structure of the leading corrections to the perturbative one-loop finite-temperature effective potential for $\lambda\Phi^4$ theory using CJT formalism in imaginary (Euclidean) time. We obtain a finite-temperature effective potential, which is exact up to the order that includes all contributions from daisy and superdaisy graphs. Instead of dropping various finite and divergent terms, as has been done often in the recent literature, we carry out renormalization and then perform a high-temperature expansion. We show explicitly that the effective potential must be calculated up to two loops in order to generate all daisy and superdaisy graphs which appear in perturbation theory [16]. Moreover, we find subtle cancellations of leading corrections between the improved one- and two-loop contributions. If it is indeed possible to determine the order of the electroweak phase transition by calculating the improved high-temperature effective potential, our result implies that the improved two-loop contribution could play a crucial role. We plan to present our calculations in gauge theories in a future publication.

II. CJT COMPOSITE OPERATOR EFFECTIVE POTENTIAL

In this section we shall review briefly the CJT formalism following Ref. [15]. To describe field theory at finite temperature T, we shall use Euclidean time τ, which is restricted to the interval $0 \le \tau \le \beta \equiv 1/T$. The Feynman rules are the same as those at zero temperature, except that the momentum-space integral over the time component k_4 is replaced by a sum over discrete frequencies $k_4 = 2\pi n/\beta$:

$$\int \frac{d^4k}{(2\pi)^4} \to \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \ , \tag{2.1}$$

where n is even (odd) for bosons (fermions).

We introduce a partition function $Z_\beta(J,K)$ in the presence of the sources J and K defined by

$$Z_\beta(J,K) \equiv \int D\Phi \exp\left[-\left\{ I(\Phi) + \int d^4x \, \Phi(x)J(x) + \tfrac{1}{2}\int d^4x \, d^4y \, \Phi(x)K(x,y)\Phi(y) \right\} \right] . \tag{2.2a}$$

(We have set $c=\hbar=1$.) The Φ integration is functional, and Φ satisfies the periodic boundary conditions $\Phi(\beta/2,\mathbf{x}) = \Phi(-\beta/2,\mathbf{x})$. $I(\Phi)$ is the classical Euclidean action, which may be written as

$$I(\Phi) = \int d^4x \, d^4y \, \Phi(x) D_0^{-1}(x-y)\Phi(y) + \int d^4x \, L_{\text{int}}(x) \ , \tag{2.2b}$$

where $D_0(x-y)$ is the free propagator,

$$D_0^{-1}(x-y) = -(\Box + m^2)\delta^4(x-y) \ , \tag{2.2c}$$

and the interaction Lagrangian L_{int} is at least cubic in Φ.

The effective action $\Gamma_\beta(\phi,G)$ is obtained by a double Legendre transformation of $W_\beta(\phi,G) \equiv \ln Z_\beta(\phi,G)$, which is analogous to the free energy. We define

$$\frac{\delta W_\beta(J,K)}{\delta J(x)} = \phi(x) \ , \qquad \frac{\delta W_\beta(J,K)}{\delta K(x,y)} = \frac{1}{2}[G(x,y) + \phi(x)\phi(y)] \ , \tag{2.3}$$

and eliminate J and K in favor of ϕ and G:

$$\Gamma_\beta(\phi,G) = W_\beta(J,K) - \int d^4x \, \phi(x)J(x) - \tfrac{1}{2}\int d^4x \, d^4y \, \phi(x)K(x,y)\phi(y) - \tfrac{1}{2}\int d^4x \, d^4y \, G(x,y)K(y,x) \ . \tag{2.4}$$

It follows that

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)} = J(x) - \int d^4y \, K(x,y)\phi(y) \ , \tag{2.5}$$

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} = -\frac{1}{2}K(x,y) \ . \tag{2.6}$$

Since the physical processes correspond to vanishing sources J and K, Eqs. (2.5) and (2.6) provide the stationary requirement of (1.1) and (1.2). $\Gamma_\beta(\phi,G)$ obtained in this way is the generating functional in ϕ for two-particle-irreducible Green's functions expressed in terms of the full propagator G.

