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Quantum rolling down out of equilibrium

D. Boyanovsky
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

H. J. de Vega
Laboratoire de Physique Theorique et Hautes Energies, Universite Pierre et Marie Curie, Tour 16, 1er etage 4, Place Jussieu, 75252 Paris CEDEX 05, France
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In a scalar field theory, when the tree-level potential admits broken-symmetry ground states, the quantum corrections to the static effective potential are complex. (The imaginary part is a consequence of an instability towards phase separation and the static effective potential is not a relevant quantity for understanding the dynamics.) Instead, we study here the equations of motion obtained from the one-loop effective action for slow rollover out of equilibrium. We consider the case in which a scalar field theory undergoes a rapid phase transition from $T_i > T_c$ to $T_f < T_c$. We find that, for slow-rollover initial conditions (the field near the maximum of the tree-level potential), the process of phase separation controlled by unstable long-wavelength fluctuations introduces dramatic corrections to the dynamical evolution of the field. We find that these effects slow the rollover even further, thus delaying the phase transition, and increasing the time that the field spends near the "false vacuum." Moreover, when the initial value of the field is very close to zero, the dynamics becomes *nonperturbative*.

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I. INTRODUCTION AND MOTIVATION

Inflationary cosmological models provide very appealing scenarios to describe the early evolution of our Universe [1]. Since the original model proposed by Guth [2], several alterative scenarios have been proposed to overcome some of the difficulties with the original proposal.

Among them, the new inflationary model [3–6] is perhaps one of the most attractive. The essential ingredient in the new inflationary model is a scalar field that undergoes a second-order phase transition from a high-temperature symmetric phase to a low-temperature broken-symmetry phase. The expectation value (or thermal average) of the scalar field ϕ serves as the order parameter. Initially, at high temperatures, the scalar field is assumed to be in thermal equilibrium and $\phi \approx 0$. The usual field-theoretic tool to study the phase transition is the effective potential [7–9]. At high temperatures, the global minimum of the effective potential is at $\phi = 0$, whereas at low temperatures there are two degenerate minima.

The behavior of the phase transition in the new inflationary model is the following: As the Universe cools down, the expectation value of the scalar field remains close to zero until the temperature becomes smaller than the critical temperature. Below the critical temperature, when the effective potential develops degenerate minima away from the origin, the scalar field begins to "roll down the potential hill." In the new inflationary scenario, the effective potential below the critical temperature is extremely flat near the maximum, and the scalar field remains near the origin—i.e., the false vacuum—for a very long time and eventually rolls down the hill very slowly. This scenario thus receives the name of "slow rollover." During the slow-rollover stage, the energy density of the Universe is dominated by the constant vacuum energy density $V_{\text{eff}}(\phi = 0)$, and the Universe evolves rapidly into a de Sitter space (see, for example, the reviews by Kolb and Turner [10], Linde [11], and Brandenberger [12]). Perhaps the most remarkable consequence of the new inflationary scenario and the slow-rollover transition is that they provide a calculational framework for the prediction of density fluctuations [13]. The coupling constant in the typical zero-temperature potentials must be fine-tuned to a very small value to reproduce the observed limits on density fluctuations [10,11].

This picture of the slow-rollover scenario is based on the *static* effective potential. The use of the static effective potential to describe a time-dependent situation has been critized by Mazenko, Unruh, and Wald [14]. These authors argued that the *dynamics* of the cooling down process is very similar to the process of phase separation in statistical mechanics. They argued that the system will form domains and that the scalar field will relax to the values at the minima of the potential very quickly.

Guth and Pi [15] performed a thorough analysis of the effects of quantum fluctuations on the time evolution. These authors analyzed the situation below the critical temperature by treating the potential near the origin as an *inverted harmonic oscillator*. They recognized that the instabilities associated with these upside-down oscillators lead to an exponential growth of the quantum fluctuations at long times and to a classical description of the probability distribution function. Guth and Pi also recognized that the *static* effective potential is not appropriate to describe the dynamics, which must be treated as a time-dependent process.

Subsequently, Weinberg, and Wu [16] have studied the effective potential, particularly in the situation when the tree-level potential allows for broken-symmetry ground states. These authors carefully analyzed the contributions to the effective potential and showed that the imaginary part of the one-loop effective potential is the result of an analytic continuation of the unstable modes (inverted oscillators) studied by Guth and Pi. That the effective potential develops an imaginary part was realized in the early treatments by Dolan and Jackiw [8]. This imaginary part in fact conceals the growth of these unstable modes and a time-dependent situation that cannot be described by equilibrium statistical mechanics. The imaginary part of the effective potential was related to the lifetime of a particular initial quantum state.

There have been several attempts to study the time evolution of the scalar field either in flat spacetime or de Sitter space [17–21], but to our knowledge the influence of the instabilities that are responsible for domain growth and phase separation on the dynamics of the scalar field has not yet been elucidated.

In this article we study the quantum dynamics of the scalar field by analyzing the situation in which the system originally in equilibrium at a temperature higher than the critical temperature evolves *out of equilibrium* at a temperature below the critical temperature. We provide a detailed description of the dynamics out of equilibrium, concentrating on the instabilities that drive the phase transition and phase separation, which is the unstable growth of long-wavelength fluctuations. Our approach consists in obtaining the nonequilibrium evolution equation obtained from the one-loop effective action. These equations turn out to be nonlocal and nonintegrable [22]. We provide a qualitative discussion and a consistent numerical analysis of these evolution equations for a wide variety of initial conditions.

The results of our analysis show that the instabilities that trigger the growth of long-wavelength fluctuations dramatically enhance the quantum corrections and result in a further slow down of the scalar field.

For slow-rollover initial conditions, we show that even for very weak coupling (consistent with the bounds on density fluctuations), the quantum corrections become very important and slow the dynamics dramatically, and in particular for some initial conditions, in which the scalar field is very close to the "false vacuum," the quantum corrections must be treated beyond perturbation theory.

This paper is organized as follows. In the next section we review the effective potential and its shortcomings to describe the dynamics.

In Sec. III we describe the nonequilibrium formalism and our approach to obtain the equations of motion out of equilibrium for the spatial average of the scalar field. In this section, we provide an analysis of the renormalization aspects of the effective equations of motion.

In Sec. IV we study the dynamics and provide an analytic as well as numerical analysis of the evolution equations for a wide range of initial conditions, emphasizing the consequences of the unstable growth of long-wavelength fluctuations.

We conclude the paper with an analysis of the potential implications of our results on cosmological phase transitions and inflationary models.

II. COMPLEX EFFECTIVE POTENTIAL

It is already well known that when the tree-level potential allows for broken-symmetry ground states, the one-loop effective potential becomes complex in a region of field configurations. This unacceptable complex part of the effective potential has been deemed an artifact of the loop expansion, and for the most part ignored in most treatments of the effective potential.

However, as was clearly shown by Weinberg and Wu [16], the imaginary part of the effective potential has a very physical meaning, and is a consequence of the instabilities that drive phase separation.

A rather clear understanding of the imaginary part and the physics associated with it is obtained by a derivation of the effective potential within the Hamiltonian framework. The Hamiltonian for a scalar field theory quantized in a volume Ω is

$$H=\int_{\Omega} d^3x\{\tfrac{1}{2}\Pi^2(x)+\tfrac{1}{2}[\nabla\Phi(x)]^2+V(\Phi)\} . \tag{2.1}$$

Since the effective potential is a function of the zero-momentum component of the field, we separate the constant part (zero momentum) (φ) of the field and its canonical momentum;

$$\Phi(x)=\varphi+\psi(x) ,\qquad \varphi=\frac{1}{\Omega}\int_{\Omega} d^3x\, \Phi(x) ,\quad \int_{\Omega} d^3x\, \psi(x)=0 , \tag{2.2}$$

$$\Pi(x)=\frac{P}{\Omega}+\pi(x) ,\qquad P=\int_{\Omega} d^3x\, \Pi(x) ,\quad \int_{\Omega} d^3x\, \pi(x)=0 , \tag{2.3}$$

$$[\Pi(x),\Phi(y)]=-i\hbar\delta^3(x-y),\quad [P,\varphi]=-i\hbar . \tag{2.4}$$

We have kept ($\hbar$) in the above commutators to clarify the quantum corrections.

Using the fact that π,ψ do not have a zero-momentum component, the Hamiltonian becomes

$$H=\frac{P^2}{2\Omega}+\Omega V(\varphi)+\int_{\Omega} d^3x\{\tfrac{1}{2}\pi^2(x)+\tfrac{1}{2}[\nabla\psi(x)]^2 +\tfrac{1}{2}V''(\varphi)\psi^2(x)+\cdots\} , \tag{2.5}$$

where the ellipsis stands for higher-order terms.

