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Thermodynamic stability of Kerr black holes

Osamu Kaburaki
Astronomical Institute, Faculty of Science, Tohoku University, Sendai 980, Japan

Isao Okamoto and Joseph Katz*
Division of Theoretical Astrophysics, National Astronomical Observatory, Mizusawa, Iwate 023, Japan
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Isolated Kerr black holes are thermodynamically stable with respect to axisymmetric perturbations. Holes in a heat bath may be stable with large enough angular momentum. Isolated or not, at a fixed angular velocity divided by the temperature they are all unstable. The stability of the "inner horizons" has also been considered. Isolated inner horizons are stable. With different constraints, they are either stable or unstable. The analysis concerns pure black holes. Thermal radiation around black holes is treated as one of their possible circumstances. The effects of back reaction on the metric and nonconstancy of temperature in a heat bath have been neglected.

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I. INTRODUCTION

What can be said about the thermodynamic stability of uncharged black holes, if all that one knows is the equilibrium Bekenstein-Hawking entropy $S(M,J)$ [1], a function of the mass-energy M and the angular momentum J of the hole, when back-reaction effects are ignored? Much is known on the stability of Schwarzchild holes [2,3] but little is known about Kerr and Kerr-Newman holes.

A characteristic feature of the thermodynamics of self-gravitating systems is that the stability limits of different ensembles are not the same. Discussions based on the sign of the heat capacities, their zero and infinities, do not have the same meaning in gravitating systems and in classical thermodynamics, and have led to various claims about the stability of rotating and charged black holes [4–6] that were not always strongly substantiated. This has been well illustrated for stellar clusters in a box [7–9]. Stable isolated stellar clusters have positive heat capacities at constant volume C_V when they are very hot but negative C_V otherwise, and they become unstable going from $C_V<0$ on the stable to $C_V>0$ on the unstable sides. Stable clusters in a heat bath have positive C_V. Unstable clusters, isolated or in a bath, have C_V's of both signs. Thus, discussions of stability based on classical analogues and, in particular, on the sign of C_V are not reliable in gravitating systems. Conclusions based on C_V must be supported by other means.

The situation with black-hole thermodynamics is even more delicate than with star clusters. This is because while very hot clusters behave practically like a perfect gas with negligible self-gravitational energy, black holes have no nonrelativistic limit at all. A more secure method for testing stability is that of a linear series which uses precisely and only thermodynamic functions such as the one we are given here: $S(M,J)$. This method, with all its limitations, has the advantage of giving precious information at very little cost on a subject in which few solid results are yet known.

II. THE TURNING POINT METHOD

Poincaré [10] invented a powerful method for separating stable and unstable configurations in a one-parameter series of equilibria. The method has been used by Jeans [11], Lyttelton [12], and Ledoux [13] in their basic treatises. Wheeler [14] rederived Poincaré's method and found stable configurations of cold stars (see also Thorne [15]). The same result can be derived by the turning point method in a more simple way [16]. Bardeen, Thorne, and Meltzer [17] described a method similar to Wheeler's and valid for hot isentropic stellar models. Lynden-Bell and Wood [7] were actually the first to apply the Poincaré method to thermodynamic systems and found stable isothermal spheres of gravitating particles.

Suppose one has a series of equilibrium configurations characterized by a parameter y (a linear series), Poincaré shows that changes of equilibrium occur only where two or more series cut each other (at bifurcations) or where they merge (turning points). Thus, along the linear series, if one point represents a stable configuration, all the points of the branch between two turning points or bifurcations are stable. Along a line, stable configurations may become unstable, unstable ones become either stable or more unstable. The nature in the change of stability is obtained from an eigenvalue equation. There are, however, exceptions where the eigenvalue equation need not be solved. For instance, Wheeler noted that for cold stars the linear series also gave the number of unstable modes of each equilibrium. A similar situation exists in hot isentropic models. Katz [18] has shown that the number of unstable modes of an equilibrium may, in many cases of astrophysical interest, be deduced from topological

*Permanent address: The Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel.

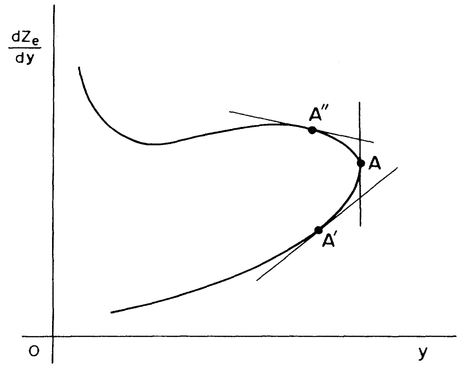

FIG. 1. Example of stability curves with turning points. A change of stability can only occur at a point A with a vertical tangent.

properties of the linear series without having to solve an eigenvalue equation. The principle is based on plotting the appropriate quantities. Let us review the method when there are only turning points; this is the case in black-hole thermodynamics. Proofs are given in the original references.

