Quantization of the Skyrmion

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We apply the Kerman-Klein method of quantization, an approach based on Heisenberg matrix mechanics, to the Skyrme model. In this approach the operator equations of motion and kinematical constraints are evaluated within an appropriately chosen Hilbert space, and the resulting set of c-number equations is solved to determine the values of matrix elements of the field operators. These values permit predictions for physical observables. The Kerman-Klein method allows symmetries to be maintained throughout the computation, a property shared with methods based on variation after projection techniques. In this report we concentrate on the quantization of the rotational zero modes of a Skyrmion. We show that the restoration of rotational symmetry leads to a Δ state that is larger than the nucleon and to a modification of the values of observables.

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I. INTRODUCTION

It has been almost a decade since, in a revival of interest in the Skyrme model [1], it was demonstrated [2, 3] that this model could fit observed properties of the baryons to an accuracy of about 30%. This rebirth of attention was stimulated by the belief that some such model is the long-wavelength limit of QCD, as reviewed, e.g., in Refs. [4–6].

In the intervening period, there have been a large number of works extending the range of applications, modifying and extending the model, and improving the way in which consequences are drawn from it. Among the further applications, the most prominent have been to pion-nucleon scattering (the relevant literature can be traced from Refs. [7–11]) and to the two-nucleon problem [12–18]. The basic Skyrme model has been extended in various directions. Excluding the mention of models that contain quarks explicitly, one encounters in the literature models with higher-order terms involving the same fields [19–23] and models in which vector mesons have been added [8, 24–27], as well as extensions to include strange [28–31] and even charmed mesons [32]. All these models are first presented as classical field theories, since one can do much physics using only selected classical solutions. The need to address the problem of quantization is, however, manifest in the intrinsic properties of the classical solution. It is to the problem (and problems) of quantization that this paper is addressed. Though the ideas to be presented could have been developed within the framework of many of the extended models, we have chosen, initially, to work with the original Skyrme model.

The capability of extracting interesting physics from the Skyrme model is grounded on the existence of a special solution of the classical field theory, the hedgehog Skyrmion. Like all interesting classical (or mean field) solutions, it breaks some of the symmetries of the underlying Lagrangian. The hedgehog Skyrmion violates translation, spatial rotation, and isospatial rotation symmetry. The restoration of these symmetries requires, at the very least, the quantization of the generators of the symmetry transformations and of the associated canonically conjugate collective coordinates. As a consequence maximum attention has been paid to this aspect of the problem of quantization (for a recent discussion, see Ref. [33]). (In addition, to study pion-baryon scattering, it is necessary to discuss quantization of the small oscillations of the pion field [7, 8, 10]. There have also been some discussions of radial oscillations [34–36] in connection with problems of stability that we shall discuss in Sec. II.)

There is major difference in methodology between all previous work on the quantum theory of Skyrme's model and that to be presented in this paper. There is also a difference in potential scope between our work and almost all previous papers, which quantize either the collective degrees of freedom, as in the original papers, or the pion fluctuations, but not both at the same time. A major exception is the work of Verschelde [37], who has developed a general method of quantization of all degrees of freedom, termed the method of "nonrigid" quantization, in which the collective coordinates are treated as redundant variables.

This paper is devoted to the application of the so-called Kerman-Klein method of quantization [38], which in its application to field theories [39–43] involves, first of all, formal quantization of the entire classical field. The special signature of the method, however, is in its approach to the study of both the equations of motion and the kinematical constraints, involving, by means of an analysis of the structure of Hilbert space, the definition of a sequence of symmetry-conserving approximations. By means of this scheme one can recover the classical solution as a limiting case, but at the same time, at least for a renormalizable field theory, one can proceed as far

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toward a better solution as one's analytic and computational powers permit.

Questions may be raised concerning the justification for quantizing the Skyrme Lagrangian at all, since it is, in no sense, a fundamental field theory, but rather a classical model that results from taking the limit of such a theory, including only some of the degrees of freedom of the original theory. This means that even if a formal requantization could be carried out unambiguously, which we shall see below is not the case here, we could hardly insist on the validity of all the consequences of such a reconstituted quantum theory. Nevertheless, there is a rich experience from the nonrelativistic many-body problem, for example from nuclear physics [44–46], suggesting the validity of such an approach for the study of collective properties at low energies.

An undertaking to quantize the Skyrme model by the Kerman-Klein procedure confronts a number of problems, some of which are shared with other methods. The first such problem is the choice of field variables. We attempted, initially, to quantize a set of independent angle-valued fields, perhaps most closely related to the chiral hedgehog field, and then applied the resulting formulation to the restoration of translational symmetry [47]. In both the formulation, on theoretical grounds, and in the application, for practical reasons, there are difficulties which led us to abandon this effort and to replace it by the choices made in this paper. We chose the formulation in terms of four fields, a scalar-isoscalar meson (σ meson) and a pseudoscalar-isovector meson ($\boldsymbol{\pi}$ meson), which satisfy the constraint $\sigma^2 + \boldsymbol{\pi}^2 = f_\pi^2$ at each point. The reason for this choice is that the Kerman-Klein method is most conveniently implemented when the equations of motion are polynomial in the fundamental variables. The variables chosen satisfy this criterion, at least at the classical level.

As a consequence of the above choice of variables, we have to follow the Dirac method [48, 49] of constrained quantization. This, by itself, is not a special source of difficulty. The main source of difficulty lies in the circumstance, shared with previous work in the field, that the form of the kinetic energy implies that the field theory is defined on a curved field space. For such a theory, the problem of operator ordering precludes a unique solution to the problem of quantization. The choices made in this paper have been guided both by theoretical considerations and practical advantages. Though other choices may be equally justifiable, our selection involves an intriguing novelty that may prove useful for solving other classes of problems. Once made, the bulk of the effort reported in this paper, based on the work of one of us [50], is concerned with the working out of a symmetry-conserving coupled-channel approach. Guided by the hedgehog solution we restrict the study to states with equal spin and isospin, with special emphasis on the nucleon and Δ.

Turning to the actual content of this work, in Sec. II, we discuss selected aspects of the classical Skyrme model. In addition to a review of standard material, novel aspects include a new suggestion for stabilization of the rotating ("cranked") hedgehog, one that later proves its usefulness in the quantized theory, and the application of Dirac's method of modified Poisson brackets for the case of the redundant fields chosen for the present study.

In Sec. III we describe the method of quantization utilized in this work, involving both an account of the distinguishing features of the Kerman-Klein method and our resolution of the special problems associated with trying to quantize a field in curved space. In particular, we adopt a quantization procedure based on a c-number variational principle, the trace variational principle, which is shown to be completely equivalent to canonical quantization in cases without ordering ambiguities. Though such a proof of equivalence is missing in our case, the method adopted does define a quantum theory, has the correct classical limit, and in addition provides the basis for a powerful computational method.

The application of the quantization procedure to the infinite tower of baryon-number-1 states contained in the hedgehog solution is described in Sec. IV. By introducing a closure approximation suggested by projecting the hedgehog solution onto states of good spin and isospin, we reduce the problem to a coupled-channel calculation for the nucleon and Δ. It is shown that the classical solution is contained as a limiting case, and new symmetry-preserving solutions are found with the help of an algorithm tied to the trace variational principle. Calculation of some observables is carried out and comparison with previous results included, without producing any striking differences with the latter. In the summary and discussion, it is emphasized, however, that because of the limited number of states that have been included in the scheme thus far, the most important aspect of our contribution is that we may have shown the feasibility of a new method for studying the quantum theory of Skyrme-like models.

A. Definition of conventions

It is convenient to summarize here the notational conventions that are used in this paper. The indices $a, b, c, \ldots \in \{1, 2, 3\}$ represent the three components of a vector in isospace, the indices $\alpha, \beta, \gamma, \ldots \in \{0, 1, 2, 3\}$, at the head of the Greek alphabet, represent an isoscalar mode in addition to the three isovector ones. The indices $i, j, k, \ldots$ indicate spatial directions, and we use the subscript i to denote a derivative with respect to the coordinate x_i. The Greek letters $\mu, \nu, \lambda, \ldots$ represent axes in Minkowski space. Summation conventions are used for all these indices. We use a set of units ("Skyrme units") that are related to the natural units ($\hbar = c = 1$) by rescaling time and distance by $f_\pi g_\pi$ and rescaling the units of energy and momentum by $(f_\pi g_\pi)^{-1}$; an exception to this convention is the radius r, which is related to the corresponding dimensionless quantity x by $r = x/(f_\pi g_\pi)$. Carets over spatial coordinates ($\hat{x}$) signify a unit vector; carets over variables representing magnitudes of angular momenta ($\hat{l}$) denote the numerical factor $\sqrt{2l+1}$, while other appearances of carets indicate an operator. Conventions for coupling angular momenta are contained in Ref. [51]. In all other respects, such as the choice of Lorentz metric, we follow Ref. [54].

II. CLASSICAL SKYRME MODEL

In the Skyrme model, there are various ways of representing the constituent meson fields. Since it is convenient for the quantization discussed below, we select a form involving one scalar (σ) and three pseudoscalar (π_a) meson fields constrained such that the sum of the squares of the field values is a constant, $\sigma^2 + \pi_a\pi_a = f_\pi^2$. This constraint dictates that there are only three fundamental field degrees of freedom. It is convenient to rescale these fields by f_π and introduce the four symbols ϕ_α, defined by

$$\phi_\alpha \equiv \begin{cases} \sigma/f_\pi & \text{if } \alpha = 0, \\ \pi_a/f_\pi & \text{if } \alpha \in \{1,2,3\}. \end{cases} \quad (2.1)$$

This allows the constraint to be written as $\phi_\alpha\phi_\alpha = 1$. In terms of the field quaternion $U \equiv \phi_0 + i\tau_a\phi_a$ (where τ_a are the three Pauli matrices) the constraint is $U^\dagger U = 1$. Although this constraint could be automatically satisfied by using three independent fields defined by $U \equiv e^{i\tau_a\theta_a}$, we have not found that such a choice leads to a convenient form of the quantum theory.