In order to obtain a series expansion of $\Gamma_\beta(\phi,G)$, we introduce the functional operator $D^{-1}(\phi;x,y)$:

$$D^{-1}(\phi;x,y) = \frac{\delta^2 I(\phi)}{\delta\phi(x)\delta\phi(y)} \ . \tag{2.7}$$

The required series obtained by CJT is then

$$\Gamma_\beta(\phi,G) = I_{\text{cl}}(\phi) + \tfrac{1}{2}\text{Tr}\ln D_0 G^{-1} + \tfrac{1}{2}\text{Tr}[D^{-1}G - 1] + \Gamma_\beta^{(2)}(\phi,G) \ , \tag{2.8}$$

where the ϕ-independent terms are chosen so that the overall normalization is consistent with the conventional effective action $\Gamma_\beta(\phi)$ according to (1.3). The quantity $\Gamma_\beta^{(2)}(\phi)$ is computed as follows. In the classical action $I(\Phi)$, shift the field Φ by ϕ. Then $I(\Phi+\phi)$ contains terms cubic and higher in Φ that define $I_{\text{int}}(\phi;\Phi)$ where the vertices depend on ϕ. $\Gamma_\beta^{(2)}(\phi)$ is given by all the two-particle-irreducible (2PI) vacuum graphs in the theory with vertices given by $I_{\text{int}}(\phi;\Phi)$ and propagators set equal to $G(x,y)$.

From the stationary requirement (1.2), we obtain the gap equation for G:

$$G^{-1}(x,y)=D^{-1}(x,y)-2\frac{\delta\Gamma_\beta^{(2)}(\phi,G)}{\delta G(x,y)} . \tag{2.9}$$

When one is interested in translationally invariant solutions, the generalized effective potential $V_\beta(\phi,G)$ can be obtained using (1.4) and (2.8).

III. $\lambda\Phi^4$ THEORY

A. Effective potential $V_\beta(\phi,G)$

In this section we calculate the finite-temperature effective potential for the single scalar field with $\lambda\Phi^4$ interaction. The Euclidean Lagrangian density is given by

$$L=\frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi)+\frac{1}{2}m^2\Phi^2+\frac{\lambda}{4!}\Phi^4 . \tag{3.1}$$

The propagator defined in (2.7) is

$$D^{-1}(\phi;x,y)=-\left[\Box+m^2+\frac{\lambda}{2}\phi^2\right]\delta^4(x-y) , \tag{3.2}$$

and the vertices of the shifted theory are given by

$$L_{\rm int}(\phi;\Phi)=\frac{\lambda}{6}\phi\Phi^3+\frac{\lambda}{4!}\Phi^4 . \tag{3.3}$$

In Fig. 2 the diagrams contributing to $\Gamma_\beta^{(2)}(\phi,G)$ are shown up to three loops: Each line represents the propagator $G(x,y)$, and there are two kinds of vertices.

In order to determine how to truncate the loop expansion so that all daisy and superdaisy graphs of perturbation theory are included, we first observe that upon dropping $\Gamma_\beta^{(2)}$ altogether, the gap equation (2.9) gives

$$G^{-1}(x,y)=D^{-1}(x,y) , \tag{3.4}$$

and $\Gamma(\phi,D)$ is simply the ordinary one-loop effective potential. Therefore a nontrivial G can be obtained only if we retain some of the graphs in $\Gamma_\beta^{(2)}$. Next we observe that among the graphs in Fig. 2 only the two-loop graph of $O(\lambda)$ will include contributions from daisy and superdaisy graphs of ordinary perturbation theory. Therefore we shall truncate the series at $O(\lambda)$. This is known as the Hartree-Fock [17] approximation. The effective action is then

$$\Gamma_\beta(\phi,G)=I_{\rm cl}(\phi)+\tfrac{1}{2}{\rm Tr}\ln D_0G^{-1}+\tfrac{1}{2}{\rm Tr}[D^{-1}G-1] + \frac{3}{4!}\lambda\int d^4x\, G(x,x)G(x,x) . \tag{3.5}$$