To this order the Hamiltonian for the field $\psi(x)$ is quadratic, with a mass term that depends on φ. In the representation with creation and annihilation operators for the harmonic oscillators, the Hamiltonain finally becomes (in the discrete momentum representation)

$$H=\frac{P^2}{2\Omega}+\Omega V(\varphi)+\sum_k (a_k^\dagger a_k+\tfrac{1}{2})\hbar\omega(k,\varphi) , \tag{2.6}$$

$$\omega(k,\varphi)=[k^2+V''(\varphi)]^{1/2} . \tag{2.7}$$

When all the oscillators are in their ground state, the zero-point energy for the oscillators is then recognized as the first-order quantum correction to the effective potential. Taking the large volume limit, the effective potential to this order is then

$$V_{\rm eff}(\varphi)=V(\varphi)+\frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}[k^2+V''(\varphi)]^{1/2}\ . \quad (2.8)$$

As usual, renormalization is carried out in the standard manner. From now on, we set $\hbar=1$. Consider for simplicity the case in which

$$V(\varphi)=-\tfrac{1}{2}\mu_0^2\varphi^2+\frac{\lambda_0}{4!}\varphi^4\ . \quad (2.9)$$

Then for all values of φ such that $V''(\varphi)<0$, the effective potential acquires an imaginary part given by

$$\mathrm{Im}V_{\rm eff}(\varphi)=\pm\frac{\hbar}{4\pi^2}\int_0^{k_{\rm max}}k^2[|V''(\varphi)|-k^2]^{1/2}dk\ , \quad (2.10)$$

$$(k_{\rm max})^2=|V''(\varphi)|\ . \quad (2.11)$$

This imaginary part is finite (independent of renormalization), and arises because for these values of φ the modes for which $k^2<(k_{\rm max})^2$ are unstable and the frequencies of the oscillators in (2.6) are imaginary. For these modes the potential is that of an *inverted oscillator.*

These inverted oscillators for which $\omega^2(k,\varphi)<0$ do not have stationary-state solutions and the imaginary part of the zero-point energy arises from the *analytic continuation* of harmonic-oscillator wave functions from positive to negative ω^2. The sign of the imaginary part depends on the analytic continuation. The quantum mechanics of these inverted harmonic oscillators has been thoroughly studied by Guth and Pi [15] and Weinberg and Wu [16].

There being no stationary states associated with these inverted oscillators, an initial state must be specified. If, for example, a Gaussian wave packet is prepared initially centered at the origin of these inverted oscillators, it will spread and the width of the packet will increase exponentially in time [15,16]. This exponential growth is manifest in the equal-time two-point function [15,16]

$$\langle\psi_k(t)\psi_{-k}(t)\rangle\sim e^{2|\omega(k,\varphi)|t}\ .$$

This is the two-point Green's function computed in this Gaussian state and measures the fluctuations of the operator ψ. In particular, the growth of these unstable modes is precisely the mechanism that Mazenko, Unruh, and Wald [14] suggest.

The importance of the scalar field fluctuations in the initial stages of cosmological phase transitions was pointed out by Linde [11].

One notes, however, that the two-point Green's function calculated in the *analytically continued state* is *time independent* and given by

$$\langle\psi_k(t)\psi_{-k}(t)\rangle=\frac{1}{2\omega(k,\varphi)}\ . \quad (2.12)$$

This result is purely imaginary, but again this imaginary part is obtained after the analytic continuation of these Gaussian states, and is actually concealing a *time-dependent situation.*

Another critical observation is that it is this two-point function evaluated in these Gaussian states that gives the one-loop correction to the effective potential. Thus it becomes clear that the imaginary part, resulting from an analytic continuation of the unstable modes, is in fact hiding an unstable time evolution. At finite temperature, the imaginary part is signaling a nonequilibrium situation.

This situation is well understood in statistical mechanics; perhaps the earliest example is the van der Walls equation of state and its unphysical isotherms. In the unphysical region, the situation must be studied out of equilibrium. In this region there is coexistence of different phases that may not be studied within equilibrium statistical mechanics with a free energy for homogeneous field configurations. An *ad hoc* remedy in this situation is the Maxwell construction that replaces the unphysical region of the isotherms by a straight line. Its physical interpretation is that the nonequilibrium state may be found, in the coexistence region, as a mixture of phases with arbitrary concentrations of each phase. This situation is also typical of binary mixtures in statistical mechanics [23,24]. The Maxwell constructed effective potential or free energy is *irrelevant* for the dynamical nonequilibrium description of the system.

The exponential growth of the unstable modes (inverted oscillators) signals the growth of domains and the onset of the process that triggers the phase transition, i.e., phase separation. This is very similar to the process of spinodal decomposition in statistical mechanics [23,24]. An attempt to describe spinodal decomposition within the context of field theory has been described by Calzetta [27].

Clearly the approximation of inverted oscillators is crude as it neglects nonlinear effects. The growth of these unstable modes will eventually slow down when the nonlinearities become important; this is the process of coarsening. As we will show later, a clear understanding of the physics of coarsening in the regime where the nonlinearities become important may require departing from perturbative treatments.

Usually, in order to understand the dynamics of the scalar field in this type of situation, the *static effective potential* is used in the equations of motion, resulting in a typical evolution equation (in flat spacetime)

$$\frac{d^2\varphi}{dt^2}+\frac{dV_{\rm eff}(\varphi)}{d\varphi}=0\ . \quad (2.13)$$

After the arguments presented above, it becomes clear that this equation is inappropriate for the study of the dynamical evolution. The imaginary part of $V_{\rm eff}(\varphi)$ signals an unstability and the static effective potential is not a suitable quantity to study the dynamics. That is, one must consider the effective action for *time-dependent fields* and not merely the effective potential which holds for constant fields.

We turn to this study in the next section.

III. NONEQUILIBRIUM TIME EVOLUTION

As explained in the previous section, one must depart from the usual description in terms of the *static* effective potential, and treat the dynamics with the full time evolution. The time evolution of the system will be determined once the evolution Hamiltonian and *the initial state* are prescribed.

In order to understand the dynamics of the phase transition and the physics of the instabilities mentioned above, let us consider the situation in which for time $t<0$ the system is in *equilibrium* at an initial temperature $T_i > T_c$, where T_c is the critical temperature. At time $t=0$ the system is rapidly "quenched" to a final temperature below the critical temperature $T_f < T_c$ and evolves thereafter out of equilibrium.

What we have in mind in this situation is a cosmological scenario with a period of rapid inflation in which the temperature drops very fast compared to typical relaxation times of the scalar field. In particular this situation should correspond to the case $h \gg \tau^{-1}$, with h Hubble's constant and τ a typical relaxation time. At high temperatures and weakly coupled theories we would expect $\tau^{-1} \approx \lambda^2 T$ (λ is the coupling). When $T \approx T_c \propto \lambda^{-1/2}$ relaxation times become very large and the dynamics of the long-wavelength modes (the only relevant modes for the phase transition) becomes critically slowed down. Precisely because of this critical slowing down, we conjecture that an inflationary period at temperatures near the critical temperature may be described in this "quenched" approximation. Another situation that may be described by this approximation is that of a scalar field again at $T_i > T_c$ suddenly coupled to a "heat bath" at a much lower temperature (below the transition temperature) and evolving out of equilibrium. The heat bath may be other fields at a different temperature. The influence of a heat bath in an inflationary universe has been studied in the linearized approximation by Cornwall and Bruinsma [26].

One then would expect that a "quenching out-of-equilibrium scenario" may be an appropriate description near the critical temperature for these situations. Certainly this is only a plausibility argument; a deeper understanding of the initial conditions must be pursued to obtain a more precise knowledge of the cooling down process.

We do not envisage in this article to study the case of an inflationary cosmology or the detailed dynamics of the mechanism that produces the "quenching" below the critical temperature and departure from equilibrium. We expect to report on these investigations in forthcoming articles.