Suppose $Z(x^i,y)$ is a distribution function whose extrema $\partial Z/\partial x^i=0$ define equilibrium configurations that are stable if the extremal value of Z is a maximum. Stable solutions have a matrix $-(\partial^2 Z/\partial x^i \partial x^j)_e$ with a positive spectrum of eigenvalues $0<\lambda_1<\lambda_2<\cdots$. The x^i's may be functions with compact support, in which case the spectrum is also discrete. If it is one dimensional as in spherical systems, the spectrum is nondegenerate.

Now consider the equilibrium value $Z_e(y) = Z[X^i(y),y]$, $X^i(y)$ is a solution of $\partial Z/\partial x^i=0$. Suppose the derivative dZ_e/dy, plotted versus y has the topology shown in Fig. 1. Changes of stability will only occur at points such as A where tangents are vertical. The branch with *negative* slopes *near* A, such as point A'', is a branch of *unstable* configurations. The branch with a *positive* slope *near* A, such as A', is one of the *more stable* configurations. If the lower branch is stable then $\lambda_1>0$ and on the upper branch, $\lambda_1<0<\lambda_2<\lambda_3<\cdots$. This method found useful applications in engineering [19] and in astrophysics [20] and as been refined in several ways [21].

A statistical mechanics of black holes has been developed by Zurek and Thorne [22], and Thorne, Price, and Macdonald [23]; see also York [24]. This means that they have a $Z(x^i,M,J,Q)$ for black holes whose maximum value in Planck units $c=G=\hbar=8\pi k_B=1$ is the Bekenstein-Hawking entropy $S(M,J,Q)$.

III. THE STABILITY OF SCHWARZSCHILD HOLES

It is most indicated to apply the turning point method to Schwarzschild holes first. Here we know results from other, more complete methods, and comparison among them will emphasize the limitations of our standpoint and also value of the Poincaré method in this case.

A. Isolated systems

If a system under consideration is isolated from its surroundings the total entropy S of the system is the function Z_e which is extremum in equilibrium configurations. A thermodynamic variable in this case is the total mass-energy E of the system. The conjugate variable of E with respect to S is the derivative

$$\beta=\partial S/\partial E\ , \tag{1}$$

the inverse temperature of the system ($\beta=T^{-1}$). The differentiation is performed with other variables fixed.

1. Pure holes

In order to make a Schwarzschild hole in a vacuum a completely isolated system, one has to suppress the particles evaporating from the hole. As a thought experiment, this can be achieved by covering just above the event horizon with a perfectly reflecting, spherical mirror. In this case, the total mass-energy E of the system is that of the hole M and [1]

$$S=\tfrac{1}{2}M^2\ . \tag{2}$$

The conjugate variable is M as calculated from Eqs. (1) and (2). The stability curve is thus a straight line at 45° [line (a) in Fig. 2]. It has no vertical tangent. Accordingly we say that a pure isolated hole is thermodynamically stable with respect to spherically symmetric perturbations since the line represents all the equilibrium configurations within our approximation and no other equilibrium configurations are available.

2. Hole and radiation

Hawking [2] has considered the stability of a Schwarzschild hole immersed in a thermal bath of finite size from a viewpoint of microcanonical ensemble. This is because the canonical ensemble of Schwarzschild holes (the holes in an infinite bath) is shown to always be unstable. We reproduce here his results by using the turning

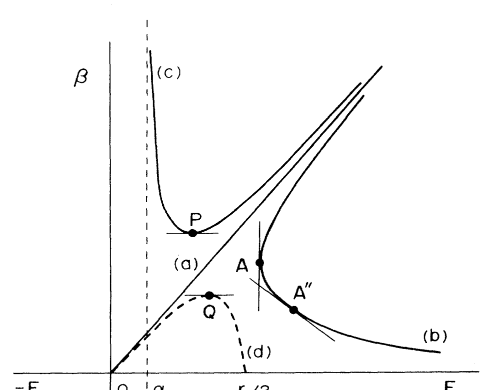

FIG. 2. Stability curves of a Schwarzschild hole in different circumstances and different approximations: (a) a pure hole, (b) a hole with radiation, (c) a hole with back reaction, and (d) a hole in a heat bath whose temperature is fixed at the wall. All curves are only topologically correct.

point method.