By stationarizing Γ_β with respect to G, we obtain the gap equation in the Hartree-Fock approximation:

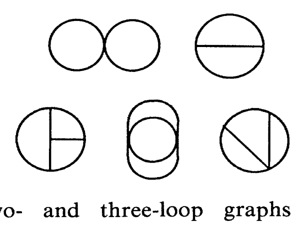

FIG. 2. Two- and three-loop graphs contributing to $\Gamma_\beta^{(2)}(\phi,G)$. They are two-particle irreducible, and their lines represent the full propagator G.

$$G^{-1}(x,y)=D^{-1}(x,y)+\frac{\lambda}{2}G(x,x)\delta^4(x-y) . \tag{3.6}$$

It is straightforward to show by iteration that Eq. (3.6) generates all daisy and superdaisy graphs that contribute to the full propagator in ordinary perturbation theory. [Equations (3.5) and (3.6) have the same structure as those in the leading large-N approximation since in both cases n-point functions are expressed in terms of one- and two-point functions. However, as discussed in Ref. [15], in the large-N approximation those daisy and superdaisy diagrams which are of $O(1/N)$ are dropped and therefore the coefficients of λ in (3.5) and (3.6) become smaller by a factor of 3.]

In order to obtain the effective potential $V_\beta(\phi,G)$ for translation-invariant field configurations, we define the Fourier-transformed propagators

$$D(k)=\int\frac{d^4k}{(2\pi)^4}D(x-y)e^{ik(x-y)} =\frac{1}{k^2+m^2+(\lambda/2)\phi^2} , \tag{3.7}$$

$$G(k)=\int\frac{d^4k}{(2\pi)^4}G(x-y)e^{ik(x-y)}=\frac{1}{k^2+M^2} , \tag{3.8}$$

where we have taken an ansatz for $G(k)$ using an effective mass M. Since the correction to the gap equation in this approximation is given by $\lambda G(x,x)/2$, M^2 can be taken to be momentum independent. The effective potential is then

$$V(\phi,M)=\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln[k^2+M^2] -\frac{1}{2}\left[M^2-m^2-\frac{\lambda}{2}\phi^2\right]G(x,x) +\frac{\lambda}{8}G(x,x)G(x,x) . \tag{3.9}$$

From (3.9) the stationary requirements are

$$\frac{\partial V_\beta(\phi,M)}{\partial\phi}=\phi\left[m^2+\frac{\lambda}{6}\phi^2+\frac{\lambda}{2}G(x,x)\right]=0 , \tag{3.10}$$

$$\frac{\partial V_\beta(\phi,M)}{\partial M^2}=-\frac{1}{2}\frac{\partial G(x,x)}{\partial M^2}\left[M^2-m^2-\frac{\lambda}{2}\phi^2 -\frac{\lambda}{2}G(x,x)\right]=0 . \tag{3.11}$$

The conventional effective potential is obtained by evaluating $V_\beta(\phi,M)$ at the solution $M(\phi)$ of (3.11). It is composed of three terms—the classical (V^0), one-loop ($V^{\rm I}$), and two-loop ($V^{\rm II}$) contributions:

$$V_\beta(\phi,M(\phi))=V^0+V^{\rm I}+V^{\rm II} , \tag{3.12a}$$

$$V^0=\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4 , \tag{3.12b}$$

$$V^{\rm I}=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln[k^2+M^2(\phi)] , \tag{3.12c}$$

$$V^{\rm II}=-\frac{\lambda}{8}G(x,x)G(x,x)\ , \tag{3.12d}$$

where

$$M^2(\phi)=m^2+\frac{\lambda}{2}\phi^2+\frac{\lambda}{2}G(x,x)\ , \tag{3.12e}$$

and now $G(x,x)$ is given by

$$G(x,x)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+M^2(\phi)}\ . \tag{3.12f}$$