Here we just assume that such a mechanism takes place and simplify the situation by introducing a Hamiltonian with a *time-dependent mass term* to describe this situation:

$$H(t) = \int_\Omega d^3x \left[\tfrac{1}{2}\Pi^2(x) + \tfrac{1}{2}[\nabla\Phi(x)]^2 + \tfrac{1}{2}m^2(t)\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x) \right], \tag{3.1}$$

$$m^2(t) = m^2\Theta(-t) + (-\mu^2)\Theta(t), \tag{3.2}$$

where both m^2 and μ^2 are positive. We assume that for all times $t<0$ there is thermal equilibrium at temperature T_i, and the system is described by the density matrix

$$\hat{\rho}_i = e^{-\beta_i H_i}, \tag{3.3}$$

$$H_i = H(t<0). \tag{3.4}$$

In the Schrödinger picture, the density matrix evolves in time as

$$\hat{\rho}(t) = U(t)\hat{\rho}_i U^{-1}(t), \tag{3.5}$$

with $U(t)$ the time evolution operator.

An alternative and equally valid interpretation (and the one that we like best) is that the initial condition being considered here is that of a system in equilibrium in the symmetric phase, and evolved in time with a Hamiltonian that allows for broken-symmetry ground states, i.e., the Hamiltonian (3.1) and (3.2) for $t>0$.

The expectation value of any operator is thus

$$\langle \mathcal{O} \rangle(t) = \mathrm{Tr}\, e^{-\beta_i H_i} U^{-1}(t)\mathcal{O}U(t) / \mathrm{Tr}\, e^{-\beta_i H_i}. \tag{3.6}$$

This expression may be written in a more illuminating form by choosing an arbitrary time $T<0$ for which $U(T)=\exp[-iTH_i]$. Then we may write

$$\exp[-\beta_i H_i] = \exp[-iH_i(T-i\beta_i-T)] = U(T-i\beta_i, T).$$

Inserting in the trace $U^{-1}(T)U(T)=1$, commuting $U^{-1}(T)$ with $\hat{\rho}_i$, and using the composition property of the evolution operator, we may write (3.6) as

$$\langle \mathcal{O} \rangle(t) = \mathrm{Tr}\,U(T-i\beta_i t)\mathcal{O}U(t,T) / \mathrm{Tr}\,U(T-i\beta_i, T). \tag{3.7}$$

The numerator of the above expression has a simple meaning: Start at time $T<0$, evolve to time t, insert the operator $\mathcal{O}$, and evolve backwards in time from t to $T<0$ and along the negative imaginary axis from T to $T-i\beta_i$. This operation is depicted in Fig. 1(a). The denominator just evolves along the negative imaginary axis from T to $T-i\beta_i$. The contour in the numerator may be extended to an arbitrary large positive time T' by inserting $U(t,T')U(T',t)=1$ to the left of $\mathcal{O}$ in (3.7), thus becoming

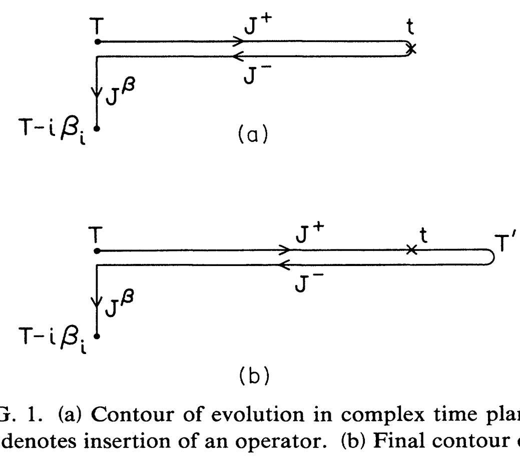

FIG. 1. (a) Contour of evolution in complex time plane; the cross denotes insertion of an operator. (b) Final contour of evolution, eventually $T' \to \infty, T \to -\infty$.

$$\langle \mathcal{O}\rangle(t)=\mathrm{Tr}U(T-i\beta_i,T)U(T,T')U(T',t)\times\mathcal{O}U(t,T)/\mathrm{Tr}U(T-i\beta_i,T)\ . \tag{3.8}$$

The numerator now represents the process of evolving from $T<0$ to t, inserting the operator $\mathcal{O}$, evolving further to T', and backwards from T' to T, and down the negative imaginary axis to $T-i\beta_i$. This process is depicted in the contour of Fig. 1(b). Eventually we take $T\to-\infty$, $T'\to\infty$. It is straightforward to generalize to real-time correlation functions of Heisenberg picture operators.

This formalism allows us also to study the general case in which both the mass and the coupling depend on time, and furthermore, by taking the zero-temperature limit, we can study the situation in which a particular state is prepared at time $t=0$ and evolved in time.

For example, by switching off the coupling for $t<0$ one is preparing a Gaussian density matrix at $t=0$ or, in the zero-temperature limit, a Gaussian wave functional. This density matrix or Gaussian functional will then be evolved in time, and in this time-evolved state (or density matrix) we compute expectation values of operators, or correlation functions. We then see that the above formalism permits us to study these situations in great generality.

As mentioned before, another point of view that one may take on the "quenching" below the critical temperature is that a definite state or density matrix describing the symmetric phase is prepared as an initial condition for $t<0$ and evolved in time with the Hamiltonian that allows for broken-symmetry states. One then studies the dynamics of the phase transition, and how the system evolves in time from the initially symmetric state towards the asymmetric states.

As usual, the insertion of an operator may be achieved by inserting sources in the time evolution operators, defining the generating functionals and eventually taking functional derivatives with respect to these sources. Note that we have three evolution operators, from T to T', from T', back to T (inverse operator), and from T to $T-i\beta_i$. Since each of these operators has interactions and we want to use perturbation theory and generate the diagrammatics from the generating functionals, we use *three different sources:* a source J^+ for the evolution $T\to T',J^-$ for the branch $T'\to T$, and finally J^β for $T\to T-i\beta_i$. The denominator may be obtained from the numerator by setting $J^+=J^-=0$. Finally the generating functional

$$Z[J^+,J^-,J^\beta]=\mathrm{Tr}U(T-i\beta_i,T;J^\beta)U(T,T';J^-)\times U(T',T;J^+)$$

may be written in term of path integrals as (here we neglect the spatial arguments to avoid cluttering of notation)

$$Z[J^+,J^-,J^\beta]=\int D\Phi D\Phi_1 D\Phi_2\int\mathcal{D}\Phi^+\mathcal{D}\Phi^-\mathcal{D}\Phi^\beta\exp\left[i\int_T^{T'}\{\mathcal{L}[\Phi^+,J^+]-\mathcal{L}[\Phi^-,J^-]\}\right]\times\exp\left[i\int_T^{T-i\beta_i}\mathcal{L}[\Phi^\beta,J^\beta]\right]\ , \tag{3.9}$$

with the boundary conditions $\Phi^+(T)=\Phi^\beta(T-i\beta_i)=\Phi$, $\Phi^+(T')=\Phi^-(T')=\Phi_2$, $\Phi^-(T)=\Phi^\beta(T)=\Phi_1$. As usual, the path integrals over the quadratic forms may be done and one obtains the final result for the partition function:

$$Z[J^+,J^-,J^\beta]=\exp\left[i\int_T^{T'}dt\,[\mathcal{L}_{\mathrm{int}}(-i\delta/\delta J^+)-\mathcal{L}_{\mathrm{int}}(i\delta/\delta J^-)]\right]\exp\left[i\int_T^{T-i\beta_i}dt\,\mathcal{L}_{\mathrm{int}}(-i\delta/\delta J^\beta)\right]\times\exp\left[\frac{i}{2}\int_c dt_1\int_c dt_2 J_c(t_1)J_c(t_2)G_c(t_1,t_2)\right]\ , \tag{3.10}$$

where J_c are the currents defined on the contour of Fig. 1(b), $J^\pm,J^\beta$ [25], and G_c is the Green's function on the contour (see below), and again the spatial arguments have been suppressed.

In the two contour integrals (on t_1,t_2) in (3.10) there are altogether nine terms, corresponding to the combination of currents in each of the three branches. However, in the limit $T\to-\infty$, the contributions arising from the terms in which one current is on the $(+)$ or $(-)$ branch and another on the imaginary time segment (from T to $T-i\beta_i$) go to zero when computing correlation functions in which the external legs are at finite *real time.* For this *real-time correlation function* there is no contribution from the J^β terms that cancel between numerator and denominator, and the information on finite temperature is encoded in the boundary conditions on the Green's functions (see below). Then for the calculation of finite *real-time* correlation functions the generating functional simplifies to [27,28]

$$Z[J^+,J^-]=\exp\left\{i\int_T^{T'}dt[\mathcal{L}_{\mathrm{int}}(-i\delta/\delta J^+)-\mathcal{L}_{\mathrm{int}}(i\delta/\delta J^0)]\right\}\exp\left\{\frac{i}{2}\int_T^{T'}dt_1\int_T^{T'}dt_2 J_a(t_1)J_b(t_2)G_{ab}(t_1,t_2)\right\}\ , \tag{3.11}$$

with $a,b=+,-$.