The radiation energy in a bath is proportional to T^4, say σVT^4 or $\sigma V\beta^{-4}$, where V is the volume and σ a positive number. The total mass energy is thus

$$E=\beta+\sigma V\beta^{-4} . \tag{3}$$

The entropy in this case is a function of E and V. The stability line $\beta(E)$ has the topology of curve (b) in Fig. 2. This has a vertical tangent at A. Hotter holes $(\beta<\beta_A)$ such as A'' are unstable because the tangent of the stability curve is negative. Cooler ones are stable. This means that the instability of a Schwarzchild hole in radiation is stabilized if the faction of the energy in radiation is sufficiently small. For β high enough the curve tends to the straight line $\beta=E$. Thus, the stability curve of pure holes is approximately valid also for holes in radiation as long as temperatures for a given V are low enough so that radiation effects can be neglected.

3. Stability of black holes with back-reaction effects

The accurate approximation found by Page [25] for the expectation value of the renormalized thermal equilibrium stress-energy tensor of a free conformal field in a Schwarzschild black-hole background was used by York [26] to calculate the back-reaction effect on the metric. A similar attempt was also made by Balbinot and Barletta [27]. Here we quote the latter result for a vector-boson dominated case:

$$\beta=\frac{M}{1-\alpha^2/M^2} , \tag{4}$$

where α is a positive number. The curve $\beta(M)$ is shown in Fig. 2 as (c). This has no vertical tangent, though it deviates completely from the uncorrected curve (a) at small M. Therefore, there is no change of stability. Since the curve tends to the stable line (a) in the large mass region, the whole curve represents stable equilibria. Thus, an isolated Schwarzschild hole is stable even if the back-reaction effects of the thermal atmosphere [23] are taken into account.

The stability of black holes with back-reaction effects will be further discussed elsewhere [28].

B. Holes in a heat bath

The effects of thermal radiation around black holes can also be examined from a standpoint of canonical ensemble. The radiation is then treated as a heat reservoir, a circumstance specifying black-hole states. In this case, instead of M, β becomes a control parameter provided that a reservoir is ideal and can provide or absorb heat which is needed by any change of state of the hole under consideration, without varying its own temperature. In this sense, reservoirs constituting a canonical ensemble are required to have infinite heat capacities.

The entropy is no more the appropriate distribution function which takes its maximum at a stable equilibrium. It is rather $S-\beta M=-\beta F$, with F the Helmholtz free energy. Thus, here Z_e is $S-\beta M$, the parameter y is β, and the conjugate parameter

$$d(S-\beta M)/d\beta=-M . \tag{5}$$

The stability curve is thus $-M(\beta)$ which is just Fig. 2 rotated 90° clockwise (E should be interpreted as M in this case). Looking at this rotated Fig. 2, we can see that without back-reaction effects [curve (a)] there is no vertical tangent. It is evident that there is at least one unstable mode which is associated with the negative heat capacity [2]. Therefore, we conclude that a Schwarzschild black hole in an infinite heat bath is unstable at any temperature. However, the rotated curve (c) has a vertical tangent at P. The lower mass branch has positive tangents near P while the higher mass branch has negative tangents and is hence unstable. Thus, we see that the back reaction of Hawking radiation has a stabilizing effect [27] for a Schwarzschild hole in a heat bath.

One may legitimately argue like York [3] that β is not a homogeneous parameter in a heat bath owing to the background metric of Schwarzschild holes. One should rather fix the temperature at the surface of a spherical box of radius r_0 containing a hole. Then one has

$$\beta_{\text{York}}=M\left[1-\frac{2M}{r_0}\right]^{1/2} . \tag{6}$$

York has found along this line a new branch which is stable in a heat bath. This branch, which is different from the stable one found by Hawking [2], is again beyond the scope of our present treatment. However, his result can also be interpreted by the turning point method. The $-M(\beta_{\text{York}})$ curve [curve (d) in Fig. 2 with β and E interpreted as β_{York} and M, respectively] has a turning point Q, and his unstable branch approaches curve (a) in the high-temperature limit. If we let the radius of his box be infinitely large, we recover our inverse temperature β. In this case, his unstable branch exactly coincides with curve (a) and the stable one disappears to infinity. The combined effects of back-reaction and local temperature on the stability of Schwarzschild holes have been discussed by Zaslavskii [29].