B. Renormalizing the effective potential

The expression of $V_\beta(\phi,M(\phi))$ in (3.12) contains divergent integrals. Moreover, because our approximation is nonperturbatively self-consistent, reflecting the nonlinearity of the full theory, $M(\phi)$, the argument of V_β, is not well defined because of the infinities in $G(x,x)$. Therefore we shall first obtain a well-defined finite expression for $M(\phi)$ by a renormalization. [In the rest of this paper M refers to the solution of (3.12e).] We define renormalized parameters m_R and λ_R as

$$\pm\frac{m_R^2}{\lambda_R}=\frac{m^2}{\lambda}+\frac{1}{2}I_1\ , \tag{3.13a}$$

$$\frac{1}{\lambda_R}=\frac{1}{\lambda}+\frac{1}{2}I_2(\mu)\ , \tag{3.13b}$$

where $I_{1,2}$ are divergent integrals,

$$I_1\equiv\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}=\lim_{\Lambda\to\infty}\frac{\Lambda^2}{8\pi^2}\ , \tag{3.13c}$$

$$I_2\equiv\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2|\mathbf{k}|}-\frac{1}{2(|\mathbf{k}|^2+\mu^2)^{1/2}}\right]=\lim_{\Lambda\to\infty}\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2}\ . \tag{3.13d}$$

μ is the renormalization scale, and Λ is the ultraviolet momentum cutoff. (The same renormalization prescription has been used also in the large N approximation [18].)

When the sum on n in k_4 is carried out as in Ref. [2], $G(x,x)$ becomes

$$G(x,x)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k(\exp[\beta\omega_k]-1)}\equiv G(M(\phi))+I_1-M^2(\phi)I_2(\mu)\ , \tag{3.14}$$

where $\omega_k\equiv[|\mathbf{k}|^2+M^2(\phi)]^{1/2}$ and $G(M(\phi))$ is the finite part of $G(x,x)$, given by

$$G(M(\phi))\equiv\frac{M^2(\phi)}{16\pi^2}\ln\frac{M^2(\phi)}{\mu^2}+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k(\exp[\beta\omega_k]-1)}\ . \tag{3.15}$$

In the limit $T=0$, the first term in $G(M(\phi))$ survives, but the second term vanishes.

It is straightforward to see that $M^2(\phi)$ is finite and cutoff independent in terms of m_R and λ_R:

$$M^2(\phi)=\pm m_R^2+\frac{\lambda_R}{2}\phi^2+\frac{\lambda_R}{2}G(M(\phi))\ . \tag{3.16}$$

(In this paper we shall choose the negative sign which allows spontaneous symmetry breaking.) It is convenient for our later discussions to define the tree-level effective mass $\tilde m^{\,2}(\phi)$:

$$\tilde m^{\,2}(\phi)=-m_R^2+\frac{\lambda_R}{2}\phi^2\ . \tag{3.17}$$

With this finite M^2, we are ready to discuss the divergences in $V_\beta(\phi,M)$. First, carrying out the sum on n in $V^{\rm I}$, we obtain the familiar one-loop finite-temperature formula of Ref. [2], where the tree-level effective mass $\tilde m(\phi)$ is replaced by $M(\phi)$:

$$\begin{aligned}V^{\rm I}(M)&=\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega_k+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-\exp[\beta\omega_k])\\&=\frac{M^4}{64\pi^2}\left[\ln\frac{M^2}{\mu^2}-\frac{1}{2}\right]+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-\exp[\beta\omega_k])-\frac{M^4}{4}I_2(\mu)\ .\end{aligned} \tag{3.18}$$

At $T=0$ the first term of $V^{\rm I}$ survives and provides the zero-temperature one-loop contribution, and the second term vanishes. The last term is the divergence in $V^{\rm I}$.