As has been discussed in Refs. [1, 2], it is well known that the structure of the meson fields implies the existence of a conserved quantity called topological charge. This charge has been identified with baryon number, with a value that can be determined by integrating the time component of the baryon current,

$$\mathcal{B}^\mu(\mathbf{x}) = -\frac{1}{12\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\lambda\rho}\phi_\alpha\partial_\nu\phi_\beta\partial_\lambda\phi_\gamma\partial_\rho\phi_\delta, \quad (2.2)$$

over all space.

A. Lagrangian and field equations

In terms of the chosen field variables and of the units defined at the end of the Introduction, the basic Skyrme model is described by the Lagrangian

$$L = \frac{1}{g_\pi^2}\int d^3x \left[\frac{1}{2}\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha + \beta_\pi^2(\phi_0 - 1) + \frac{\lambda}{2}(\phi_\alpha\phi_\alpha - 1) - \frac{1}{4}(\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha\partial_\nu\phi_\beta\partial^\nu\phi_\beta - \partial_\mu\phi_\alpha\partial^\nu\phi_\alpha\partial_\nu\phi_\beta\partial^\mu\phi_\beta)\right], \quad (2.3)$$

where β_π is the mass of the pion in Skyrme units and λ is a Lagrange multiplier field introduced to impose the constraint. This Lagrangian can also be written as

$$L = \int d^3x \left[\frac{1}{2}\dot\phi_\alpha\mathcal{M}_{\alpha\beta}\dot\phi_\beta - \mathcal{V} + \frac{\lambda}{2g_\pi^2}(\phi_\alpha\phi_\alpha - 1)\right], \quad (2.4)$$

where the inertia density matrix and the potential energy density are

$$\mathcal{M}_{\alpha\beta} = \frac{1}{g_\pi^2}\{\delta_{\alpha\beta}[1 + (\partial_j\phi_\gamma\partial_j\phi_\gamma)] - (\partial_j\phi_\alpha\partial_j\phi_\beta)\}, \quad (2.5)$$

$$\mathcal{V} = \frac{1}{2g_\pi^2}\left\{(\partial_j\phi_\alpha\partial_j\phi_\alpha) - \beta_\pi^2(\phi_0 - 1) + \frac{1}{2}\left[(\partial_j\phi_\alpha\partial_j\phi_\alpha)^2 - \sum_{jk}(\partial_j\phi_\alpha\partial_k\phi_\alpha)^2\right]\right\}. \quad (2.6)$$

The associated equations of motion are

$$\frac{\delta L}{\delta\phi_\gamma} = -\frac{d}{dt}\left(\mathcal{M}_{\alpha\gamma}\dot\phi_\alpha\right) - \frac{1}{2}\partial_i\left(\dot\phi_\alpha\frac{\partial\mathcal{M}_{\alpha\beta}}{\partial\phi_{\gamma,i}}\dot\phi_\beta\right) - \frac{\delta\mathcal{V}}{\delta\phi_\gamma} + \frac{\lambda}{g_\pi^2}\phi_\gamma = 0. \quad (2.7)$$

Multiplying this expression by ϕ_γ and summing over the index γ (and using the constraint condition) allows the value of the Lagrange multiplier field to be found. Substituting the value of this field back into the equations of motion transforms them into

$$\left[\frac{d}{dt}\left(\mathcal{M}_{\alpha\delta}\dot\phi_\alpha\right) + \frac{1}{2}\partial_i\left(\dot\phi_\alpha\frac{\partial\mathcal{M}_{\alpha\beta}}{\partial\phi_{\delta,i}}\dot\phi_\beta\right) + \frac{\delta\mathcal{V}}{\delta\phi_\delta}\right] \times(\delta_{\gamma\delta} - \phi_\gamma\phi_\delta) = 0. \quad (2.8)$$

These equations are not all independent, as becomes apparent if we multiply by ϕ_γ and sum on γ, which gives zero. Furthermore, by using the definitions of $\mathcal{M}_{\alpha\beta}$ and $\mathcal{V}$ given in Eqs. (2.5) and (2.6), it is straightforward to see that the field equations are polynomial in the fields,

$$[\Box\phi_\delta - \partial_\mu(\partial^\mu\phi_\delta\partial_\nu\phi_\alpha\partial^\nu\phi_\alpha) + \partial_\mu(\partial^\nu\phi_\delta\partial_\nu\phi_\alpha\partial^\mu\phi_\alpha) - \beta_\pi^2\delta_{\delta 0}](\delta_{\gamma\delta} - \phi_\gamma\phi_\delta) = 0, \quad (2.9)$$

as we require for the quantization method to be utilized in this paper.

Though the aim of this research is to go beyond the standard classical results, they are, nevertheless, used in a fundamental way for guiding our study. Thus it is important to recall a few features of the well-known "hedgehog" ansatz, which is characterized by a rigid coupling of the isospace direction of the fields with the radial direction: i.e.,

$$U = \phi_0(x) + i\tau_a\phi_a(x) \to \cos\theta_H(x) + i\frac{\tau_a x_a}{x}\sin\theta_H(x). \quad (2.10)$$

This simplification reduces the equations of motion to a single ordinary differential equation for the radial function θ_H. The boundary conditions can be fixed by assuming that the field ϕ_a vanishes at infinity and that the primary configurations of interest are characterized by unit baryon number. These conditions are satisfied if $\theta_H(0) = \pi$ and $\lim_{x\to\infty} \theta_H(x) = 0$. The differential equations characterizing this solution can be found by minimizing the potential energy, as given by

$$V[\theta_H] = \frac{4\pi}{g_\pi^2} \int dx \left\{ \frac{x^2(\theta_H')^2}{2} + \sin^2\theta_H \left[1 + (\theta_H')^2\right] + \frac{\sin^4\theta_H}{2x^2} + \beta_\pi^2 x^2 (1 - \cos\theta_H) \right\}. \quad (2.11)$$

For stationary configurations, this quantity is equal to the mass of the hedgehog Skyrmion, $M = V[\theta_H]$. The function $\theta_H(x)$ has been determined numerically in Ref. [2]. This mean-field solution allows values for many observables to be predicted, some of which are reproduced in Sec. IV for purposes of comparison with our results.

B. Rotational modes of the hedgehog Skyrmion

The hedgehog ansatz breaks a number of symmetries that are respected by the Skyrme Lagrangian, such as translational, rotational, and isorotational invariance. Since the purpose of this article is to restore rotational and isorotational invariance, it is useful at this juncture to examine the effects of rotations on the hedgehog solution. This is done by applying the self-consistent cranking technique (for a general discussion see [52], for an application in a related field see [53]), in which one looks for a solution of the field equations describing a rotating and shape-altered hedgehog form. Such a solution where the rotation occurs with fixed frequency ω about the z axis is given by the expression

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t & 0 \\ 0 & -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_{H;\omega} \\ \frac{x_1}{x}\sin\theta_{H;\omega} \\ \frac{x_2}{x}\sin\theta_{H;\omega} \\ \frac{x_3}{x}\sin\theta_{H;\omega} \end{pmatrix}, \quad (2.12)$$

where $\theta_{H;\omega}$ is the hedgehog function whose shape is to be determined. By inserting this modified ansatz into the equations of motion as given by Eq. (2.9), the self-consistently determined hedgehog function is found to behave as

$$\lim_{x\to\infty} \theta_{H;\omega}(x) = \frac{A_\omega}{x^2}(1 + \mu x)e^{-\mu x}, \qquad \mu^2 \equiv \beta_\pi^2 - \frac{2\omega^2}{3}. \quad (2.13)$$

Since the nature of the solution changes as μ^2 changes sign, it is useful to define a critical frequency $\omega_c \equiv \sqrt{1.5}\beta_\pi \approx 2.57 \times 10^{23}$ Hz which corresponds to $\mu^2 = 0$. For a rotational frequency below the critical one, $\omega < \omega_c$, the resulting profile exhibits exponentially damped behavior. In contrast, the hedgehog function oscillates about zero within a $1/x$ envelope for a supercritical rotational frequency.

Although such a solution has unit baryon number, it also has an unbounded value for the energy; therefore the distribution cannot describe a stable, physical particle. This result has been interpreted as a classical manifestation of pion radiation, occurring when the energy of a rotating state exceeds that of the static hedgehog by more than the pion mass. Of course, this same problem arises in a quantum theory, where it is perfectly obvious that we must treat the Δ as a resonance in the pion-nucleon continuum. Unlike standard radiation problems, here the same field describes the bound state and the radiation. We have chosen to make an approximate decomposition of the two parts, to preserve as much as possible the picture of a tower of stable spin-isospin states inherent in the hedgehog solution. It is only with an understanding of this aim that the discussion that follows makes any physical sense, since it is a discussion of means of suppressing the decay so as to have stable excited configurations.

Several approximations designed to stabilize rotating configurations have been studied, as discussed further below, including determining the hedgehog function in a non-self-consistent manner, and attempting to separate the terms leading to radiation from those describing the core of the baryonlike excitation. The simplest approach is the adiabatic approximation, which suppresses the requirement that the shape of the hedgehog be frequency dependent. As carried out by Refs. [2–4] and many later articles, this is done by using the hedgehog function determined in the zero-frequency limit. Though a number of ambitious attempts to maintain self-consistency and achieve stability have been reported in the literature, such as those described in Refs. [10, 35, 55–57], none of these has so far proved useful for the quantum theory developed in this paper, and therefore we shall not discuss them in detail. In this paper we have utilized a new method that has a natural generalization to the full quantum theory. This method is based on an approximate separation between contributions from radiation terms and those of the core of the baryonlike state. This is implemented by making the substitution

$$\dot\theta_{H;\omega}(x) \to \dot\theta_{H;\omega}(x)\Theta(x_{\max} - x). \quad (2.14)$$

The Θ function takes a value of unity within the cutoff radius, and vanishes outside $x_{\max}$, which effectively suppresses the radiation and stabilizes the system. The restoration of the radiation can then be carried out as a second step (e.g., by treating the neglected term as a perturbation), but this has not been done in this paper. [The procedure described by Eq. (2.14) introduces a time-dependent elastic energy for the Skyrmion in the classical theory which is ignored in the numerical determinations that are given in Sec. IV. However, due to the different role that "rotation" plays in the quantum theory, additional elastic energy terms do not arise in a corresponding quantum theory.]