This formulation in terms of time evolution along a contour in complex time has been used many times in nonequilibrium statistical mechanics. To our knowledge the first to use this formulation were Schwinger [29] and Keldysh [30] (see also Mills [31]). There are many articles in the literature using these techniques to study time-dependent problems. Some of the more clear articles are by Niemi and Semenoff [25], Landsman and van Weert [32], Semenoff and Weiss [33], Jordan [34], Kobes and Kowalski [35], Calzetta and Hu [28], and Paz [36] and references therein.

At first sight one seems to have complicated the situation enormously by doubling the number of fields. However, this doubling is a natural consequence of dealing with a time evolution of a *density matrix* and in general with probabilities, instead of amplitudes. Rather than computing in-out amplitudes, we are here computing expectation values or correlation functions in the time-evolved in state or density matrix [29,34].

The Green's functions that enter in the integrals along the contours in (3.10) and (3.11) are given by (see above references)

$$G^{++}(t_1,t_2)=G^{>}(t_1,t_2)\Theta(t_1-t_2)+G^{<}(t_1,t_2)\Theta(t_2-t_1)\ , \tag{3.12}$$

$$G^{--}(t_1,t_2)=G^{>}(t_1,t_2)\Theta(t_2-t_1)+G^{<}(t_1,t_2)\Theta(t_1-t_2)\ , \tag{3.13}$$

$$G^{+-}(t_1,t_2)=-G^{<}(t_1,t_2)\ , \tag{3.14}$$

$$G^{-+}(t_1,t_2)=-G^{>}(t_1,t_2)=-G^{<}(t_2,t_1)\ , \tag{3.15}$$

$$G^{<}(T,t_2)=G^{>}(T-i\beta_i,t_2)\ . \tag{3.16}$$

As usual $G^{<},G^{>}$ are homogeneous solutions of the quadratic form with appropriate boundary conditions. We will construct them explicitly later. The condition (3.16) is recognized as the periodicity condition in imaginary time [Kubo-Martin-Schwinger (KMS) condition] [37]. It is straightforward to show, using the above Green's functions, that $Z[J,J]=1$, as it must.

Although most of the details presented above on the nonequilibrium formalism are available in the literature, we included them here for self-consistency and with the intention to clarify some issues that are usually glossed over in most treatments.

We are now in condition to obtain the evolution equations for the average of the scalar field in the case when the potential is suddenly changed, to account for a sudden change in temperature from above to below the critical temperature as described by the model Hamiltonian (3.1) with (3.2). For this purpose we use the tadpole method [9], and write

$$\Phi^{\pm}(\mathbf{x},t)=\phi(t)+\Psi^{\pm}(\mathbf{x},t)\ , \tag{3.17}$$

where, again, the $\pm$ refers to the branches for forward and backward time propagation. The reason for shifting both ($\pm$) fields by the *same* classical configuration is that ϕ enters in the time evolution operator as a background c-number variable, and time evolution, both forward and backwards, is now considered in this background.

The evolution equations are obtained with the tadpole method by expanding the Lagrangian around $\phi(t)$ and considering the *linear*, cubic, quartic, and higher-order terms in $\Psi^{\pm}$ as perturbations and requiring that

$$\langle \Psi^{\pm}(\mathbf{x},t)\rangle=0\ .$$

It is a straightforward exercise to see that this is equivalent to extremizing the one-loop effective action in which the determinant (in the logdet) incorporates the boundary condition of equilibrium at time $t<0$ at the initial temperature.

To one loop we find the equation of motion,

$$\frac{d^2\phi(t)}{dt^2}+m^2(t)\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}(-i)G_k(t,t)=0\ , \tag{3.18}$$

where $G_k(t,t)=G_k^{<}(t,t)=G_k^{>}(t,t)$ is the spatial Fourier transform of the equal-time Green's function.

At this point, we would like to remind the reader that

$$-iG_k(t,t)=\langle \Psi_{\mathbf{k}}^{+}(t)\Psi_{-\mathbf{k}}^{+}(t)\rangle$$

is a *positive-definite quantity* (because the field Ψ is real) and as we argued before (and will be seen explicitly shortly) this Green's function grows in time because of the instabilities associated with the phase transition and domain growth [15,16].

These Green's functions are constructed out of the homogeneous solutions to the operator of quadratic fluctuations,

$$\left[\frac{d^2}{dt^2}+\mathbf{k}^2+M^2(t)\right]\mathcal{U}_k^{\pm}=0\ , \tag{3.19}$$

$$M^2(t)=\left[m^2+\frac{\lambda}{2}\phi_i^2\right]\Theta(-t)+\left[-\mu^2+\frac{\lambda}{2}\phi^2(t)\right]\Theta(t)\ . \tag{3.20}$$

The boundary conditions on the homogeneous solutions are

$$\mathcal{U}_k^{\pm}(t<0)=e^{\mp i\omega_{<}(k)t}\ , \tag{3.21}$$

$$\omega_{<}(k)=\left[\mathbf{k}^2+m^2+\frac{\lambda}{2}\phi_i^2\right]^{1/2}\ , \tag{3.22}$$

where ϕ_i is the value of the classical field at time $t<0$ and is the initial boundary condition on the equation of motion. Truly speaking, starting in a fully symmetric phase will force $\phi_i=0$, and the time evolution will maintain this value; therefore, we admit a small explicit symmetry-breaking field in the initial density matrix to allow for a small ϕ_i. The introduction of this initial condition seems artificial since we are studying this situation of cooling down from the symmetric phase.

However, we recognize that the phase transition from the symmetric phase occurs via formation of domains (in the case of a discrete symmetry), inside which the order parameter acquires nonzero values. The domains will have the same probability for either value of the field and the volume average of the field will remain zero. These domains will grow in time; this is the phenomenon of the phase separation and spinodal decomposition familiar in condensed matter physics. Our evolution equations presumably will apply to the coarse-grained average of the scalar field inside each of these domains. This average will only depend on time. Thus we interpret φ_i as corresponding to the coarse-grained average of the field in each of these domains. The question of initial conditions on the scalar field is also present (but usually overlooked) in the slow-rollover scenarios, but as we will see later, it plays a fundamental role in the description of the evolution.

The identification of the initial value φ_i with the average of the field in each domain is certainly a plausibility argument to justify an initially small asymmetry in the scalar field which is necessary for the further evolution of the field, and is consistent with the usual assumption within the slow-rollover scenario.

We are currently studying the dynamics of the phase transition from the symmetric phase by looking at the composite operator $\Phi^2(\mathbf{x},t)$ which measures the fluctuations, and will report on our studies in a forthcoming article [38]. An alternative approach using this composite operator has also been proposed by Lawrie [39].

The boundary conditions on the mode functions $\mathcal{U}_k^{\pm}(t)$ correspond to "vacuum" boundary conditions of positive and negative frequency modes (particles and antiparticles) for $t<0$.

Finite temperature enters through the periodicity conditions (3.16) and the Green's functions are

$$G_k^{>}(t,t')=\frac{i}{2\omega_<(k)}\frac{1}{1-e^{-\beta_i\omega_<(k)}}\times[\mathcal{U}_k^+(t)\mathcal{U}_k^-(t')+e^{-\beta_i\omega_<(k)}\mathcal{U}_k^-(t)\mathcal{U}_k^+(t')] , \tag{3.23}$$

$$G_k^{<}(t,t')=G^{>}(t',t) . \tag{3.24}$$

Summarizing, the effective equations of motion to one loop that determine the time evolution of the scalar field are

$$\frac{d^2\phi(t)}{dt^2}+m^2(t)\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{\mathcal{U}_k^+(t)\mathcal{U}_k^-(t)}{2\omega_<(k)}\coth\left[\frac{\beta_i\omega_<(k)}{2}\right]=0 , \tag{3.25}$$

$$\left[\frac{d^2}{dt^2}+\mathbf{k}^2+M^2(t)\right]\mathcal{U}_k^{\pm}=0 , \tag{3.26}$$

with (3.20) and (3.21).

This set of equations is too complicated to attempt an analytic solution; we will study this system numerically shortly.

However, before studying numerically these equations, one recognizes that there are several features of this set of equations that reveal the basic physical aspects of the dynamics of the scalar field.