C. Summary

If we neglect the effects of the back reaction on the metric and of inhomogeneous temperature in a heat bath, the turning point method shows all isolated holes to be stable and those in a large enough heat bath to be unstable. We now apply the method to rotating holes with these limitations in mind [30]. Other limitations may be present in fast rotating holes since rotational velocities should not exceed that of light. This would put limitations on the volume of heat reservoirs which may conflict, in some cases, with the requirement for our β to be observed at infinity.

IV. ROTATING BLACK HOLES

A. The thermodynamic functions

By virtue of the no-hair theorem [31], the equilibrium entropy S of Kerr black holes are specified by two independent variables, the mass M and the angular momentum J. The equation of state $S(M,J)$ is most conveniently written in terms of the rotation parameter h introduced by Okamoto and Kaburaki [32]:

$$h \equiv \frac{a}{r_H} , \tag{7}$$

where $a = J/M$ represents the specific angular momentum and $r_H = M + \sqrt{M^2 - a^2}$ the "radius" of the event horizon. Thus,

$$S = \frac{1}{2(1+h^2)} M^2 , \quad J = \frac{2h}{1+h^2} M^2 . \tag{8}$$

The first law of thermodynamics reads

$$dS = \beta dM - \alpha dJ , \tag{9}$$

where $\beta = T^{-1}$ is the inverse temperature and $\alpha = \Omega/T$, with Ω the angular velocity:

$$\beta = \frac{1}{T} = \frac{1}{1-h^2} M , \quad \alpha = \frac{\Omega}{T} = \frac{1}{2} \frac{h}{1-h^2} . \tag{10}$$

All other thermodynamic functions [32,33] can be obtained from Eqs. (8) and (10). Among general Kerr black holes $0 \le h \le 1$, those of $h = 0$ correspond to nonrotating Schwarzschild holes while $h = 1$ correspond to extreme Kerr holes. Eqs. (8) may actually be considered for $h > 1$. Solving the latter of Eqs. (8) for h in terms of J/M^2, one readily finds two solutions for h, h_+, and h_-, corresponding to the outer (r_+) and inner (r_-) horizons, $r_\pm = a/h_\pm$. Equations (8) may thus be regarded as the thermodynamic equation of state of the inner horizon [34]. Thermodynamic properties of the inner horizons have also been discussed by Curir [35].

Corresponding to the heat capacities at constant pressure and volume in usual thermodynamics, those at constant angular velocity and angular momentum are defined for Kerr holes, respectively, by the relations [32]

$$C_\Omega = T \left(\frac{\partial S}{\partial T} \right)_\Omega = -\frac{2(1-h^2)}{(1+h^2)^2} S , \tag{11}$$

$$C_J = T \left(\frac{\partial S}{\partial T} \right)_J = -\frac{2(1-h^2)(1+h^2)}{1-6h^2-3h^4} S . \tag{12}$$

Notice that C_Ω is negative for $0 \le h < 1$, positive for $h > 1$, and tends to 0 as $h \to \infty$. C_J diverges at $h = (1/\sqrt{3} - 1)^{1/2} \equiv h_c$ going from $-\infty$ to $+\infty$ when h crosses h_c from below. C_J is negative for $h > 1$ and tends to 0 as $h \to \infty$. At $h = 0$ and 1, the two heat capacities are equal (see Fig. 3).

B. Massieu functions for rotating holes and their conjugate pairs of variables

For stable isolated black holes the entropy is maximum. In Gibbs' statistical mechanics, isolated systems are treated in terms of the microcanonical ensembles. Other ensembles have different thermodynamic functions that are maxima for stable equilibrium; they are the Massieu functions [36]. The equilibrium values of the Massieu functions for Kerr holes in various ensembles are as follows: for the microcanonical ensemble, S is given by Eq. (8) or by

$$S(M,J) = \frac{M^2}{4} \left[1 \pm \left(1 - \frac{J^2}{M^4} \right)^{1/2} \right] , \tag{13}$$

where the upper sign is for $0 \le h \le 1$ and the lower for $1 \le h$; for the canonical ensemble which is in contact with a heat bath