In the semiclassical cranking approach, one makes contact with the quantum theory by assuming a relation between spin and rotational frequency, namely,

$$\sqrt{J(J+1)} = \Lambda[\theta_{H;\omega}]\omega. \quad (2.15)$$

Since there is now a method of converting from angular frequency to angular momentum, it is permissible to label the moment of inertia and potential energy by J, the value of the angular momentum. Within the semiclassical approximation, the energy of a rotating hedgehog Skyrmion can be written as

$$E_J = \frac{J(J+1)}{2\Lambda(J)} + V_J. \tag{2.16}$$

In Fig. 1, the dependence of energy upon the value for the angular momentum is shown for subcritical values of angular momentum. This energy can be interpreted as the mass of a configuration that has intrinsic angular momentum J and isospin $I = J$, since this last equality is inherent in the use of the hedgehog solution. The masses of the nucleon ($I = 1/2$) and Δ ($I = 3/2$) states can therefore be written as

$$\begin{aligned} M_N = E_{I=J=\frac{1}{2}} &= \frac{3}{8\Lambda_{J=\frac{1}{2}}} + V_{J=\frac{1}{2}}, \\ M_\Delta = E_{I=J=\frac{3}{2}} &= \frac{15}{8\Lambda_{J=\frac{3}{2}}} + V_{J=\frac{3}{2}}. \end{aligned} \tag{2.17}$$

The masses of the nucleon states can be directly determined within either the adiabatic approach or the self-consistent approach (without imposing additional restrictions). However, the calculation of a finite result for M_Δ within the self-consistent approach requires the previously discussed technique for removing oscillations, since the angular frequency is supercritical.

C. Modified Poisson bracket formalism

One of the aims of this work is to quantize the basic Skyrme model by a canonical method. Because we have chosen to work with redundant field variables, we cannot consistently quantize the Poisson brackets. Therefore it is necessary to replace the standard Poisson bracket by a modified bracket, such as the one due to Dirac [48, 49], which takes proper account of the constraints. In this section we describe briefly the application of this method to the Skyrme model.

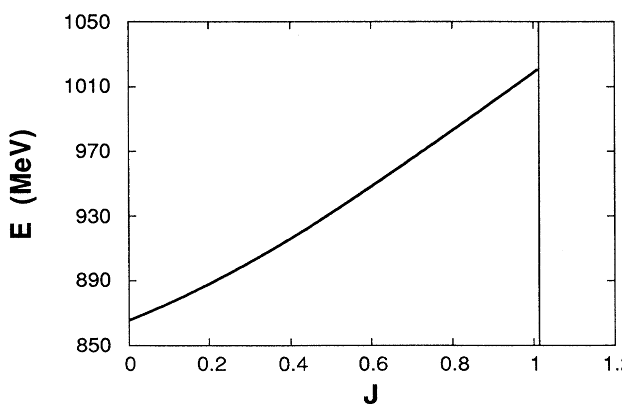

FIG. 1. Dependence of the energy of a spinning hedgehog Skyrmion (2.16) on the angular momentum for subcritical values of J.

In the usual Hamiltonian formulation, one derives the field equations by taking Poisson brackets of the fields and field momenta with the Hamiltonian, evaluating these expressions by assuming that the fundamental Poisson brackets take their canonical form. This procedure appears to work also for the constrained theory, if the constrained Hamiltonian (the transform of the constrained Lagrangian) is utilized and if the evaluation also includes the imposition of the constraint condition

$$\chi_1 \equiv \tfrac{1}{2}\left(\phi_\alpha\phi_\alpha - 1\right) = 0 \tag{2.18}$$

after the evaluation of the Poisson brackets. The further requirement that the condition $\chi_1 = 0$ not vary in time can be satisfied by imposing a secondary constraint

$$\chi_2 \equiv \phi_\alpha\pi_a = \phi_\alpha \mathcal{M}_{ab}\dot{\phi}_\beta = 0. \tag{2.19}$$

The two constraints involving χ_1 and χ_2 are said to be "second class" because the Poisson bracket of the fields χ_1 and χ_2 is nonvanishing. Furthermore, it can be shown that this secondary constraint is also independent of time, using the canonical equations of motion.

Nevertheless, if one carries out a canonical quantization by replacing the fundamental Poisson brackets by the usual commutators, trouble ensues from the fact that $[\chi_1, \chi_2]_{\rm PB} \neq 0$. The quantum expression of the constraints is that every vector in Hilbert space must be annihilated by the constraint operators $\hat{\chi}_1$ and $\hat{\chi}_2$. It follows trivially that every vector must also be annihilated by their commutator, and this conclusion is inconsistent since the commutator in question is itself nonvanishing in the canonical quantization.

One resolution of this difficulty is to introduce modified classical brackets, the Dirac brackets, which share with the Poisson brackets all its basic algebraic properties, but are designed so that the Dirac bracket of any pair of second class constraints vanishes, in our case, $[\chi_1, \chi_2]_{\rm DB} = 0$. As a consequence of its definition, given below, it follows that the Dirac brackets of *any* dynamical variable with the constraints vanish. If A and B are dynamical variables and $\{\chi_i\}$ is the complete set of second-class constraints, the definition that satisfies these conditions is

$$\begin{aligned} [A,B]_{\rm DB} &= [A,B]_{\rm PB} - [A,\chi_i]_{\rm PB}\,\Delta_{ij}^{-1}\,[\chi_j,B]_{\rm PB}\,, \\ \Delta_{ij} &= [\chi_i,\chi_j]_{\rm PB}\,, \end{aligned} \tag{2.20}$$

where the set of expressions $\{\chi_i = 0\}$ contains all the second-class constraints. For our particular problem, the basic Dirac brackets involving the fields and momenta are

$$[\phi_\alpha(\mathbf{x}), \phi_\beta(\mathbf{y})]_{\rm DB} = 0, \tag{2.21}$$

$$[\phi_\alpha(\mathbf{x}), \pi_\beta(\mathbf{y})]_{\rm DB} = \delta^3(\mathbf{x}-\mathbf{y})\left(\delta_{\alpha\beta} - \phi_\alpha\phi_\beta\right), \tag{2.22}$$

$$[\pi_\alpha(\mathbf{x}), \pi_\beta(\mathbf{y})]_{\rm DB} = \delta^3(\mathbf{x}-\mathbf{y})\left(\pi_\alpha\phi_\beta - \phi_\alpha\pi_\beta\right). \tag{2.23}$$

The Dirac brackets replace the Poisson brackets for determining the time evolution of relevant quantities. This replacement also eliminates the need to introduce a Lagrange multiplier field. The appropriate Hamiltonian for deriving the evolution of the meson fields is simply

$$H = \int d^3x \left(\frac{1}{2}\pi_\alpha \mathcal{M}^{-1}_{\alpha\beta}\pi_\beta + \mathcal{V}\right). \tag{2.24}$$

The time development of the fields and the momenta can be found by taking the Dirac bracket of these quantities with the Hamiltonian. This leads to the equation

$$\left[\dot{\pi}_\delta - \frac{1}{2}\partial_i\left(\pi_\alpha \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial \phi_{\delta,i}}\pi_\beta\right) + \frac{\delta \mathcal{V}}{\delta \phi_\delta}\right](\delta_{\gamma\delta} - \phi_\gamma\phi_d) = 0, \tag{2.25}$$

which is equivalent to that derived using the Lagrangian formalism [Eq. (2.8)].

III. QUANTIZATION OF THE SKYRME MODEL

The major part of the research into the quantum mechanics of the Skyrmion has been based on the quantization of an effective classical Hamiltonian written in terms of collective coordinates and harmonic fluctuations about the hedgehog solution, as exemplified by Ref. [2]. The Kerman-Klein method is an alternative approach to quantization, usually based on a formal canonical quantization of the entire classical field, where the restriction of study to special degrees of freedom enters through assumptions about the composition of Hilbert space. In this approach, based on Heisenberg's matrix mechanics, we express the equations of motion and the kinematical constraints as nonlinear equations for matrix elements by taking the expectation value of the operator equations of motion between unknown eigenstates, while using the completeness relation for any matrix element of a product of operators. It is turned into a useful calculus by recognizing how the structure of Hilbert space permits a series of justified truncations of the infinite sum over intermediate states in the equations, based on the classification of matrix elements according to degree of collectivity, i.e., according to relative orders of magnitude.

We shall illustrate the ideas underlying this method by a review of the one-dimensional ϕ^4 model, using the canonical method. An equivalent method, formulated in terms of c-number equations, the trace variational formalism, is also described. The ideas involved in truncation of the equations to manageable size are then described. Unfortunately, when we turn to the Skyrme model, we encounter new difficulties associated with the nonuniqueness in the quantization of the "kinetic energy." We have chosen to describe the dynamics by a trace variational principle, but we must still choose operators to represent other observables. As an example, in Sec. IV, we exhibit an energy operator and a symmetrized operator form of the conserved topological current.

A. Quantization and the trace variational principle: One-dimensional model

Before describing quantization of the Skyrme model, it is useful to review how the one-dimensional ϕ^4 model can be quantized and studied using both the canonical method and the Kerman-Klein approach. By using standard techniques to derive operator equations of motion, and evaluating them within the complete Hilbert space, one obtains a set of c-number relations involving matrix elements. However, these equations can be reproduced by taking variations of a single c-number functional. This motivates the definition of an alternative method of quantization, the trace variational approach, which turns out to lead to equivalent results when the quantum theory does not contain any inherent ordering uncertainties. (Most of the results presented in this subsection have been derived previously in Refs. [39, 40].)