(i) The effective evolution equations are *real*. The mode functions $\mathcal{U}_k^{\pm}(t)$ are a complex conjugate of each other as may be seen from the time-reversal symmetry of the equations and the boundary conditions (3.21). This situation must be contrasted with the expression for the effective potential for the *analytically continued modes.*

(ii) Consider the situation in which the initial configuration of the classical field is near the origin $\phi_i\approx 0$; for $t>0$, the modes for which $\mathbf{k}^2<(k_{\max})^2$, $(k_{\max})^2=\mu^2-(\lambda/2)\phi_i^2$ are *unstable.*

In particular, for early times ($t>0$), when $\phi_i\approx 0$, these unstable modes behave approximately as

$$\mathcal{U}_k^+(t)=A_k e^{W_k t}+B_k e^{-W_k t} , \tag{3.27}$$

$$\mathcal{U}_k^-(t)=[\mathcal{U}_k^+(t)]^* , \tag{3.28}$$

$$A_k=\frac{1}{2}\left[1-i\frac{\omega_<(k)}{W_k}\right],\quad B_k=A_k^* , \tag{3.29}$$

$$W_k=\left[\mu^2-\frac{\lambda}{2}\phi_i^2-\mathbf{k}^2\right]^{1/2} . \tag{3.30}$$

Then the early time behavior of $-iG_k(t,t)$ is given by

$$-iG_k(t,t)\approx\frac{1}{2\omega_<(k)}\left[1+\frac{1}{2}\frac{\mu^2+m^2}{\mu^2-\frac{\lambda}{2}\phi_i^2-k^2}\times[\cosh(2W_kt)-1]\right]\times\coth[\beta_i\omega_<(k)/2] . \tag{3.31}$$

This early time behavior coincides with the Green's function of Guth and Pi [15] and Weinberg and Wu [16] for the inverted harmonic oscillators when our initial state (density matrix) is taken into account.

Our evolution equations, however, permit us to go beyond the early time behavior and to incorporate the nonlinearities that will eventually shut off the instabilities.

These early-stage instabilities and subsequent growth of fluctuations and correlations are the hallmark of the process of phase separation and precisely the instabilities that trigger the phase transition.

It is clear from the above equations of evolution that the description in terms of inverted oscillators will only be valid at very early times.

At early times, the *stable* modes for which $\mathbf{k}^2>(k_{\max})^2$ are obtained from (3.27), (3.28), and (3.29) by the analytic continuation

$$W_k\to -i\omega_>(k)=\left[\mathbf{k}^2-\mu^2+\frac{\lambda}{2}\phi_i^2\right]^{1/2} .$$

For $t<0, \mathcal{U}_k^+(t)\mathcal{U}_k^-(t)=1$, and one obtains the usual result for the evolution equation,

$$\frac{d^2\phi(t)}{dt^2}+\frac{dV_{\text{eff}}(\phi)}{d\phi}=0 ,$$

with $V_{\text{eff}}(\phi)$ the finite-temperature effective potential, but for $t<0$ there are no unstable modes.

It becomes clear, however, that for $t>0$ there are no *static* solutions to the evolution equations for $\phi(t)\neq 0$.

(iii) Coarsening: As the classical expectation value $\phi(t)$ "rolls down" the potential hill, $\phi^2(t)$ increases and

$$[k_{\max}(t)]^2=\mu^2-\frac{\lambda}{2}\phi^2(t)$$

decreases, and only the very long-wavelength modes remain unstable, until, for a particular time t_s,

$[k_{\max}(t_s)]^2=0$. This occurs when $\phi^2(t_s)=2\mu^2/\lambda$; this is the inflexion point of the tree-level potential. In statistical mechanics this point is known as the "classical spinodal point" and t_s as the "spinodal time" [23,24]. When the classical field reaches the spinodal point, all instabilities shut off. From this point on, the dynamics is oscillatory and this period is identified with the "reheating" stage in cosmological scenarios [11,12].

It is clear from the above equations of evolution that the description in terms of inverted oscillators will only be valid at small positive times, as eventually the unstable growth will shut off.

The value of the spinodal time depends on the initial conditions of $\phi(t)$. If the initial value ϕ_i is very near the classical spinodal point, t_s will be relatively small and there will not be enough time for the unstable modes to grow too much. In this case, the one-loop corrections for small coupling constant will remain perturbatively small. On the other hand, however, if $\phi_i \approx 0$, and the initial velocity is small, it will take a very long time to reach the classical spinodal point. In this case the unstable modes *may grow dramatically, making the one-loop corrections non-negligible even for small coupling.* These initial conditions of small initial field and velocity are precisely the "slow-rollover" conditions that are of interest in cosmological scenarios of "new inflation."

(iv) Renormalization: As argued above, for $t>0$, there are no static solutions to the equation of motion for the scalar field. The mode functions $\mathcal{U}_k^{\pm}(t)$ depend *implicitly* on the field $\phi(t)$. As in the usual situation, one expects ultraviolet divergences in the one-loop correction. It is not clear from the equation of motion for the scalar field whether these divergences may be absorbed in a redefinition of the mass and coupling constant, or canceled by *local* counterterms. Since we want to study the time evolution and be able to extract meaningful information, we must first understand the renormalization aspects of the effective equation of motion.

The ultraviolet divergences must be absorbed in $m^2(t)$ and λ, whose coefficients in the equation of motion are $\phi(t)$ and $\phi^3(t)$, respectively, i.e., *time dependent.* Since, as mentioned previously, for $t<0$ the situation corresponds to the usual case of the static effective potential, renormalization proceeds in the standard manner. However, for $t>0$ the situation is different and to understand it we need the large k behavior of the mode functions. We study the short-wavelength behavior by a WKB-type analysis: We define $\epsilon=1/k$ and divide the equation for the mode functions by k^2, thus obtaining the equation

$$\left[\epsilon^2\frac{d^2}{dt^2}+1+\epsilon^2M^2(t)\right]\mathcal{U}_k^{\pm}(t)=0\ , \tag{3.32}$$

with the boundary conditions (3.21). Let us define the functions $\mathcal{V}_k^{\pm}(t)$ as the two linearly independent solutions for $t>0$ to Eq. (3.32) with the boundary conditions $\mathcal{V}_k^{\pm}(0^+)=1$. The mode functions $\mathcal{U}_k^{\pm}(t)$ solutions to (3.32) with the boundary conditions (3.21) are written as

$$\mathcal{U}_k^{+}(t)=c_k\mathcal{V}_k^{+}(t)+d_k\mathcal{V}_k^{-}(t)\ , \tag{3.33}$$

$$\mathcal{U}_k^{-}(t)=[\mathcal{U}_k^{+}(t)]^*\ . \tag{3.34}$$

The coefficients c_k, d_k are obtained from the matching conditions at $t=0$. We propose a WKB ansatz for the mode functions $\mathcal{V}_k^{\pm}(t)$:

$$\mathcal{V}_k^{\pm}(t)=e^{iS^{\pm}(t)/\epsilon}\ , \tag{3.35}$$

$$S^{\pm}(t)=\sum_{n=0}\epsilon^nS_n^{\pm}(t)\ . \tag{3.36}$$

Inserting this ansatz in (3.32) and comparing powers of ϵ we find the asymptotic behavior for large k to be

$$\mathcal{V}_k^{\pm}(t)=\exp\left[\mp i\left[kt+\frac{1}{2k}\int_0^t M^2(t')dt'\right]\right]\times\left[1-\frac{1}{4k^2}[M^2(t)-M^2(0^+)]\right]+\cdots\ . \tag{3.37}$$

The leading behavior for large k of the coefficients is found to be

$$c_k=\frac{1}{2}\left[1+\frac{\omega_<(k)}{k+M^2(0^+)/2k}\right]+\cdots\ , \tag{3.38}$$

$$d_k=\frac{1}{2}\left[1-\frac{\omega_<(k)}{k+M^2(0^+)/2k}\right]+\cdots\ . \tag{3.39}$$

Inserting these results for the large k behavior in the one-loop contribution, it is straightforward to find that the divergent terms are independent of temperature and we obtain

$$\int\frac{d^3k}{(2\pi)^3}\frac{\mathcal{U}_k^{+}(t)\mathcal{U}_k^{-}(t)}{2\omega_<(k)}\coth\left[\frac{\beta_i\omega_<(k)}{2}\right]=\frac{1}{8\pi^2}\Lambda^2-\frac{1}{8\pi^2}\left[-\mu^2+\frac{\lambda}{2}\phi^2(t)\right]\ln\left[\frac{\Lambda}{\kappa}\right]+\text{finite}\ , \tag{3.40}$$

where Λ is an upper momentum cutoff, κ a renormalization scale, and the finite part is time, temperature, and κ dependent.