Divergences in the two-loop contribution $V^{\rm II}$ come from $G(x,x)$ and its square. Finiteness of $V_\beta(\phi,M(\phi))$ can be shown by first combining V^0 and $V^{\rm II}$ using the unrenormalized form of the gap equation. When the combined expression is written in terms of renormalized parameters, the remaining divergent integral is canceled by that of $V^{\rm I}$ in (3.18). This is another indication that the two-loop contribution must be included for a finite self-consistent approximation. The resulting finite expression for $V_\beta(\phi,M(\phi))$ is

$$V_\beta(\phi,M(\phi))=(V^0+V^{\rm II})+V^{\rm I}\ , \tag{3.19a}$$

$$V^0+V^{\rm II}=\frac{M^4}{2\lambda_R}-\frac{1}{2}M^2G(M)-\frac{\lambda}{12}\phi^4\ , \tag{3.19b}$$

$$V^{\rm I}=\frac{M^4}{64\pi^2}\left[\ln\frac{M^2}{\mu^2}-\frac{1}{2}\right]+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-\exp[\beta\omega_k])\ , \tag{3.19c}$$

[A constant term $m^4/(2\lambda)$ has been adjusted to obtain (3.19) from (3.12).] However, in order to compare the high-temperature effective potential with and without the two-loop contribution in our later discussion, we still have to extract $V^{\rm II}$ from (3.19b). Observing that V^0 must be a function of ϕ only and that in our approximation $V^{\rm II}$ does not depend on ϕ explicitly [since the graph of $O(\lambda)$ in Fig. 2 does not involve any vertices which depend on ϕ], we obtain, by using the renormalized gap equation,

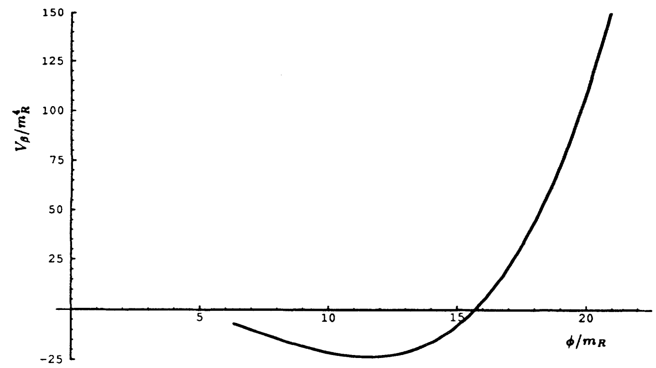

FIG. 3. $V_\beta(\phi,M(\phi))/m_R^4$ as a function of ϕ/m_R at $T=0$. V_β illustrated in the figure corresponds to $\lambda_R=0.05$ and $\ln(\Lambda^2/m_R^2)=16\pi^2$. V_β becomes imaginary for small ϕ/m_R.

$$V^0+V^{\rm II}=\left[\frac{\lambda_R}{8}\left[\phi^2-2\frac{m_R^2}{\lambda_R}\right]-\frac{\lambda}{12}\phi^4\right]-\frac{\lambda_R}{8}G(M)G(M)\ . \tag{3.20}$$

Clearly, the last term in (3.20) is the two-loop contribution. The quantity in the brackets is the classical contribution after renormalization is carried out. It is cutoff dependent because of the term $-\lambda\phi^4/12$, which did not get renormalized because of the structure of the gap equation. But the renormalization prescription (3.13) tells us that if λ_R is held fixed as $\Lambda\rightarrow\infty$, λ approaches 0_-. (A necessary condition in a renormalized $\lambda\phi^4$ theory is $\lambda<0$.) As shown in the large-N studies [18], such theory is intrinsically unstable. On the other hand, holding $\lambda>0$ implies $\lambda_R\rightarrow 0$ as $\Lambda\rightarrow\infty$. For $\lambda>0$ a sensible theory can be obtained for a fixed small $\lambda_R>0$ as an effective low-energy theory, if Λ is kept fixed at a large but finite value. Such a theory requires

$$\frac{\lambda_R}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}<1\ , \tag{3.21}$$

in order to have $\lambda>0$, and all momenta, temperature, and any other physical mass scale must be much smaller than Λ.