The quantum mechanical Hamiltonian operator for the ϕ^4 theory can be derived in an unambiguous manner from the classical Hamiltonian. The quantum operator

$$\hat{H} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\hat{\phi}')^2 + \frac{1}{2\lambda}\left(m^2 - \lambda\hat{\phi}^2\right)^2\right] \tag{3.1}$$

is simply the classical Hamiltonian with field values replaced by field operators. The standard commutation relations

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x-y) \tag{3.2}$$

can be used to determine two operator equations of motion:

$$\dot{\hat{\phi}} = -i[\hat{\phi}, \hat{H}] = \hat{\pi}, \tag{3.3}$$

$$\dot{\hat{\pi}} = -i[\hat{\pi}, \hat{H}] = \hat{\phi}'' + 2m^2\hat{\phi} - 2\lambda\hat{\phi}^3. \tag{3.4}$$

In the sequel, we find it useful to eliminate both the field momentum and the explicit occurrence of the time derivative of the field in favor of the commutator of the field with the Hamiltonian, both ends achieved with the aid of Eq. (3.3).

These substitutions allow the operator equations of motion and the canonical commutation relations to be written as

$$[[\hat{\phi}, \hat{H}], \hat{H}] + \hat{\phi}'' + 2m^2\hat{\phi} - 2\lambda\hat{\phi}^3 = 0, \tag{3.5}$$

$$[\hat{\phi}(x), [\hat{\phi}(y), \hat{H}]] = -\delta(x-y). \tag{3.6}$$

These equations are next converted to c-number equations by taking matrix elements between states $|\psi\rangle$, $|\psi'\rangle, \ldots$ that are a complete set of eigenstates of the energy, and by evaluating the matrix element of a product of operators by means of the completeness relation. We thus obtain the following sets of c-number equations:

$$\left[(E_{\psi'} - E_\psi)^2 + \nabla^2 + 2m^2\right]\langle\psi|\hat{\phi}|\psi'\rangle - 2\lambda\sum_{\tilde{\psi},\tilde{\psi}'}\langle\psi|\hat{\phi}|\tilde{\psi}\rangle\langle\tilde{\psi}|\hat{\phi}|\tilde{\psi}'\rangle\langle\tilde{\psi}'|\hat{\phi}|\psi'\rangle = 0, \tag{3.7}$$

$$\sum_{\tilde{\psi}}\left(E_\psi + E_{\psi'} - 2E_{\tilde{\psi}}\right)\langle\psi|\hat{\phi}(x)|\tilde{\psi}\rangle\langle\tilde{\psi}|\hat{\phi}(y)|\psi'\rangle = -\delta_{\psi\psi'}\delta(x-y). \tag{3.8}$$

What has emerged from our considerations is thus a sum-rule formulation of Heisenberg's matrix mechanics. This is the formal structure of the so-called Kerman-

Klein quantization, which will be discussed both in this subsection and in the succeeding one. To be useful for any particular problem, however, we must adjoin an analysis that justifies replacing, in the sum rules above, the full Hilbert space of states, $|\psi\rangle$, by a subspace of states, $|\chi\rangle$, that defines a leading approximation. This analysis should also specify the order in which the discarded elements of the full Hilbert space are to be restored for more ambitious approximations.

Before turning to these matters (in the next subsection), we wish to discuss another essential property of this method. It is important to emphasize that Eq. (3.7) can be derived from a variational principle as the variation with respect to $\langle\psi'|\hat{\phi}|\psi\rangle$ of a functional, given by

$$F_\phi = \frac{1}{\mathrm{Tr}[1]}\mathrm{Tr}[\hat{L}], \tag{3.9}$$

$$\hat{L} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2\lambda}\left(m^2 - \lambda\hat{\phi}^2\right)^2\right], \tag{3.10}$$

where again a time derivative is replaced by a commutator with the Hamiltonian, and the latter is *not varied* inside the functional. The operator $\hat{L}$ may be regarded as a "quantum Lagrangian," since it is a symmetrized, operator-valued generalization of the classical Lagrangian; for the model under discussion, $\hat{L}$ is essentially unique.

Since the trace variational principle is equivalent to the sum rules derived from the equations of motion, we may choose this principle plus the kinematical sum rules obtained from the commutation relations as the definition of the method of quantization adopted in this work. The approximate sum rules that follow from choosing a subspace of the Hilbert space may also be considered as an approximate method of quantization for those degrees of freedom included in the subspace.

The reason for emphasizing these seemingly elementary points is that in contrast with the models illustrated here, for the Skyrme model the process of canonical quantization (or better Dirac quantization) is not unique because of the multitude of choices of a quantum Hamiltonian associated with the classical one. Each such choice leads to a different set of equations of motion at order $\hbar^2$. Furthermore, we have not been able to associate the equations of motion that follow from any of these choices with a variational principle. We have therefore taken the radical and simplifying step of defining a quantum theory for the Skyrme model by choosing as equations of motion those that follow from the trace variational method. This choice commends itself above all others for resolving the ambiguities involved in a quantization of the Skyrme model, because it also provides, as we report later in this paper, a simplified and powerful technique for solving the equations of motion.

Quantization by the trace variational principle does not, however, resolve all problems involved in the quantization. We still have to choose, separately, forms for observables such as the energy and the currents. The former enters as part of the solution algorithm to be applied, and both are needed for comparison with experiment. We shall return to these points in Sec. III C.

B. Kerman-Klein quantization

We have already described the essential formal aspects of the Kerman-Klein method, namely, the expression of Heisenberg matrix mechanics as a set of nonlinear equations of both dynamical and kinematical origin obtained by the use of completeness. This method was, however, first introduced for the restoration of broken symmetries in the nonrelativistic many-body problem [38]; it was first applied to the ϕ^4 model discussed above by Goldstone and Jackiw [39] in order to solve a problem of broken translational invariance. The application described in this paper belongs to the same class, aiming in this case to restore rotational invariance in ordinary space and in isospace, at the same time recognizing that this cannot be done without an improvement in the dynamical description of the system.

The application of the Kerman-Klein method to a many-body problem, be it relativistic or nonrelativistic, has proved to be of value for systems with mean-field or classical solution describing either a stable many-particle system (nontopological soliton) or, as in the problem under study, a topological soliton. The latter can be treated as a classical particle when its size is large compared to its Compton wavelength, or equivalently its mass is large compared to that of the elementary boson in the problem, where the latter is described by fields that appear directly in the Lagrangian.

The mean-field or classical solution always breaks some symmetry of the Lagrangian. The elaboration of which symmetries are broken, together with the remarks about relative scales in the previous paragraph, is sufficient to teach us how to implement the Kerman-Klein method, namely, how to elaborate the necessary elements of the Hilbert space, and how to subdivide those elements into subspaces that define successive approximations.

To be specific, let us review the situation for the one-dimensional model. The existence of a localized solitary-wave solution informs us that there is a heavy-particle sector in Hilbert space. The minimal way to restore the broken translational invariance and, incidentally, the broken Lorentz invariance, is to restrict the sum rules given above to a subset of states $\{|\chi\rangle\} \in \{|\psi\rangle\}$ that describe all possible states of linear momentum of the heavy particle. Since the number of heavy particles is a (topologically) conserved quantum number, the missing pieces of Hilbert space contain one or more light particles (mesons) in addition to the single heavy particle. The fact that there is approximate decoupling of any subspace containing one heavy particle and a fixed number of light particles arises from the property of the theory that a quantum fluctuation such as the emission or absorption of a light particle is proportional to the square root of the ratio of the mass of the light to that of the heavy particle, which thus serves the role of a small parameter in the theory and demonstrates the convergence of the so-called loop expansion for the model. Formally, let $\hat{A}$ and $\hat{B}$ represent field operators whose matrix elements are the objects that one would like to study. The evaluation of the matrix element of the product of these two operators between soliton states of different momentum can thus be written

$$\left\langle \chi \left| \hat{A}\hat{B} \right| \chi' \right\rangle = \sum_{\psi} \left\langle \chi \left| \hat{A} \right| \psi \right\rangle \left\langle \psi \left| \hat{B} \right| \chi' \right\rangle \cong \sum_{\chi''} \left\langle \chi \left| \hat{A} \right| \chi'' \right\rangle \left\langle \chi'' \left| \hat{B} \right| \chi' \right\rangle, \quad (3.11)$$

where the most important missing piece comes from intermediate states containing one additional meson, the one-loop contribution. As already implied by our previous discussion, this evaluation method can be applied to the computation of any matrix element of a finite product of operators to produce a result involving matrix elements of single operators.

Similar but more complicated considerations apply to the Skyrme model. The topological solution in this case breaks translational (and boost) invariance, as in the one-dimensional case. We have already treated this subject [47] using a choice of quantum fields different from the ones used in the present paper. This choice led to some technical difficulties that can be avoided in the present work. Of more physical interest for us is that the Skyrmion also breaks rotational and isorotational invariance; i.e., it is "deformed" in space and isospace. As is well known, this implies that the model predicts an infinite tower of states with spin J equal to isospin T. Restoration of the rotational invariance of the model at the quantum level requires that we choose this tower of states as the *minimal* set defining the collective subspace that plays the role of the set $|\chi\rangle$ in Eq. (3.11) above. Further discussion of how to treat this case is, of course, one of the essential elements of this work and will be taken up in context.

In general terms, we shall seek, within the collective subspace, some tractable set of c-number equations that can be solved for the matrix elements of the basic field operators between states of interest. The same approximate sum-rule technique is subsequently employed to evaluate operators associated with observables to predict values that can be compared with experiments. It is straightforward to use this approach for evaluating polynomial expressions, and this is why we have chosen to carry out the quantization with constrained field variables.

C. Formal quantization of the Skyrme model

Although in the end we have studied a slightly different form of the quantum theory than that obtained by slavishly following the Dirac method of quantization, we shall nevertheless begin by describing, briefly, the results of this procedure.