It is clear that these divergences may be canceled by local counterterms of the usual form, where the mass counterterm depends (locally) on time and changes suddenly at $t=0$. The coupling constant is renormalized in the usual manner.

It is important to point out that the integral for the one-loop correction may be split into the contribution from the unstable modes $k^2<[k_{\max}(t)]^2$ and that of the stable modes $k^2>[k_{\max}(t)]^2$. It is only the latter that requires renormalization and where the divergences reside. The contribution from the unstable modes is *finite* and does not require renormalizations. The renormalization of the effective action has been done in an alternative manner using dimensional regularization by Avan and de Vega [41].

IV. ANALYSIS OF THE EVOLUTION

As mentioned previously within the context of coarsening, when the initial value of the scalar field $\phi_i \approx 0$, and the initial temporal derivative is small, the scalar field slowly rolls down the potential hill. But during the time while the scalar field remains smaller than the "spinodal" value, the unstable modes grow and the one-loop contribution grows consequently. For a "slow-rollover" condition, the field remains very small $[\phi^2(t) \ll 2\mu^2/\lambda]$ for a long time, and during this time the unstable modes grow exponentially. The stable modes, on the other hand, give an oscillatory contribution which is bound in time, and for weak coupling remains perturbatively small at all times.

Then, for a "slow-rollover" situation and for weak coupling, only the unstable modes will yield to an important contribution to the one-loop correction. Thus, in the evolution equation for the scalar field, we will keep only the integral over the *unstable modes* in the one-loop correction.

Phenomenologically, the coupling constant in these models is bound by the spectrum of density fluctuations to be within the range $\lambda_R \approx 10^{-12} - 10^{-14}$ [11]. The stable modes will *always* give a *perturbative* contribution, whereas the unstable modes grow exponentially in time, thus raising the possibility of giving a non-negligible contribution.

With the purpose of numerical analysis of the effective equations of motion, it proves convenient to introduce the following dimensionless variables:

$$\tau = \mu_R t, \quad q = k/\mu_R , \tag{4.1}$$

$$\eta^2(t) = \frac{\lambda_R}{6\mu_R^2}\phi^2(t), \quad L^2 = \frac{m_R^2 + \frac{1}{2}\lambda_R \phi_i^2}{\mu_R^2} , \tag{4.2}$$

and to account for the change from the initial temperature to the final temperature $(T_i > T_c, T_f < T_c)$ we parametrize [40]

$$m^2 = \mu_R(0)\left[\frac{T_i^2}{T_c^2} - 1\right] , \tag{4.3}$$

$$\mu_R = \mu_R(0)\left[1 - \frac{T_f^2}{T_c^2}\right] , \tag{4.4}$$

where the subscript R stands for renormalized quantities, $-\mu_R(0)$ is the renormalized zero-temperature "negative mass squared," and $T_c^2 = 24\mu_R^2(0)/\lambda_R$. Furthermore, because $[k_{\max}(t)]^2 \le \mu_R^2$ and $T_i > T_c$, for the unstable modes $T_i \gg [k_{\max}(t)]$ and we can take the high-temperature limit $\coth[\beta_i \omega_<(k)/2] \approx 2T_i/\omega_<(k)$. Finally, the effective equations of evolution for $t > 0$, keeping in the one-loop contribution only the unstable modes as explained above $(q^2 < [q_{\max}(\tau)]^2)$, become, after using $\omega_<^2 = \mu_R^2(q^2 + L^2)$,

$$\frac{d^2}{d\tau^2}\eta(\tau) - \eta(\tau) + \eta^3(\tau) + g\eta(\tau)\int_0^{q_{\max}(\tau)} q^2 \frac{\mathcal{U}_q^+(\tau)\mathcal{U}_q^-(\tau)}{q^2 + L^2} dq = 0 , \tag{4.5}$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - [q_{\max}(t)]^2\right]\mathcal{U}_q^{\pm}(\tau) = 0 , \tag{4.6}$$

$$[q_{\max}(\tau)]^2 = 1 - 3\eta^2(\tau) , \tag{4.7}$$

$$g = \frac{\sqrt{6\lambda_R}}{2\pi^2}\frac{T_i}{T_c[1 - T_f^2/T_c^2]} . \tag{4.8}$$

For $T_i \ge T_c$ and $T_f \ll T_c$ the coupling (4.8) is bound within the range $g \approx 10^{-7} - 10^{-8}$. The dependence of the coupling with the temperature reflects the fact that at higher temperatures the fluctuations are enhanced. It is then clear that the contribution from the stable modes is *always perturbatively small,* and only the unstable modes may introduce important corrections if they are allowed to grow for a long time.

From (4.5) we see that the quantum corrections act as a *positive dynamical renormalization* of the "negative mass" term that drives the rolling down dynamics. It is then clear that the quantum corrections tend to *slow down the evolution.*

In particular, if the initial value $\eta(0)$ is very small, the unstable modes grow for a long time before $\eta(\tau)$ reaches the spinodal point $\eta(\tau_s) = 1/\sqrt{3}$, at which point the instabilities shut off. If this is the case, the quantum corrections introduce substantial modifications to the classical equations of motion, thus becoming nonperturbative. If $\eta(0)$ is closer to the classical spinodal point, the unstable modes do not have time to grow dramatically and the quantum corrections are perturbatively small.

Thus we conclude that the initial conditions on the field determine whether or not the quantum corrections are perturbatively small.

Although the system of equations (4.5) and (4.6) are coupled, nonlinear, and integro-differential, they may be integrated numerically. Figures 2, 3, and 4 depict (a) the solutions for the classical [42] (solid lines) and quantum (dashed lines) evolution; (b) the quantum correction, i.e., the fourth term in (4.5) *including the coupling g;* (c),(d) the classical (solid lines) and quantum (dashed lines) velocities $d\eta(\tau)/d\tau$ and $[q_{\max}(\tau)]^2$. For the numerical integration, we have chosen $L^2 = 1$; the results are only weakly dependent on L, and taking $g = 10^{-7}$, we have varied the initial condition $\eta(0)$ but used $[d\eta(\tau)/d\tau]_{\tau=0} = 0$.

We recall from a previous discussion that $\eta(\tau)$ should be identified with the average of the field within a domain. We are considering the situation in which this average is very small, according to the usual slow-rollover hypothesis, and for which the instabilities are stronger.

In Fig. 2(a), $\eta(0) = 2.3 \times 10^{-5}$. We begin to see that the quantum corrections become important at $t \approx 10/\mu_R$ and slow down the dynamics. By the time that the *classical* evolution reaches the minimum of the classical potential at $\eta = 1$, the quantum evolution has just reached the classical spinodal $\eta = 1/\sqrt{3}$. We see in Fig. 2(b) that the quantum correction becomes large enough to change the sign of the "mass term." The field continues its evolution towards larger values, however, because the velocity is different from zero, attaining a maximum [Fig. 2(c)] around the time when the quantum correction attains its maximum. As η gets closer to the classical spinodal point, the unstability shuts off as is clearly seen in Fig. 2(d) and the quantum correction arising from the unstable modes becomes perturbatively small. From the spinodal point onwards, the field evolves towards the minimum and begins to oscillate around it; the quantum correction will be perturbatively small, as all the instabilities had shut off. Higher-order corrections will introduce a damping term as quanta may decay in elementary excitations of the true vacuum.

Figures 3(a)–3(d) show a more marked behavior for $\eta(0)=2.27\times10^{-5}$, $d\eta(0)/d\tau=0$; note that the classical evolution of the field has reached beyond the minimum of the potential at the time when the quantum evolution has just reached the classical spinodal point. Figure 3(b) shows that the quantum correction becomes larger than 1, and dramatically slows down the evolution; again because the velocity is different from zero [Fig. 3(c)] the field continues to grow. The velocity reaches a maximum and begins to drop. Once the field reaches the spinodal again the instabilities shut off [Figs. 3(b) and 3(d)] and from this point the field will continue to evolve towards the minimum of the potential, but the quantum corrections will be perturbatively small.