$$\Psi(\beta, J) \equiv S - \beta M = -\beta F , \tag{14}$$

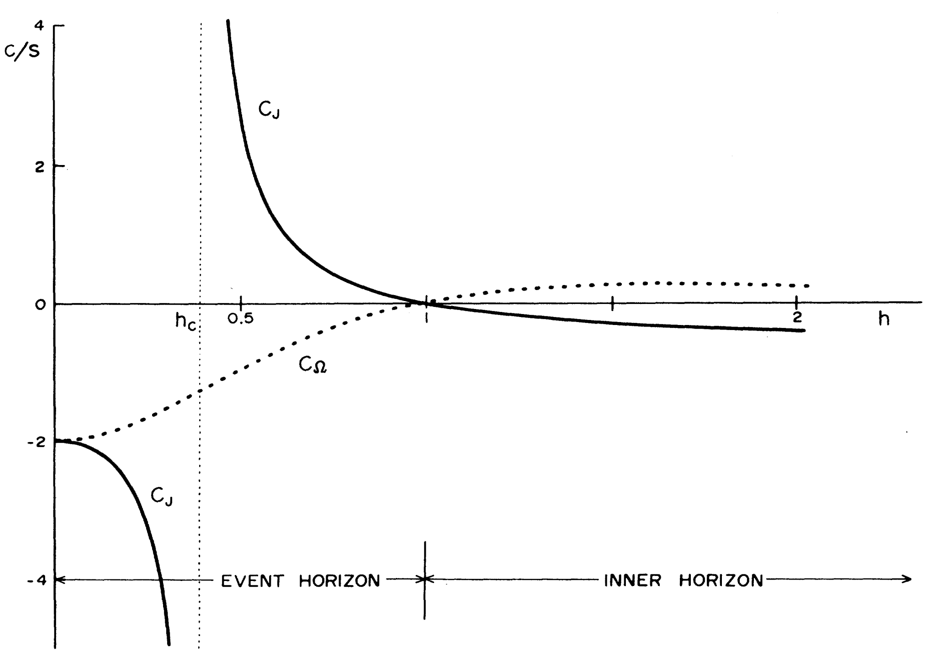

FIG. 3. Comparison of the heat capacities at constant J and Ω. C_J is shown with solid curves while C_Ω, with dotted curve.

where $F = M - TS$ is the Helmholtz free energy; for the "α-microcanonical" ensemble which is in contact with an "α bath,"

$$\Sigma(M,\alpha) \equiv S + \alpha J \ , \tag{15}$$

and for the "α-canonical" ensemble which is in contact with both heat and "α" baths,

$$\Lambda(\beta,\alpha) \equiv S - \beta M + \alpha J = -\beta G \ , \tag{16}$$

where $G = F - \Omega J$ is the Gibbs free energy. Here we have used the correspondence of variables $J \leftrightarrow V$ and $\Omega \leftrightarrow -p$ between the black hole and usual thermodynamics, where V and p denote volume and pressure, respectively. The α-microcanonical ensemble, whose counterpart in usual thermodynamics is artificial (p/T is fixed but T itself is not), is considered together since, in general, the various ensembles give complementary information [9]. There is a hierarchy in the constraints from the most constrained one (microcanonical) to the least constrained one (α-canonical ensemble). A more constrained system cannot be less stable than a less constrained system. Instead of Massieu functions one may use the thermodynamic potentials M, F, $-\beta^{-1}(S + \alpha J)$, and G which are minimum for stable equilibria. But then the stability curves have to be read in a slightly different way from the one described in Sec. II.

Information on *stability changes* comes from plotting the *conjugate* of each parameter versus the parameter itself [18]. For an isolated hole the conjugate parameter of M at fixed J is the derivative of S with respect to M:

$$\left[\frac{\partial S}{\partial M}\right]_J = \beta(M) \ . \tag{17}$$

Similarly, the conjugate parameter of J at fixed M is

$$\left[\frac{\partial S}{\partial J}\right]_M = -\alpha(J) \ . \tag{18}$$

For the canonical ensemble the relevant figures will come from plotting $-M(\beta)$ at J constant and $-\alpha(J)$ at β constant since

$$\left[\frac{\partial \Psi}{\partial \beta}\right]_J = -M(\beta) \ , \quad \left[\frac{\partial \Psi}{\partial J}\right]_\beta = -\alpha(J) \ , \tag{19}$$

and so on. Table I gives the Massieu functions and the conjugate pairs of the four ensembles. The linear series of conjugate variables are readily obtained from Eqs. (13)–(16) and are given in Figs. 4–7. Notice in Table I that the function $-M(\beta)$ at fixed J is the function $\beta(M)$ for a fixed J rotated 90° clockwise; the same is true for $-M(\beta)$ and $\beta(M)$ for fixed α. At fixed M, $J(\alpha)$ is $-\alpha(J)$ rotated 90° counterclockwise; the same is true with $J(\alpha)$ and $-\alpha(J)$ with fixed β.