As discussed previously, the existence of a constraint condition requires that the methods of canonical quantization be modified. The Dirac formalism provides the most convenient approach for treating the constraint condition, dictating that the commutation relations take a noncanonical form determined by the Dirac brackets [Eq. (2.23)],

$$[\hat{\phi}_\alpha(\mathbf{x}), \hat{\phi}_\beta(\mathbf{y})] = 0, \quad (3.12)$$

$$[\hat{\phi}_\alpha(\mathbf{x}), \hat{\pi}_\beta(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) \left(\delta_{\alpha\beta} - \hat{\phi}_\alpha\hat{\phi}_\beta\right), \quad (3.13)$$

$$[\hat{\pi}_\alpha(\mathbf{x}), \hat{\pi}_\beta(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})\frac{1}{2}\left\{\hat{\pi}_\gamma, \hat{B}_\gamma^{\alpha\beta}\right\}, \quad (3.14)$$

$$\hat{B}_\gamma^{\alpha\beta} \equiv \delta_{\alpha\gamma}\hat{\phi}_\beta - \delta_{\beta\gamma}\hat{\phi}_\alpha. \quad (3.15)$$

The constraint conditions involve the operators

$$\hat{\chi}_1 \equiv \frac{1}{2}\left(\hat{\phi}_\alpha\hat{\phi}_\alpha - 1\right), \quad (3.16)$$

$$\hat{\chi}_2 \equiv \frac{1}{2}\left\{\hat{\phi}_\alpha, \hat{\pi}_\alpha\right\}, \quad (3.17)$$

which must annihilate all physically meaningful states. It is straightforward to show that the operators $\hat{\chi}_i$ do not evolve in time, so that an appropriate choice of initial conditions leads to a satisfaction of the constraints for all values of the time.

The dependence of the inertia density $\mathcal{M}_{\alpha\beta}$ on the field values implies that the Hamiltonian operator carries with it ordering uncertainties that lead to ambiguities at relative order $O(\hbar^2)$. One choice for the Hamiltonian operator is

$$\hat{H} = \int d^3x \left(\frac{1}{8}\left\{\hat{\pi}_\alpha, \left\{\hat{\pi}_\beta, \hat{\mathcal{M}}^{-1}_{\alpha\beta}\right\}\right\} + \hat{\mathcal{V}}\right), \quad (3.18)$$

where the inertia density matrix and the potential energy density are elevated to operators. This Hamiltonian, together with the commutation relations given above, leads to operator equations of motion that can be combined to form the operator equations

$$\begin{aligned}\frac{1}{4}&\left\{\frac{d}{dt}\left\{\dot{\hat{\phi}}_\epsilon, \hat{\mathcal{M}}_{\delta\epsilon}\right\}, \left(\delta_{\gamma\delta} - \hat{\phi}_\gamma\hat{\phi}_\delta\right)\right\} \\ &+\frac{1}{8}\partial_i\left\{\dot{\hat{\phi}}_\alpha, \left\{\dot{\hat{\phi}}_\beta, \frac{\partial\hat{\mathcal{M}}_{\alpha\beta}}{\partial\phi_{\delta,i}}\left(\delta_{\gamma\delta} - \hat{\phi}_\gamma\hat{\phi}_\delta\right)\right\}\right\} \\ &-\frac{1}{8}\left\{\dot{\hat{\phi}}_\alpha, \left\{\dot{\hat{\phi}}_\beta, \frac{\partial\hat{\mathcal{M}}_{\alpha\beta}}{\partial\phi_{\delta,i}}\partial_i\left(\delta_{\gamma\delta} - \hat{\phi}_\gamma\hat{\phi}_\delta\right)\right\}\right\} \\ &+\frac{\delta\hat{\mathcal{V}}}{\delta\phi_\delta}\left(\delta_{\gamma\delta} - \hat{\phi}_\gamma\hat{\phi}_\delta\right) = \hat{Q}_\gamma, \quad (3.19)\end{aligned}$$

where $\hat{Q}_\gamma$ is a quantum force that is $\mathcal{O}(\hbar^2)$,

$$\begin{aligned}\hat{Q}_\gamma = \frac{1}{4}\hat{\mathcal{M}}_{\alpha\beta}\hat{\phi}_\delta\left[\dot{\hat{\phi}}_\alpha, \left[\dot{\hat{\phi}}_\beta, \hat{\phi}_\gamma\hat{\phi}_\delta\right]\right] + \frac{1}{8}\left(\delta_{\delta\varphi} - \hat{\phi}_\delta\hat{\phi}_\varphi\right)\Bigg\{&\left(\partial_i\frac{\partial\hat{\mathcal{M}}_{\alpha\beta}}{\partial\phi_{\varphi,i}}\right)\left[\dot{\hat{\phi}}_\alpha, \left[\dot{\hat{\phi}}_\beta, \hat{\phi}_\gamma\hat{\phi}_\delta\right]\right] \\ &+\frac{\partial\hat{\mathcal{M}}_{\alpha\beta}}{\partial\phi_{\varphi,i}}\left(\partial_i\left[\dot{\hat{\phi}}_\alpha, \left[\dot{\hat{\phi}}_\beta, \hat{\phi}_\gamma\hat{\phi}_\delta\right]\right] - \left[\dot{\hat{\phi}}_\alpha, \left[\dot{\hat{\phi}}_\beta, \partial_i\left(\hat{\phi}_\gamma\hat{\phi}_\delta\right)\right]\right]\right)\Bigg\}. \quad (3.20)\end{aligned}$$

When the quantum force term is neglected, these equations are polynomial in the field operators $\hat{\phi}_\alpha$ and $\dot{\hat{\phi}}_\beta$. Even with the quantum force term, the equations are polynomial in the fields, provided we do not carry through the equal-time commutators. On the other hand, we have not succeeded in deriving these equations, in whichever form, from a variational principle. Since the existence of such a principle has proved to be of inestimable value in obtaining numerical solutions after making suitable closure approximations on intermediate-state sums, we have chosen to study the set of field equations that follow from a trace variational principle.

The procedure is simply to define a quantum Lagrangian that is a fully symmetrized (Hermitized) version of the classical Lagrangian [Eq. (2.3)]. Thus the appropriate functional for determining values of the matrix elements is

$$\begin{aligned} F_{\rm Sk} &= \frac{1}{\mathrm{Tr}[1]}\mathrm{Tr}\left[\hat{L}\right], \\ \hat{L} &= \frac{1}{g_\pi^2}\int d^3x\left[\frac{1}{2}\partial_\mu\hat{\phi}_\alpha\partial^\mu\hat{\phi}_\alpha + \beta_\pi^2\left(\hat{\phi}_0 - 1\right)\right. \\ &\quad + \frac{1}{2}\left\{\hat{\lambda}, \frac{1}{2}(\hat{\phi}_\alpha^2 - 1)\right\} \\ &\quad - \frac{1}{32}\left\{\left\{\partial_\mu\hat{\phi}_\alpha, \partial^\mu\hat{\phi}_\alpha\right\}, \left\{\partial_\nu\hat{\phi}_\beta, \partial^\nu\hat{\phi}_\beta\right\}\right\} \\ &\quad \left. + \frac{1}{32}\left\{\left\{\partial_\mu\hat{\phi}_\alpha, \partial^\nu\hat{\phi}_\alpha\right\}, \left\{\partial_\nu\hat{\phi}_\beta, \partial^\mu\hat{\phi}_\beta\right\}\right\}\right], \\ \hat{\lambda} &= \frac{1}{2}\left\{\hat{\phi}_\alpha, \Box\hat{\phi}_\alpha - \partial_\mu\left(\frac{1}{2}\left\{\partial^\mu\hat{\phi}_\alpha, \partial_\nu\hat{\phi}_\beta\partial^\nu\hat{\phi}_\beta\right\}\right)\right. \\ &\quad \left. + \partial_\mu\left(\frac{1}{4}\left\{\partial^\nu\hat{\phi}_\alpha, \left\{\partial_\nu\hat{\phi}_\beta, \partial^\mu\hat{\phi}_\beta\right\}\right\}\right) - \beta_\pi^2\delta_{\alpha 0}\right\}. \end{aligned} \tag{3.21}$$

The determination of the matrix elements is reduced to finding stationary values of $F_{\rm Sk}$. The field equations proper that follow from this variational expression are shown in Ref. [50], but are not displayed here since they are not needed. In the next section, this approach is used directly to study matrix elements of the field operators in a collective subspace of rotational states. Other aspects of the quantization, such as the choice of an energy operator and the topological current, will also be discussed.

IV. RESTORATION OF ROTATIONAL INVARIANCE

In this section, we apply the Kerman-Klein method to the study of the tower of spin-isospin states implied by the hedgehog solution of the Skyrme model. We first summarize the elements, discussed in Sec. III, necessary to carry out this study.

(1) Definition of a collective subspace.

(2) Selection of the basic field operators and an analysis of the matrix elements included in the study.

(3) Derivation of operator equations of motion (or, equivalently, a variational functional) and auxiliary conditions such as commutation relations and constraints.

(4) Determination of values for matrix elements of the fields by solution of the nonlinear equations provided by step (3) or, in our case, by replacement of the equations of motion by a variational principle.

(5) Evaluation of observables using these values for the matrix elements.

These steps are carried out in the subsections that follow. We shall also show that the hedgehog solution is a limiting case of our equations.

A. Collective subspace

Guided by the semiclassical results found from projecting out states with good spin and isospin from a cranked hedgehog configuration, the collective subspace of Hilbert space is chosen to consist of baryonlike states having definite values for the magnitude and z component of isospin (I, m_I) and spin (J, m_J), restricted to the values $I = J$, which are designated by $|I, m_I; J, m_J\rangle$. This space of states is spanned by two commuting representations of SU(2) subject to the condition that the magnitude of the spin is equal to the magnitude of the isospin. This collective subspace does not include any states possessing free mesons, necessary to account for loop effects. Without the inclusion of such quantum fluctuations, the sum rules based on the commutation relations cannot be satisfied, and therefore these do not play any role in the following discussion.

For brevity it is useful to assign labels to the states that will play the major roles in the following discussion:

N for $I = J = \frac{1}{2}$,
Δ for $I = J = \frac{3}{2}$,
V for $I = J = \frac{5}{2}$.

In contrast with the N and Δ states, the V states may be considered to be artifacts of the large-N_c limit that are included to allow a connection with previously derived hedgehog results.