Figures 4(a)-4(d) show a dramatic behavior for $\eta(0)=2.258\times10^{-5}$, $d\eta(0)/d\tau=0$. The unstable modes have enough time to grow so dramatically that the quantum correction [Fig. 4(b)] becomes extremely large $\gg 1$ [Fig. 4(b)], overwhelming the "negative mass" term near the origin. The dynamical time-dependent potential now becomes a *minimum* at the origin and the quantum evo-

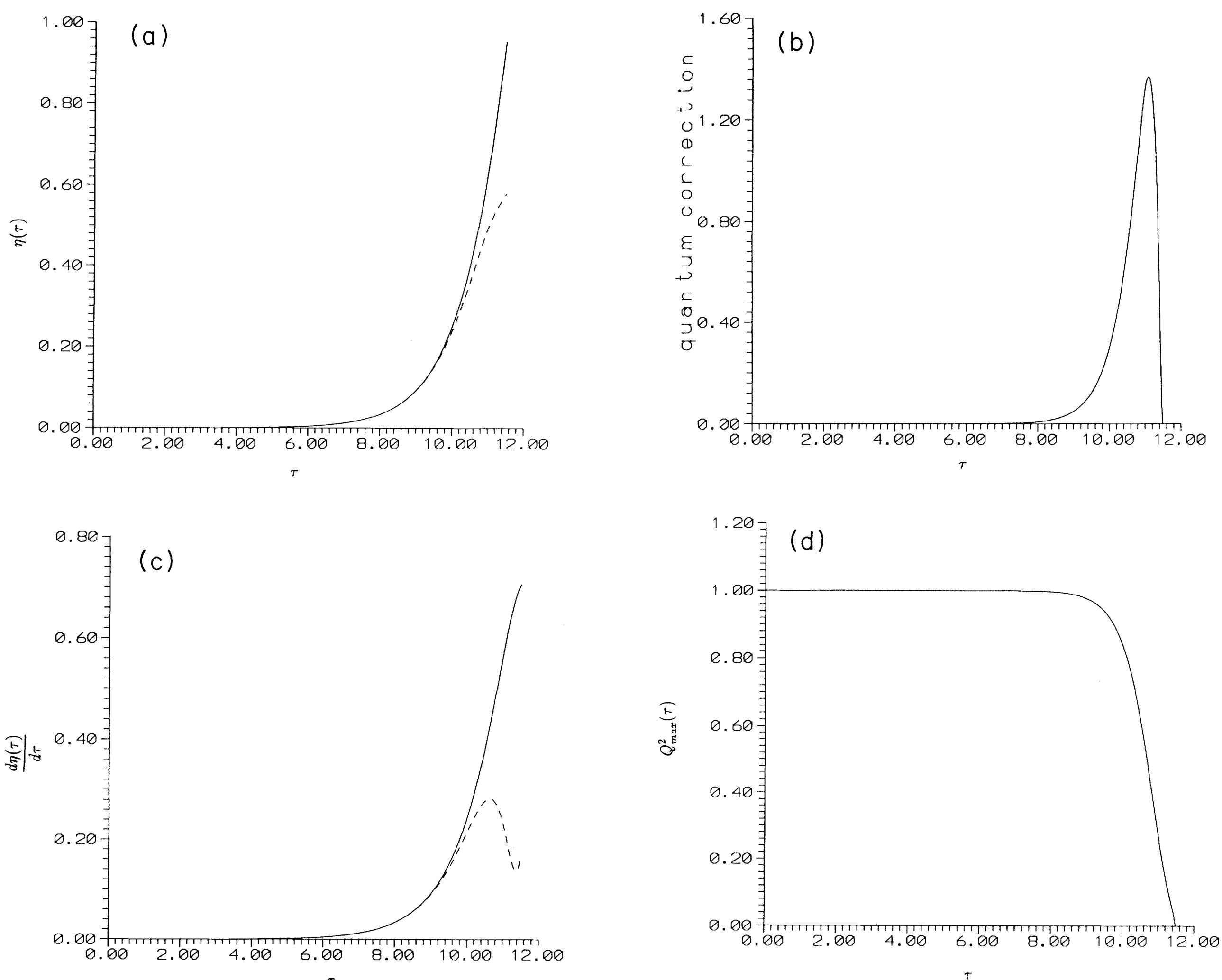

FIG. 2. (a) η vs τ (notation in the text) for $g=10^{-7}$, $\eta(0)=2.3\times10^{-5}$, $\eta'(0)=0$, $L=1$. The solid line is the classical evolution; the dashed line is the evolution from the one-loop corrected equation (4.5). (b) One-loop contribution including the coupling g for the values of the parameters used in (a). (c) Velocity $\partial\eta/\partial\tau$; same values and conventions as in (a). (d) $[q_{\max}(\tau)]^2$ vs τ for the same values as in (a).

lution begins to *oscillate* near $\eta=0$. The contribution of the unstable modes has become *nonperturbative,* and certainly our one-loop approximation breaks down.

As the initial value of the field gets closer to zero, the unstable modes grow for a very long time. At this point, we realize, however, that this picture cannot be complete. To see this more clearly, consider the case in which the initial state or density matrix corresponds exactly to the symmetric case. $\eta=0$ is necessarily, by symmetry, a fixed point of the equations of motion. Beginning from the symmetric state, the field will *always remain* at the origin, and though there will be strong quantum and thermal fluctuations, these are symmetric and will sample field configurations with opposite values of the field with equal probability.

In this situation, and according to the picture presented above, one would then expect that the unstable modes will grow indefinitely because the scalar field does not roll down and will never reach the classical spinodal point, thus shutting off the instabilities. What is missing in this picture and the resulting equations of motion is a self-consistent treatment of the unstable fluctuations, which must necessarily go beyond one loop. A more sophisticated and clearly nonperturbative scheme must be invoked that will incorporate coarsening, that is, the shift with time of the unstable modes towards longer wavelength and the eventual shutting off of the instabilities. We are currently [38] exploring a Hartree approximation that will incorporate self-consistently these features. Another possible approach would be a variational treatment as advocated in Refs. [18,19] or as proposed by Lawrie [39].

V. CONCLUSIONS

We have studied the effective equations of motion for the spatially independent average of a scalar field evolving out of equilibrium after a second-order phase transition. After pointing out the severe shortcomings in the usual description in terms of the effective potential, we considered the situation in which a scalar field theory is