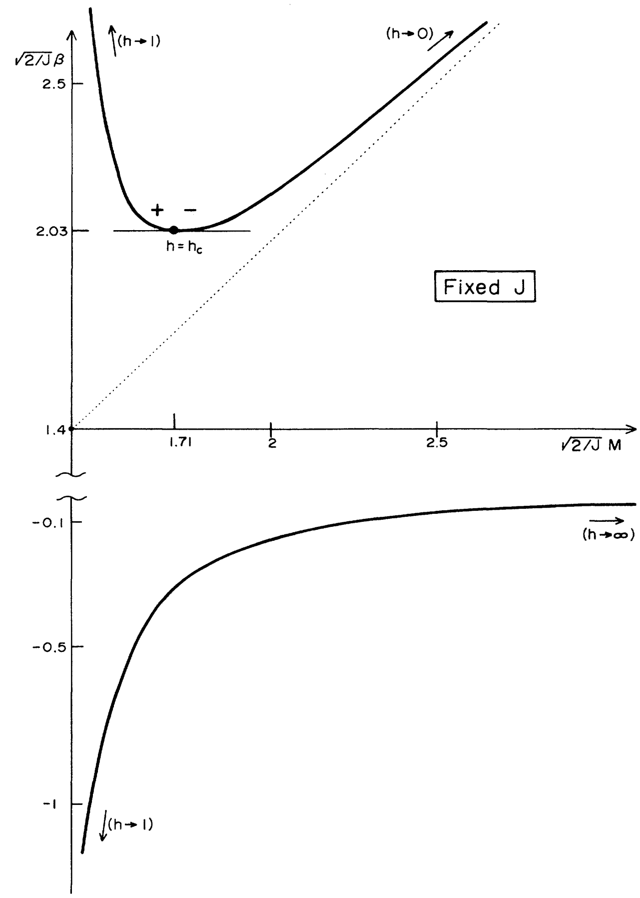

FIG. 4. The stability curves of conjugate parameters $\beta(M)$ at fixed J. Parametric equations are $\sqrt{2/J}\,\beta = (h + h^{-1})^{1/2}(1-h^2)^{-1}$ and $\sqrt{2/J}\,M = (h+h^{-1})^{1/2}$. The curve for the microcanonical ensemble shows no vertical tangent, thus no change in stability. Rotating the figure 90° clockwise represents $-M(\beta)$ at fixed J. That is the stability curve for the canonical ensemble. A vertical tangent at $\sqrt{2/J}\,\beta \simeq 2.03$, $\sqrt{2/J}\,M \simeq 1.71$ shows a loss of stability for $h < h_c \simeq 0.39$. No vertical tangent appears on the $h > 1$ curve.

V. THE STABILITY OF ROTATING HOLES

In what follows we assume that the spectrum λ_i is nondegenerate as in nonrotating holes.

TABLE I. Massieu functions and conjugate variables.

Type	Ensemble Control parameters	Massieu function	Pairs of conjugate variables Function	Fixed	Function	Fixed
Microcanonical	M, J	S	$\beta(M)$	J	$-\alpha(J)$	M
Canonical	β, J	$S-\beta M$	$-M(\beta)$	J	$-\alpha(J)$	β
α-microcanonical	M, α	$S+\alpha J$	$\beta(M)$	α	$J(\alpha)$	M
α-canonical	β, α	$S+\alpha J-\beta M$	$-M(\beta)$	α	$J(\alpha)$	β

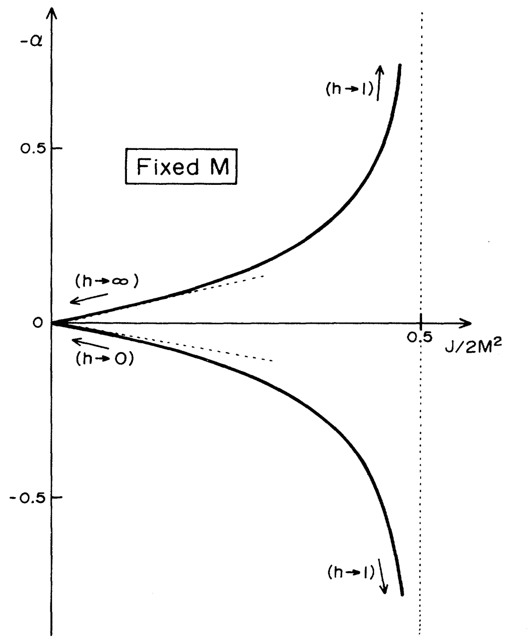

FIG. 5. The stability curves of conjugate parameters $-\alpha(J)$ at fixed M. Parametric equations are $-\alpha=-\frac{1}{2}h(1-h^2)$ and $J/2M^2=h(1+h^2)^{-1}$. The curves hold for the microcanonical ensemble and, rotated 90° counterclockwise, for the α-microcanonical ensemble. There is no vertical tangent and no change of stability at any point.