B. Field operators and matrix elements

It is convenient to choose a form for the meson operators characterized by spherical indices in isospin space, $\hat{\phi}^{T_\alpha q_\alpha}$, where T_α is the magnitude of the isospin and q_α is the z component. (In future discussion, the ± 1 values for q_α may be written as $\pm$ where they appear as indices, and the isospin values $T = \{0, 1\}$ may be expressed by $T = \{\sigma, \pi\}$.) These field operators can be written in terms of mode operators $\hat{A}_{Lm}^{Tq}(x)$ with definite values of angular momentum (operator partial-wave decomposition):

$$\hat{\phi}^{Tq}(\mathbf{x}) = \sqrt{4\pi}\sum_{L=0}\sum_{m=-L}^{L}(i)^{L+T}\hat{A}_{Lm}^{Tq}(x)Y^{Lm}(\hat{x}). \tag{4.1}$$

In practice, the unrestricted sum over L will be cut off by a restriction on the maximum value of the angular momentum of the baryon states actually included in the final calculational scheme. Since the mode operators and the states of the collective subspace possess definite values of spin and isospin, the Wigner-Eckart theorem can be used to express matrix elements of the field operators as

$$\begin{aligned}\langle I_0, m_{I_0}; J_0, m_{J_0}|\,\hat{\phi}^{Tq}\,|I_1, m_{I_1}; J_1, m_{J_1}\rangle &= \sum_{L,m} \frac{\sqrt{4\pi}(i)^{L+T}}{(2I_0+1)} C^{I_0 m_{I_0}}_{I_1 m_{I_1} Tq} C^{J_0 m_{J_0}}_{Lm J_1 m_{J_1}} A^T_L(I_0, I_1) Y^{Lm}(\hat{x}) \\ &= \sum_{L,m} \sqrt{4\pi}(i)^{L+T}(-)^{2I_0 - m_{I_0} - m_{J_0}} \begin{pmatrix} I_0 & T & I_1 \\ -m_{I_0} & q & m_{I_1} \end{pmatrix} \begin{pmatrix} J_0 & L & J_1 \\ -m_{J_0} & m & m_{J_1} \end{pmatrix} \\ &\quad \times A^T_L(I_0, I_1) Y^{Lm}(\hat{x}), \end{aligned} \tag{4.2}$$

where $A^T_L(I_0, I_1) \equiv \langle I_0|||\hat{A}^T_L(x)|||I_1\rangle$ defines a reduced matrix element that depends on the radial coordinate.

Since the σ and the pions possess definite parity, the mode operators must satisfy the relations

$$\hat{A}^{Tq}_{Lm}(x) = (-)^{T+L}\hat{A}^{Tq}_{Lm}(x), \tag{4.3}$$

which require that half of the reduced matrix elements vanish identically. In addition, since matrix elements of field operators must satisfy

$$\langle\chi|\hat{\phi}^{Tq}|\chi'\rangle = \langle\chi'|(\hat{\phi}^{Tq})^\dagger|\chi\rangle^*, \tag{4.4}$$

it is necessary that $A^T_L(I', I)^* = A^T_L(I, I')$. This implies that the "diagonal" ($I = I'$) reduced matrix elements are real. In fact, all the solutions found in this paper consist of purely real reduced matrix elements, whether diagonal or off diagonal.

Formally, the set of c-number equations that follow from taking variations of the functional $F_{\rm Sk}$ with respect to matrix elements is infinite dimensional, and does not allow a natural decoupling into manageable subsets of equations. This difficulty arises because the isovector mode operators connect states with different values of isospin. A complete solution to the c-number equations requires, for example, the self-consistent determination of the following unbounded set of reduced matrix elements:

$$\left\langle \frac{1}{2}\middle|\middle|\middle|\hat{A}^\pi_L(x)\middle|\middle|\middle|\frac{3}{2}\right\rangle, \left\langle \frac{3}{2}\middle|\middle|\middle|\hat{A}^\pi_L(x)\middle|\middle|\middle|\frac{5}{2}\right\rangle, \ldots, \left\langle I\middle|\middle|\middle|\hat{A}^\pi_L(x)\middle|\middle|\middle|I+1\right\rangle, \ldots . \tag{4.5}$$

The method that we have chosen to obtain closure and to thus reduce the number of equations to a manageable size is to use results derived in the hedgehog limit (that will be discussed later) to relate values of the reduced matrix elements involving the unwanted high-spin states to functions involving states with $I \leq I_{\rm max}$, namely,

$$A^T_L(I, I') = \begin{cases} \left(\frac{\hat{I}^2}{\hat{I}^2_{\rm max}}\right) A^T_L(I_{\rm max}, I_{\rm max}) & \text{if } I = I', \\ \left(\frac{\hat{I}\hat{I}'}{\hat{I}_{\rm max}\widehat{(I_{\rm max}-1)}}\right) \delta_{T1}\delta_{L1} A^\pi_1(I_{\rm max}, I_{\rm max}-1) & \text{if } I - 1 = I', \\ \left(\frac{\hat{I}\hat{I}'}{\hat{I}_{\rm max}\widehat{(I_{\rm max}-1)}}\right) \delta_{T1}\delta_{L1} A^\pi_1(I_{\rm max}-1, I_{\rm max}) & \text{if } I + 1 = I', \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

In addition, energy differences involving states having $I > I_{\rm max}$ are fixed by assuming that the baryon energies correspond with those found for a rigid rotor; this leads to

$$E_{I+1} - E_I = \frac{I+1}{I_{\rm max}}\left(E_{I_{\rm max}} - E_{I_{\rm max}-1}\right). \tag{4.7}$$

The relations given in Eqs. (4.6) and (4.7) are used throughout this section. The merit of this form of closure is that the remaining equations still contain a limiting solution corresponding to the classical hedgehog, but with this symmetry-preserving formulation, the possibility of a richer and more complex physics emerges, in which the remaining amplitudes deviate from their hedgehog values.

In practice, we have chosen a cutoff at $I_{\rm max} = 3/2$, so as to be able to study at least the nucleon and the Δ particles. This cutoff still leaves us with seven radial functions, namely, the reduced matrix elements

$$A^\sigma_0(N,N), A^\pi_1(N,N), A^\pi_1(N,\Delta), A^\sigma_0(\Delta,\Delta), A^\pi_1(\Delta,\Delta), A^\sigma_2(\Delta,\Delta), A^\pi_3(\Delta,\Delta). \tag{4.8}$$

C. Evaluation of operator expressions

In this subsection, operator expressions are evaluated, utilizing the approximation scheme that has now been fully defined, to produce sets of c-number equations that can be used to determine the values of the reduced matrix elements. The required operator expressions are (i) the constraint condition, (ii) the quantum Lagrangian used to define the variational functional $F_{\rm Sk}$, and (iii) the energy. If meson loops were included in this treatment, the commutation relations would also need to be considered.

For example, matrix elements of the operator constraint condition are given by

$$\sum_{T,q} \langle I_0, m_{I_0}; J_0, m_{J_0}|\,(-)^q\hat{\phi}^{Tq}(\mathbf{x})\hat{\phi}^{T-q}(\mathbf{x})\,|I_2, m_{I_2}; J_2, m_{J_2}\rangle = \delta_{I_0 I_2}\delta_{m_{I_0} m_{I_2}}\delta_{m_{J_0} m_{J_2}}. \tag{4.9}$$

This expression can be evaluated using standard rules for coupling angular momenta, as described in Ref. [51]. Remembering the isoscalar nature of the constraint condition, we thus obtain three nontrivial equations involving only nucleon and Δ external states, one from the N-N channel ($L = 0$) and two from the Δ-Δ channel ($L = 0, 2$).

These are

$$C^N_{L=0} \equiv \frac{1}{4}\left[A^\sigma_0(N,N)^2 + A^\pi_1(N,N)^2 + |A^\pi_1(N,\Delta)|^2\right] - 1 = 0, \tag{4.10}$$

$$C^\Delta_{L=0} \equiv \frac{1}{16}\Bigg[A^\sigma_0(\Delta,\Delta)^2 + A^\pi_1(\Delta,\Delta)^2 + A^\sigma_2(\Delta,\Delta)^2 + A^\pi_3(\Delta,\Delta)^2 + |A^\pi_1(\Delta,N)|^2 + |A^\pi_1(\Delta,V)|^2 + |A^\pi_3(\Delta,V)|^2\Bigg] - 1 = 0, \tag{4.11}$$

$$C^\Delta_{L=2} \equiv \frac{1}{16}\Bigg[2A^\sigma_0(\Delta,\Delta)A^\sigma_2(\Delta,\Delta) - \frac{4}{5}A^\pi_1(\Delta,\Delta)^2 + \frac{6}{5}A^\pi_1(\Delta,\Delta)A^\pi_3(\Delta,\Delta) + \frac{4}{5}A^\pi_3(\Delta,\Delta)^2 + |A^\pi_1(\Delta,N)|^2 + \frac{1}{5}|A^\pi_1(\Delta,V)|^2 + \frac{4\sqrt{6}}{5}\mathrm{Re}[A^\pi_1(\Delta,V)A^\pi_3(V,\Delta)] - \frac{1}{5}|A^\pi_3(\Delta,V)|^2\Bigg] = 0. \tag{4.12}$$

In the remainder of the text we shall refer to these three as the "c-number constraint conditions."