As shown in Fig. 3, the zero-temperature phase structure of the effective theory with finite Λ is similar to that of perturbation theory: There exists a minimum at a nonzero value of ϕ [19]. We shall consider such an effective theory and take our temperature to be $T\ll\Lambda$.

C. High-temperature effective potential

Our main interest is in the form of the high-temperature effective potential. We first find the high-temperature gap equation by expanding the integral expression of $G(M)$, i.e., the second term in (3.15) at high temperature. Since the basic mass scale in the problem is M, we consider an expansion in $M^2/T^2\ll 1$:

$$G(M)=T^2\left[\frac{1}{12}-\frac{1}{4\pi}\frac{M}{T}+O\left[\frac{M^2}{T^2}\ln T\right]\right]\ . \tag{3.22}$$

(We have chosen our renormalization scale μ to be m_R.) Then the high-temperature gap equation takes the form

$$M^2=\tilde{m}^{\,2}(\phi)+\frac{\lambda_R}{24}T^2-\frac{\lambda_R}{8\pi}MT+O(\lambda_R M^2\ln T)\ . \tag{3.23}$$

From the solution of this equation, one finds that, for a small coupling $\lambda_R\ll 1$, the condition $M^2/T^2\ll 1$ is consistent with

$$\tilde{m}^{\,2}(\phi)/T^2\ll 1\ , \tag{3.24}$$

which is exactly the required condition for high-temperature expansion of the perturbative calculation in Ref. [2].

Now we return to Eqs. (3.19) and (3.20). The high-temperature expansion of the effective potential can be obtained using the high-temperature expansion of $G(M)$ and also the high-temperature expansion of the perturbative one-loop effective potential of Ref. [2] by replacing the tree-level effective mass $\tilde{m}(\phi)$ by M:

$$V_\beta(\phi,M(\phi))=V^0+V^{\rm I}+V^{\rm II}\ , \tag{3.25a}$$

$$V^0=\frac{\lambda_R}{8}\left[\phi^2-2\frac{m_R^2}{\lambda_R}\right]^2-\frac{\lambda}{12}\phi^4\ , \tag{3.25b}$$

$$V^{\rm I}=-\frac{\pi^2}{90}T^4+\frac{M^2T^2}{24}-\frac{M^3T}{12\pi}+O\left[M^4\ln\frac{M^2}{T^2}\right]\ , \tag{3.25c}$$

$$V^{\rm II}=-\frac{\lambda_R}{8}\left[\frac{T^4}{144}-\frac{MT^3}{24\pi}+\frac{M^2T^2}{16\pi^2}+O\left[M^4\ln\frac{M^2}{T^2}\right]\right]\ . \tag{3.25d}$$

IV. DISCUSSION AND CONCLUSIONS

In order to understand the structure of the effective potential in our approximation, we shall first consider the nonlinear aspects of the gap equation (3.6), which in the high-temperature limit can be expressed as

$$M^2(\phi)=\tilde{m}^{\,2}(\phi)+\frac{\lambda_R}{24}T^2-\frac{\lambda_R}{8\pi}TM(\phi)\ . \tag{4.1}$$

Consequently, $M(\phi)$ can be expanded for small λ_R as

$$M(\phi)=M_L(\phi)\left\{1-\frac{\lambda_R T}{16\pi M_L(\phi)}+O\left[\left[\frac{\lambda_R T}{16\pi M_L(\phi)}\right]^2\right]\right\}\ , \tag{4.2a}$$

where

$$M_L(\phi)\equiv\left[\tilde{m}^{\,2}(\phi)+\frac{\lambda_R}{24}T^2\right]^{1/2} \tag{4.2b}$$

solves the linearized high-temperature gap equation, i.e., (4.1), without the last term.