A. Isolated holes

Curves at constant J are given in Fig. 4. There is no vertical tangent and therefore no change of stability for any M. Since as we have seen isolated Schwarzschild black holes are stable [37], we may infer by continuity that holes with a tiny angular momentum are equally stable. Therefore, the upper branch of Fig. 4 where $h \rightarrow 0$ is associated with stable configurations. From this we conclude that isolated Kerr holes are thermodynamically stable with respect to *axisymmetric perturbations,* since J is constant. "Inner horizons" with $h>1$ are either all stable or all unstable. Since we do not know the stability of any of them, we cannot say more.

$-\alpha(J)$ at constant M are given in Fig. 5 and tell exactly the same thing as $\beta(M)$ at constant J (see Fig. 4). This is because J/M^2 appears in one combination.

We have seen that in spite of negative heat capacities for $h<h_c$, there is no change of thermodynamic stability in the linear series $\beta(M)$, for any value of J. All isolated Kerr holes must thus be stable with respect to axially symmetric perturbations. The sign of the heat capacities is thus no indicator of stability for self-gravitating microcanonical ensemble.

Note, however, that no back reaction is taken into account in the above discussion. These results are likely to be upset by taking the back reaction and inhomogeneity of temperature into account.

If we remain in the equilibrium thermodynamics, Kerr holes in our Universe seem to be best described as isolated systems. At least in the present Universe they are not immersed in a heat bath, and the thermal relaxation time measured by their evaporation times is longer than the age of the Universe for holes of $M \geq 10^{15}$ g. This means that, unless miniblack holes ($M \ll 10^{15}$ g), reversible heat-exchange processes with the environment can be neglected. However, they are not prevented from losing mass and angular momentum through gravitational radiation. In this sense, holes in the Universe are actually open systems even if some astrophysically probable interactions with the environment such as accretion processes are neglected. It has indeed been shown [38] that extreme Kerr holes have indications that they are unstable for strongly nonaxisymmetric perturbations and radiate gravitational waves.

B. Rotating holes in a heat bath

Curves of $-M(\beta)$ at constant J are given by Fig. 4 rotated 90° clockwise. Here we see a vertical tangent for $h=h_c$. The branch of the line $\beta(M)$ for $0<h<h_c$ is thus certainly less stable than the branch for greater h's since

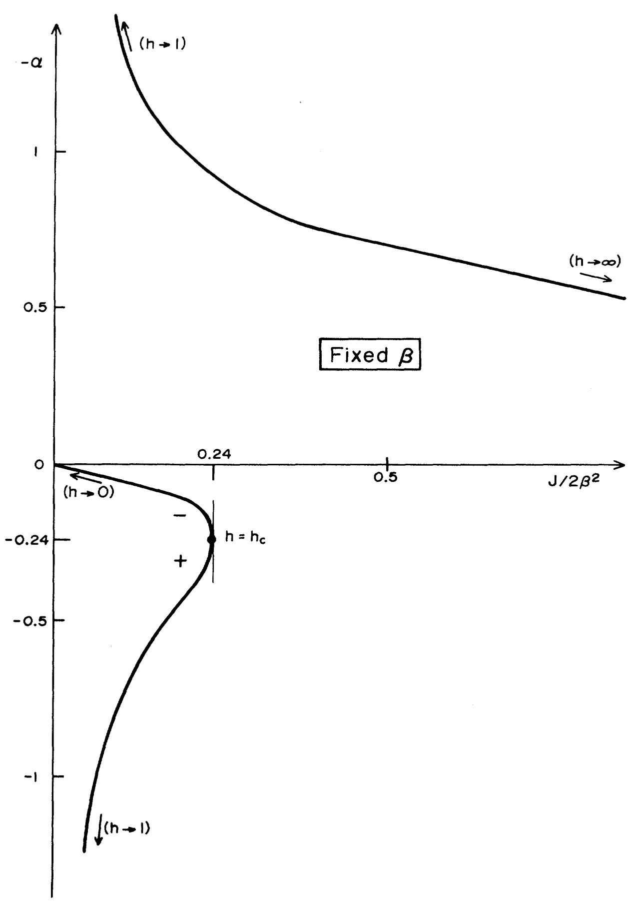

FIG. 6. The stability curve, at fixed β, for the canonical ensemble and, rotated 90° counterclockwise, for the α-canonical ensemble. There is a vertical tangent and a loss of stability for $h<h_c$ in the canonical ensemble. The α-canonical ensemble does not show any change of stability.