The variational functional $F_{\rm Sk}$ is used as the basic calculational tool for the explicit determination of values for the matrix elements. Specialization of this functional, as defined in Eq. (3.21), to the collective subspace described in the beginning of this section allows it to be written as

$$F_{\rm Sk} = \frac{1}{\sum_{I_0}(2I_0+1)^2}\sum_{I_0 m_{I_0} m_{J_0}} \langle I_0, m_{I_0}; J_0, m_{J_0}|\,\hat{L}\,|I_0, m_{I_0}; J_0, m_{J_0}\rangle\,, \tag{4.13}$$

where $\hat{L}$ is the quantum Lagrangian defined in Eq. (3.21). For example, the quadratic contribution to this functional is given by

$$\begin{aligned} F_{\rm Sk,2} &= \frac{1}{\sum_{I_0}(2I_0+1)^2}\sum_{I_0 m_{I_0} m_{J_0}} \left\langle I_0, m_{I_0}; J_0, m_{J_0}\left|\int \frac{d^3x}{2g_\pi^2}\partial_\mu\hat{\phi}_\alpha\partial^\mu\hat{\phi}_\alpha\right| I_0, m_{I_0}; J_0, m_{J_0}\right\rangle \\ &= \frac{1}{g_\pi^2\sum_{I_0}(2I_0+1)^2}\int d^3x \sum_{I_0 I_1 T L}\left\{\left[\frac{1}{2}(E_{I_1}-E_{I_0})^2 - \frac{L(L+1)}{2x^2}\right]\left|A^T_L(I_0,I_1)\right|^2 - \frac{1}{2}\left|\partial_x A^T_L(I_0,I_1)\right|^2\right\}. \end{aligned} \tag{4.14}$$

The quartic contributions lead to additional, and more complicated, functions of the reduced matrix elements, which we shall spare the reader. For the choice of subspace made above, the sums over external states should only be taken over the nucleon and Δ states, $I_0 = \{1/2, 3/2\}$. The value $I = 5/2$ then occurs as one of the intermediate states, but the matrix elements that contain this state are eliminated by our closure approximation [Eq. (4.6)], at the same time that the corresponding unknown energy difference is eliminated by the use of Eq. (4.7).

In principle, the energy operator is not needed to determine the values of the reduced matrix elements, but in practice it is necessary to calculate the energy of states to ensure self-consistency in the value $\Delta E = E_\Delta - E_N$. We have already explained that due to the nontrivial nature of the inertia density matrix $\mathcal{M}_{\alpha\beta}$, the association of a quantum energy operator with the classical one is not unique. A particular ordering that is consistent with that of the quantum Lagrange operator used in the trace variational principle is the symmetric form, as given by

$$\begin{aligned}\hat{E} = \frac{1}{g_\pi^2}\int d^3x\Bigg\{ &\frac{1}{2}\left(\dot{\hat{\phi}}_\alpha\dot{\hat{\phi}}_\alpha + \partial_i\hat{\phi}_\alpha\partial_i\hat{\phi}_\alpha\right) - \beta_\pi^2\left(\hat{\phi}_0 - 1\right) + \frac{1}{4}\left[\left\{\dot{\hat{\phi}}_\alpha\dot{\hat{\phi}}_\alpha, \partial_i\hat{\phi}_\beta\partial_i\hat{\phi}_\beta\right\} + \left(\partial_i\hat{\phi}_\alpha\partial_i\hat{\phi}_\alpha\right)\left(\partial_j\hat{\phi}_\beta\partial_j\hat{\phi}_\beta\right)\right] \\ &-\frac{1}{4}\left[\frac{1}{4}\left\{\left\{\dot{\hat{\phi}}_\alpha, \partial_i\hat{\phi}_\alpha\right\}, \left\{\dot{\hat{\phi}}_\beta, \partial_i\hat{\phi}_\beta\right\}\right\} + \left(\partial_i\hat{\phi}_\alpha\partial_j\hat{\phi}_\alpha\right)\left(\partial_i\hat{\phi}_\beta\partial_j\hat{\phi}_\beta\right)\right]\Bigg\}. \end{aligned} \tag{4.15}$$

The energy of the state $|I, m_I; J, m_J\rangle$ is thus given by

$$E_I = \langle I, m_I; J, m_J|\,\hat{E}\,|I, m_I; J, m_J\rangle\,. \tag{4.16}$$

Once again we use completeness to replace the matrix elements of products of operators by a sum over elementary matrix elements.

D. Hedgehog limit

In this subsection, we obtain the values of the reduced matrix elements that follow from spin and isospin projection of the hedgehog solution. This not only establishes a connection between the hedgehog function $\theta_H(x)$ and the reduced matrix elements, but also provides the information that was needed to define the closure approximation.

For this analysis, we introduce a set of localized states $|A\rangle$, where A denotes a set of Euler angles. We may picture these states to correspond to hedgehog field distributions that have been obtained by carrying out an isorotation A on the standard defensive hedgehog of Eq. (2.10), which thus corresponds to the identity transformation. This picture is not to be taken literally, however, since it is strictly correct only in the classical limit.

Since the hedgehog is a collective state, where the collective degrees of freedom can be separated from the non-collective internal degrees of freedom, we now assume that the overlap between a state of good spin and isospin and the rotated hedgehog can be treated like the wave function of a rigid body. To distinguish the many-body state $|A\rangle$ from the rigid-body state without internal structure, we use a round bracket $|A)$ for this last state. We take the standard result [3] for the overlap of this state with a state of good spin and isospin:

$$(A|Im_Im_J) \equiv (-)^{I+m_I}\sqrt{\frac{2I+1}{2\pi^2}}D^I_{-m_Im_J}(A). \tag{4.17}$$

Thus the approximation

$$|A\rangle = \sum_{I,m_I,m_J} |I,m_I;J,m_J\rangle\,(Im_Im_J|A) \tag{4.18}$$

implies that

$$\langle A|A'\rangle = \delta(A-A'). \tag{4.19}$$

However, the statement

$$\langle A|\hat{\phi}_\alpha(x)|A'\rangle \cong \phi_\alpha(x_A)\delta(A'-A), \tag{4.20}$$

where x_A is the point to which x is rotated by the transformation A, is a physical assumption about the model, i.e., that the left-hand side of Eq. (4.20) is so sharply peaked in the variables of relative rotation and that a Δ function approximation is valid. It is also of interest to consider corrections to this extreme approximation, but this will not be done in the present paper.

It can be shown that the successive introduction of Eqs. (4.18) and (4.20) into the equations of motion for the full tower of states reduces these to those of the classical theory possessing the hedgehog solution. This allows the identification

$$\phi_\alpha(x_A) = \begin{cases} \cos\theta_H(x_A) & \text{if } \alpha = 0, \\ \frac{1}{2}\sum_\beta \left(\frac{x_b}{x}\right) \mathrm{Tr}\left[\tau_\alpha A \tau_\beta A^\dagger\right] \sin\theta_H(x_A) & \text{otherwise,} \end{cases} \tag{4.21}$$

where these expressions reduce to the hedgehog ansatz for $A = 1$, the identity transformation.

The equations given above generate the hedgehog approximation to the symmetry-preserving matrix elements of the fields, according to a formula that once again combines Eqs. (4.18) and (4.20), namely,

$$\langle I,m_I;J,m_J|\,\hat{\phi}_\alpha(x)\,|I',m_{I'};J',m_{J'}\rangle = \int dA(Im_Im_J|A)(A|I'm_{I'}m_{J'})\phi_\alpha(x_A). \tag{4.22}$$

Evaluation of this expression produces a combination of purely geometric factors and functions of $\theta_H(x)$. By identifying common factors between this expression and a similar one involving the mode operators $\hat{A}^{Tq}_{Lm}(x)$ [Eq. (4.2)], the reduced matrix elements are found to be related to $\theta_H(x)$ by

$$A^\sigma_L(I,I') = \delta_{II'}\delta_{L0}(2I+1)\cos\theta_H, \tag{4.23}$$

$$A^\pi_L(I,I') = \begin{cases} (-)^{I-I'}\delta_{L1}\frac{\sqrt{2I+1}\sqrt{2I'+1}}{\sqrt{3}}\sin\theta_H & \text{if } |I-I'| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.24}$$

These values satisfy the closure relations presented in Eq. (4.6).

The results of inserting the hedgehog values for the reduced matrix elements into the c-number expressions derived in Sec. III are as follows.

(1) The three c-number constraint equations [Eqs. (4.10)–(4.12)] are automatically satisfied.

(2) The variational functional $F_{\rm Sk}$ reduces to

$$F_{\rm Sk} = \frac{\sum_{I_0}(\hat{I}_0)^2\frac{1}{2}\Lambda_{I_0}\omega^2_{I_0}}{\sum_{I_0}(\hat{I}_0)^2} - V_{\rm cl}, \tag{4.25}$$

$$\Lambda_I \equiv \frac{8\pi}{3g^2_\pi}\int_0^\infty dx\,x^2\sin^2\theta_H\left[1+(\theta'_H)^2 + \left(\frac{I^2+I+3}{I^2+I+\frac{3}{2}}\right)\frac{\sin^2\theta_H}{x^2}\right], \tag{4.26}$$

$$\omega^2_I \equiv \left(\frac{I^2+I+\frac{3}{2}}{I^2+2I+1}\right)(E_{I+1}-E_I)^2 = \frac{4}{9}\left(I^2+I+\frac{3}{2}\right)(E_\Delta - E_N)^2, \tag{4.27}$$

and $V_{\rm cl}$ is the classical potential energy of the Skyrmion as given in Eq. (2.11). In the limit that all baryons have the same energy (degenerate limit), this functional reduces to the negative of the hedgehog mass and we thus reproduce the classical results.

(3) The energy of the Skyrmion is given by

$$E[\theta_H;I_0] = \frac{1}{2}\Lambda_{I_0}\omega^2_{I_0} + V_{\rm cl}. \tag{4.28}$$

E. Formal analysis of c-number equations

In this subsection, the c-number equations derived previously are studied to determine how they may be solved for the reduced matrix elements. In addition, with the aid of an operator version of the topologically conserved charge, we examine the role that baryon number plays in our solutions.