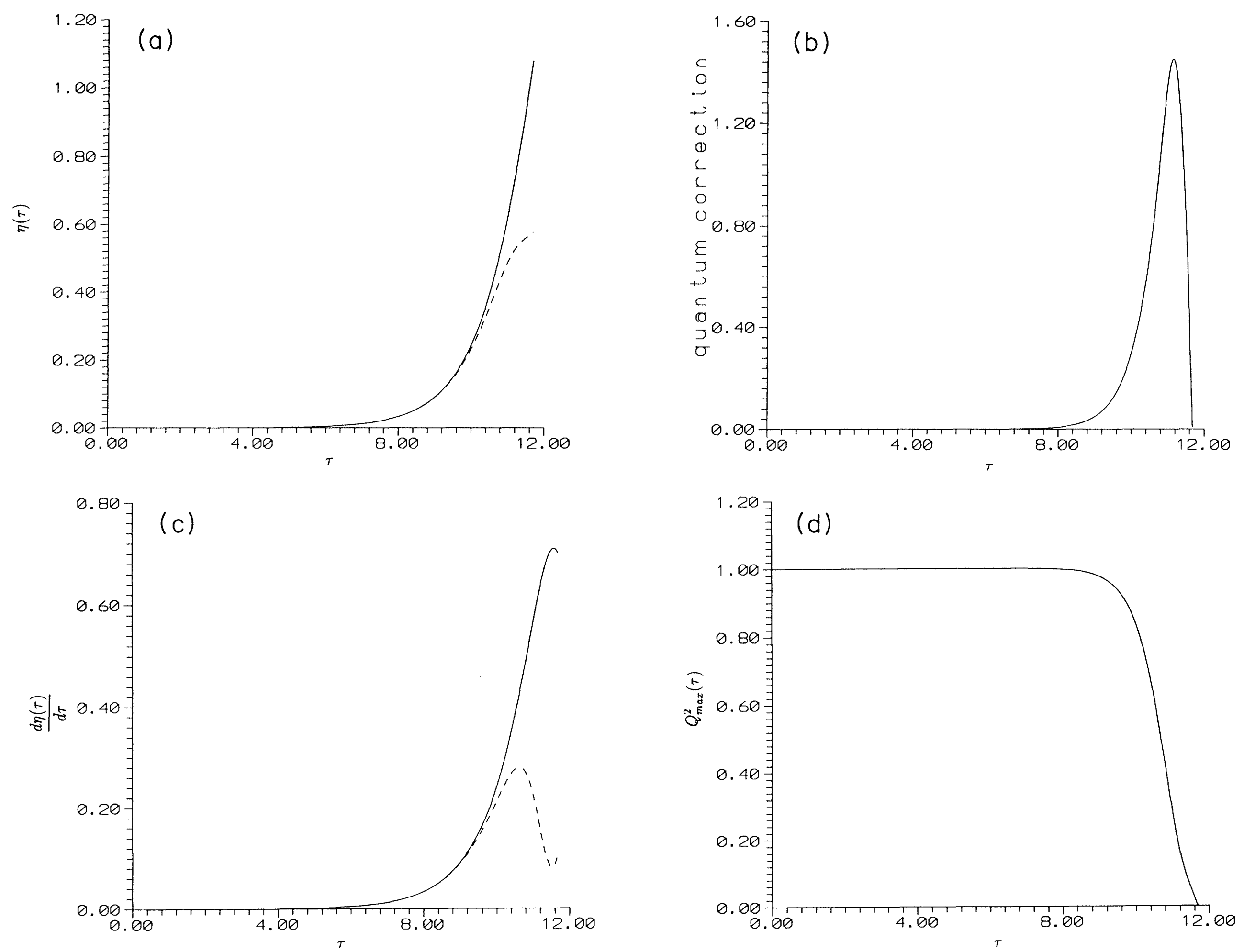

FIG. 3. (a) η vs τ with $g=10^{-7}\eta(0)=2.27\times10^{-5}$, $\eta'(0)=0$, $L=1$. Same conventions for solid and dashed lines as in Fig. 2(a). (b) One-loop contribution to the equations of motion including the coupling g for the values of the parameters used in (a). (c) Velocity $\partial\eta/\partial\tau$, same values and conventions as in (a). (d) $[q_{\max}(\tau)]^2$ vs τ for the same values as in (a).

suddenly cooled below the critical temperature from an initial state at a temperature higher than the critical temperature. We use the tools of statistical mechanics out of equilibrium to study the *real-time* dynamics during a "slow-rollover stage." The effective nonequilibrium equations of motion are studied both analytically and numerically to one-loop level. We find that the unstable growth of long-wavelength fluctuations that trigger the process of phase separation is responsible for a very marked behavior in the time evolution of the scalar field. Even for very weak couplings, consistent with the bounds from density fluctuations, for the case of "slow-rollover" initial conditions, the time evolution is dramatically *slowed down* as a consequence of this unstable growth which signals the onset of the phase transition. When the scalar field is very close to the origin (at the local maximum of the potential) the unstable modes can grow for a long time and the effect of the quantum (and thermal) corrections become very large and eventually nonperturbative. We give a qualitative and quantitative description of the coarsening process and the eventual shut off of the instabilities.

We argue that a comprehensive treatment and understanding of the late-stage dynamics of the phase transition, and time evolution of the scalar field, involves a *nonperturbative* treatment out of equilibrium. This treatment must incorporate not only the growth but also the coarsening features when the initial state is symmetric and the initial value of the scalar field is zero, thus remaining zero during the time evolution.

This is the next step in a consistent analysis of the dynamics; work on this problem is currently underway. Clearly, all this must be extended to the case of inflationary cosmology; in particular a description of the dynamics out of equilibrium in a de Sitter background is the next stage. In this case the physics of coarsening will be complicated by the redshift of wave vectors and more modes enter the unstable regime; clearly a deeper understanding of these instabilities is necessary.

However, we believe that the results presented in this

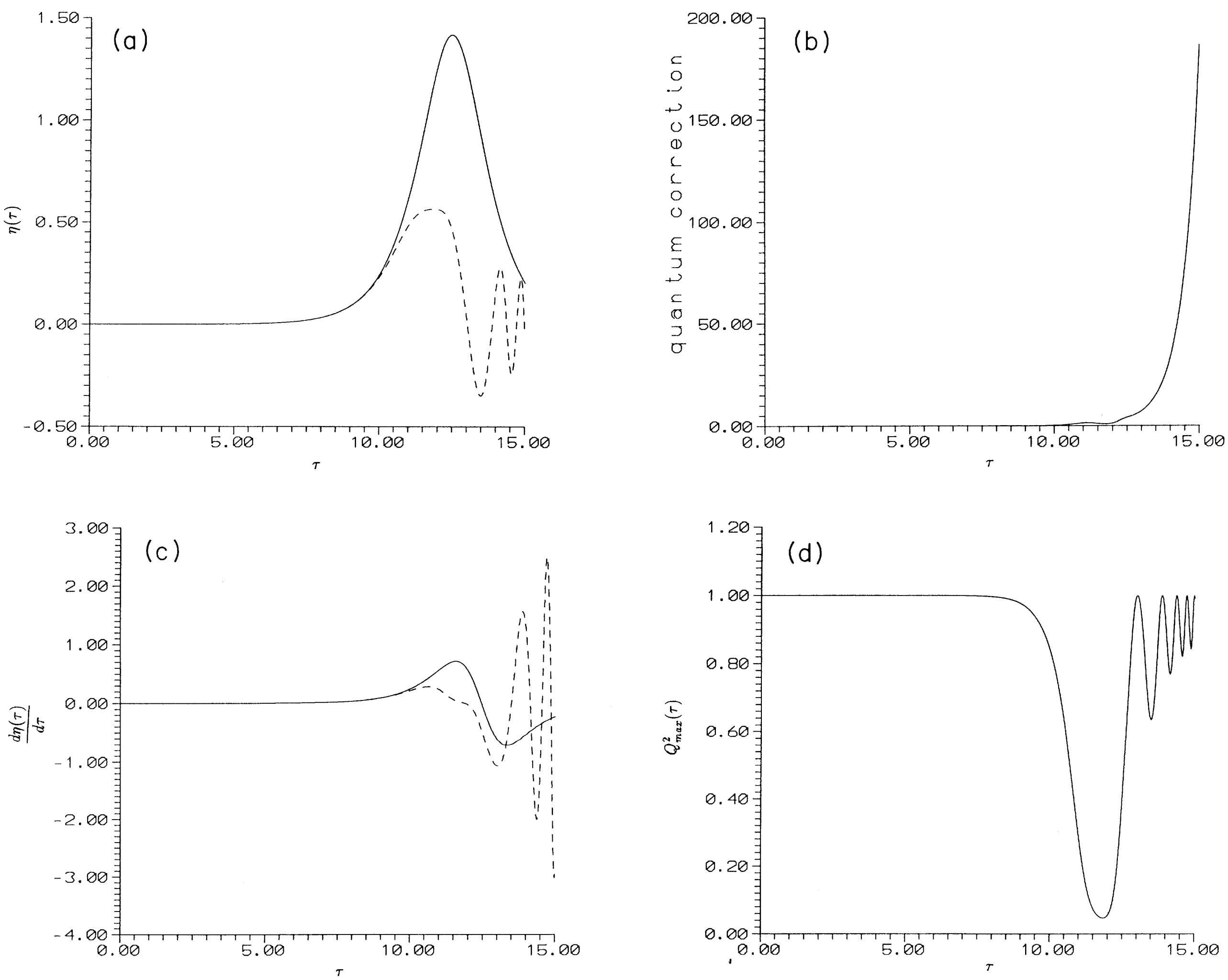

FIG. 4. (a) η vs τ with $g=10^{-7}\eta(0)=2.258\times10^{-5}$, $\eta'(0)=0, L=1$. Same conventions for solid and dashed lines as in Fig. 2(a). (b) One-loop contribution including the coupling g for the values of the parameters used in (a). (c) Velocity $\partial\eta/\partial\tau$; same values and conventions as in (a). (d) $[q_{\max}(\tau)]^2$ vs τ for the same values as in (a).

article provide a novel insight into the long-standing problem of the dynamics of a phase transition and the ensuing evolution out of equilibrium when quantum and thermal fluctuations are taken into account. It also points out that this problem must be studied as a fully time-dependent process and that one must abandon the usual treatment in terms of the effective potential.

The potential consequences for inflationary scenarios are obvious; the instabilities and growth of long-wavelength fluctuations enhance the quantum and thermal corrections that become large. As a consequence, the "slow-rollover" stage is delayed; the scalar field remains near the "false vacuum" for a longer period of time, giving rise to a longer inflationary stage and a delayed completion of the phase transition.

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[40] This may be done formally be generalizing the method proposed by S. Weinberg [9] and introducing a time-dependent counterterm in the original Hamiltonian of the form $(\lambda/24)[T_i^2\theta(-t)+T_f^2\theta(t)]$ and canceling this counterterm against the one loop contribution for $t<0$.

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