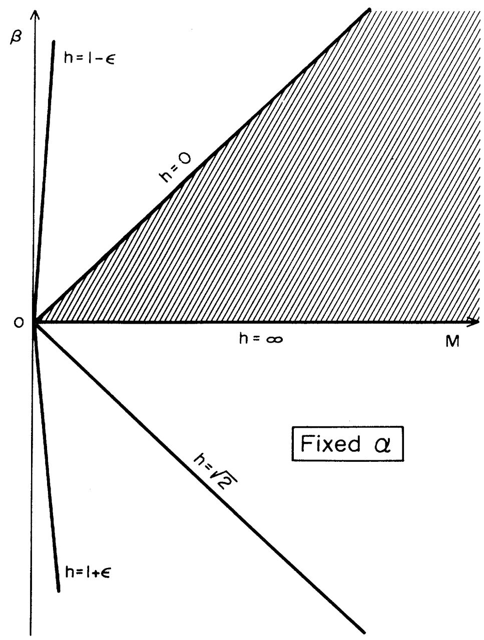

FIG. 7. The stability curves for the α-microcanonical ensemble and, rotated 90° clockwise, for the α-canonical ensemble are straight lines for $\alpha>0$. At $\alpha=0$, $\beta=\pm\infty$ for $h=1$ for any M. No change of stability shows up in these lines.

the slope goes from $-$ to $+$. This result is in line with the fact that Schwarzschild holes ($h\to 0$) in a heat bath are unstable. Fast rotation thus has a stabilizing effect. One can, however, not conclude from this analysis that fast rotating Kerr holes *are* stable because we do not know if the number of negative eigenmodes in unstable slowly rotating holes is 1 or greater than 1. We can only say that they are less unstable thermodynamically than slow holes. They are perhaps stable. $h>1$ holes have no change of stability and are thus all stable or unstable.

Curves of $-\alpha(J)$ at fixed β tell a similar story, with the same critical value of h (see Fig. 6).

There exists in the case of the canonical ensemble a simple and obvious connection between stability changes and the heat capacity C_J which goes from $+\infty$ on the stable branch to $-\infty$ on the unstable branch. The first derivative of the free energy is, however, continuous since the entropy at this point is determined uniquely. This fact may be reminiscent of a second-order phase transition. Indeed, Davies [4] claims that a phase transition must appear in Kerr holes at this point. However, the above argument is not sufficient to prove such a transition. Actually, the point of infinite C_J corresponds to the turning point of the $-M(\beta)$ curve and there is *no branching off there*. There is no stable branch to which most stable states for $h<h_c$ can go. This argument alone shows that no phase transition occurs in the black-hole states.

Similar conclusions for the nonexistence of such a phase transition have been reached by Sokołowski and Mazur [6] and Landsberg and Tranah [39] by somewhat different methods. Sokołowski and Mazur further claim that the temperature is not a fundamental parameter because the mass is a double-valued function of T. However, this is incorrect. As seen in Fig. 4, double-valuedness reflects the change of stability in the canonical ensemble.

C. Other ensembles

The α-microcanonical ensemble is associated with $\beta(M)$ at fixed α (Fig. 7) and $J(\alpha)$ at M (Fig. 5 rotated 90° counterclockwise). No change of stability occurs. Such ensembles are thus either all stable or unstable. Since we know nothing about the Schwarzschild hole with such constraints, the curves are inconclusive. However, the addition of an infinitesimal amount of charge shows that the configurations are, in fact, unstable [40] because they become *less unstable* with a lot of charge; we therefore conclude that all these configurations (with zero charge) are indeed unstable.

The α-canonical ensemble is represented by $-M(\beta)$ at fixed α (Fig. 7, 90° clockwise) and $J(\alpha)$ at fixed β (Fig. 6, 90° counterclockwise). Again, there is no change of stability. However, here we have some more information because this ensemble is less constrained than a canonical ensemble. Since some canonical ensembles are unstable we conclude that α-canonical ensembles at fixed α's are all unstable. Thus, ensembles at fixed α are all unstable.

D. Remarks about the inner horizons

The remarkable feature of the inner horizons is that no ensemble shows any change of stability. Unfortunately, stability curves at $h=1$ are always at infinity. We cannot infer by continuity from $h<1$ to $h>1$ anything about their stability except that they offer no change of stability at all for the whole range of their parameters. We shall show elsewhere [40] that this situation changes dramatically in charged holes. Notice, however, that the linear series for *isolated* holes represents all possible equilibria. The hole has nowhere else to go and must therefore be stable.

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