When the one-loop contribution $V^{\rm I}$ in (3.25c) is rewritten using the gap equation, we obtain

$$V^{\mathrm{I}}(\phi)=-\frac{\pi^2}{90}T^4+\frac{M_L^2(\phi)T^2}{24}-\frac{\lambda_R}{192\pi}M_L(\phi)T^3-\frac{M_L^3(\phi)T}{12\pi}+O\left[\frac{M^4}{64\pi^2}\ln T\right]. \quad (4.3)$$

We observe that the term linear in M, namely, the third term on the right-hand side of Eq. (4.3), arises from the nonlinearity of the gap equation, i.e., from the last term in (4.1). If we were to use the linearized gap equation without this term, the first nontrivial correction to the perturbative one-loop effective potential would be given by the term cubic in $M(\phi)$. However, at high temperatures the leading nonlinear correction is of the same order as the term cubic in $M(\phi)$ in (4.3); in fact, from (4.1) we have

$$\left[\frac{\lambda_R}{192\pi}M(\phi)T^3\right]\Bigg/\left[\frac{M^3(\phi)T}{12\pi}\right]\simeq\frac{288}{192}\sim 1\;, \quad (4.4)$$

for $T\gg\phi$, and such a term could alter the nature of the phase transition. When we include the two-loop contribution given in (3.25d), the MT^3 term disappears and the high-temperature effective potential in our approximation is (neglecting ϕ-independent contributions)

$$V(\phi)=V^0(\phi)+\left[\frac{T^2}{24}M_L^2(\phi)-\frac{T}{12\pi}M_L^3(\phi)\right][1+O(\lambda_R)]+O\left[\frac{M^4}{64\pi^2}\ln T\right]. \quad (4.5)$$

In Fig. 4 the effective potential of Eq. (4.5) is shown as a function of ϕ for five different temperatures close to the critical temperature. It is evident that there is a temperature such that there are two degenerate minima.

The above analysis of our consistent approximation shows that improving the perturbative one-loop effective potential using the nonlinear gap equation clearly leads to an erroneous result [20]. Therefore one must use a self-consistent method which relates the effective potential and the gap equation. On the other hand, we also see that because of the cancellation of the leading nonlinear effect, one can in fact obtain a consistently improved effective potential by improving the perturbative one-loop effective potential using an effective mass $M_L(\phi)$, which is the solution of the linearized gap equation. Such improved perturbation theory, using the effective mass squared shifted by a ϕ-independent amount proportional to T^2, was first suggested by Weinberg [3] and later further studied by others [21]. If the cancellation of the leading nonlinear effects in our approximation is a general feature, occurring even in gauge theories, the one-loop improved effective potential in the standard model calculated by Carrington [8] would be a consistent approximation. We plan to clarify this in a future publication.

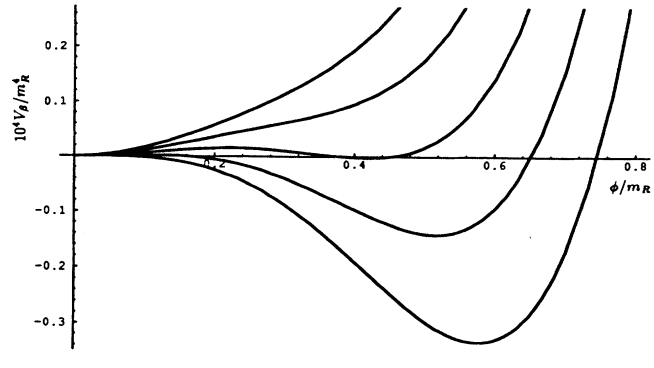

FIG. 4. High-temperature effective potential in our approximation [Eq. (4.5)] for five different temperatures close to the critical temperature. [Here we have chosen $\lambda_R=0.05$ and $\ln(\Lambda^2/m_R^2)=16\pi^2$.] In order to compare the shapes between different temperatures, we shifted the V_β's by ϕ-independent amounts so that $V_\beta=0$ for $\phi/m_R=0$. From the figure it is evident that at a certain temperature there are two degenerate minima.

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