We have found it convenient to define a parameterization of the seven basic reduced matrix elements in terms of four independent radial functions $\{\psi_2(x), \psi_3(x), \psi_4(x), \psi_5(x)\}$ and one dependent function $\psi_1(x)$:

$$\begin{aligned}
A_0^\sigma(N,N) &= 2\cos\psi_1 \cos\psi_2, \\
A_1^\pi(N,N) &= 2\cos\psi_1 \sin\psi_2, \\
A_1^\pi(N,\Delta) &= -2\sin\psi_1, \\
A_0^\sigma(\Delta,\Delta) &= 4\cos\psi_1 \cos\psi_3 \cos\psi_4 \cos\psi_5, \\
A_2^\sigma(\Delta,\Delta) &= 4\cos\psi_1 \cos\psi_3 \cos\psi_4 \sin\psi_5, \\
A_1^\pi(\Delta,\Delta) &= 4\cos\psi_1 \sin\psi_3, \\
A_3^\pi(\Delta,\Delta) &= 4\cos\psi_1 \cos\psi_3 \sin\psi_4,
\end{aligned} \tag{4.29}$$

where $\psi_1(x)$ is defined by

$$\begin{aligned}
\psi_1 &= \arcsin\sqrt{\frac{\chi}{1+\chi}}, \\
\chi &\equiv 2\sin^2\psi_3 - 3\cos\psi_3 \sin\psi_3 \sin\psi_4 \\
&\quad - \cos^2\psi_3\left(2\sin^2\psi_4 + 5\cos^2\psi_4 \cos\psi_5 \sin\psi_5\right).
\end{aligned} \tag{4.30}$$

This parametrization satisfies the three c-number constraint conditions, and simplifies the determination of values for the reduced matrix elements. In the hedgehog limit, these radial functions are given by

$$\begin{aligned}
\psi_1 &= \arcsin\left(\sqrt{\frac{2}{3}}\sin\theta_H\right), \\
\psi_2 = \psi_3 &= \arctan\left(\frac{1}{\sqrt{3}}\tan\theta_H\right), \\
\psi_4 = \psi_5 &= 0.
\end{aligned} \tag{4.31}$$

Since the pions have odd parity, the isovector mode operators must have vanishing amplitudes at the origin. In order to have a finite-energy solution, the matrix elements of the pion field should vanish far from the center of the soliton. These requirements lead to the following conditions on the reduced matrix elements involving the isovector mode operators:

$$\begin{aligned}
A_L^\pi(I,I')(0) &= 0, \\
\lim_{x\to\infty} A_L^\pi(I,I')(x) &= 0.
\end{aligned} \tag{4.32}$$

A consideration of the constraint equations as well as the standard hedgehog boundary condition leads to the following set of boundary conditions:

$$\begin{aligned}
A_0^\sigma(I,I)(0) &= -(2I+1), \\
\lim_{x\to\infty} A_0^\sigma(I,I)(x) &= +(2I+1),
\end{aligned} \tag{4.33}$$

with all other reduced matrix elements vanishing at the origin and infinity. These boundary conditions can be related to boundary conditions on the five angle-valued functions, implying that all the functions are equal to integral multiples of π at both the origin and at infinity. If it is assumed that the optimal configuration is a perturbation of the hedgehog solution, these boundary conditions take the form given in Table I. Because they are found to vanish for configurations that possess a stationary value for the functional $F_{\rm Sk}$, reduced matrix elements involving the quadrupole and octupole modes are ignored in the remainder of this subsection.

The asymptotic behaviors of the radial functions can be determined by examining the variational functional $F_{\rm Sk}$ at large radius x. The function $\psi_2(x)$ is found to behave as a damped exponential:

$$\lim_{x\to\infty}\psi_2(x) \sim \frac{e^{-\beta_\pi x}}{x}, \tag{4.34}$$

similar to the decay properties of the hedgehog function of the static Skyrmion (assuming massive pion fields). The asymptotic behavior of $\psi_3(x)$ is given by

$$\begin{aligned}
\lim_{x\to\infty}\psi_3(x) &\sim \frac{e^{-\mu x}}{x}, \\
\mu^2 &= \beta_\pi^2 - \frac{31}{21}(\Delta E)^2,
\end{aligned} \tag{4.35}$$

where $\Delta E \equiv E_\Delta - E_N$. A damped solution for $\psi_3(x)$ exists when the pion mass term is taken sufficiently large that the pion radiation threshold exceeds the energy splitting. For the experimental value of $\Delta E = 293$ MeV, the required value for β_π corresponds to a pion having a mass of approximately 356 MeV. For any value of the mass below the critical one, the functions $\psi_1(x)$ and $\psi_3(x)$ have oscillatory tails, and calculations of the energy and of $F_{\rm Sk}$ lead to divergent results. In the results given below, this unconfined nature of the radial functions is regulated using two different approaches. The simplest approach is to take the energy splitting to be vanishing, $\Delta E = 0$, which returns us to the hedgehog limit. Of course, this degeneracy was inherent in the existence of a mean-field solution. A more physical approach is to restrict the range of time variability to bounded values of the radius, such as

$$\dot{\phi}_\alpha(\mathbf{x}) \to \dot{\phi}_\alpha(\mathbf{x})\Theta(x_{\rm max} - |\mathbf{x}|). \tag{4.36}$$

The sharp cutoff used here can of course be replaced by a smoother switching function, but does not lead to any problems. Since the effects of energy splittings are eliminated at large values of the radius, there are no oscillations in the functions $\psi_i(x)$ that can lead to divergences in the values of observables. This approach effectively separates the radiation terms from the core of the baryon-like excitation, and permits a systematic expansion of the remaining radiation terms, $\dot{\phi}_\alpha(\mathbf{x})[1 - \Theta(x_{\rm max} - |\mathbf{x}|)]$.

TABLE I. Boundary conditions for the five angle variables ψ_i, introduced in Eq. (4.29).

	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$	$\psi_5(x)$
$x=0$	0	π	π	0	0
$\lim x\to\infty$	0	0	0	0	0

However, these additional terms are not studied in the present account.

Before turning to a description of the results obtained, it is necessary to describe how we dealt with the question of baryon number conservation. We have been able to define a conserved symmetrized quantum-operator version of the classical topologically conserved current. It is then natural to associate the baryon number of a given state with the expectation value of the space integral of the associated density:

$$\hat{\mathcal{B}}^0(\mathbf{x}) = \frac{1}{12\pi^2}\varepsilon^{abcd}\varepsilon^{0ijk}\mathrm{SYM}\left\{\hat{\phi}_\alpha, \partial_i\hat{\phi}_\beta, \partial_j\hat{\phi}_c, \partial_k\hat{\phi}_d\right\}. \tag{4.37}$$

When the hedgehog-limiting values of the reduced matrix elements, given by Eqs. (4.23) and (4.24), are used to evaluate the matrix elements of this operator, it is found that all Skyrmion states have unit baryon number, which is only a consistency check. The topological argument that is used to show that the baryon number in classical solutions of the Skyrme model is quantized can no longer be used for the quantum theory. In particular, we find that expressions for the baryon numbers of the nucleon and Δ states do not reduce to surface integrals. However, these computations can be simplified under the condition $\psi_2(x) = \psi_3(x)$, leading to $B_N = 1$ and $B_\Delta = 1$. In addition, it can be shown that regardless of any splittings between the two independent radial functions, all states with isospin $I > 3/2$ have unit baryon number because of the relations used to close the set of equations.

A perturbative analysis shows that the deviations of B_N and B_Δ from unity ($\delta B_I \equiv B_I - 1$), when only terms linear in the difference $\epsilon(x) \equiv \psi_3(x) - \psi_2(x)$ are retained, are given by

$$\delta B_N = \frac{2}{\pi\sqrt{3}}\int_0^\infty dx\left\{\epsilon\left[\frac{\cos\psi_3\sin^3\psi_3\psi_3'}{\left(\frac{1}{2}+\sin^2\psi_3\right)^3}\right] + O(\epsilon^2)\right\},$$
$$\delta B_\Delta = -\frac{1}{2\pi\sqrt{3}}\int_0^\infty dx\left\{\epsilon\left[\frac{\cos\psi_3\sin^3\psi_3\psi_3'}{\left(\frac{1}{2}+\sin^2\psi_3\right)^3}\right] + O(\epsilon^2)\right\}. \tag{4.38}$$

The baryon numbers characterizing the nucleon and Δ states depend sensitively on $\psi_2(x)$ and $\psi_3(x)$. One way to enforce baryon number conservation is to add appropriate constraints and Lagrange multipliers to the functional F_{Sk}. We found that the imposition of these constraints did not significantly alter the predicted values for physical observables. The deviation of the baryon number from unity is small in all cases, typically $|\delta B| < 10^{-2}$.

F. Numerical results

For a self-consistent determination of the reduced matrix elements, the following steps are carried out.

(1) Assume a particular value for $E_\Delta - E_N$.

(2) Determine values for the reduced matrix elements using the constraint condition and the variational functional (this implementation employs relaxation techniques).

(3) Compute the energies of the nucleon and Δ states, and compare the calculation of $E_\Delta - E_N$ with the assumed value; if the values disagree, this process is repeated with a different assumed value for the energy splittings.

(4) Adjust the parameters of the model to obtain self-consistency at the observed value $E_\Delta - E_N = 293$ MeV.

Before discussing the new solutions determined by the procedure outlined, it is important to point out once more that our equations also admit the hedgehog solution as a special case that we obtain by imposing the restriction that the baryons are degenerate, namely, that $E_{I+1} = E_I$ for all values of the isospin I. In this limit, all terms in the functional that depend on time derivatives vanish identically. Although the general variational problem involves four independent radial functions, the optimal solution in this case is identical to the static hedgehog results described by a single function $\theta_H(x)$. In terms of the ψ functions defined above, this is equivalent to the statements that $\psi_2(x) = \psi_3(x)$, where these quantities are related to the standard hedgehog function by $\theta_H = \arccos(\cos\psi_1\cos\psi_2)$, and that the functions $\psi_4(x)$ and $\psi_5(x)$, associated with the quadrupole and octupole modes, vanish identically.

We find it convenient not to use Skyrme units in this section. We use r to denote a radial coordinate in fm.

To obtain new solutions, we must introduce a cutoff radius to restrict the time variability of the configurations at large values of the radius, as described in connection with Eq. (4.36). Two approaches for defining a value for the cutoff radius $r_{\max}$ were utilized: (i) $r_{\max}$ was adjusted until the calculation of $E_\Delta - E_N$ yielded 293 MeV (with the values of f_π and g_π fixed), and (ii) the value of $r_{\max}$ was fixed at the Compton wavelength of the pion while the input parameters f_π and g_π were varied until the masses of the nucleon and Δ states are predicted correctly.

Consider first the case that $r_{\max}$ is varied until the value for the energy splitting between the Δ states and the nucleon states is found self-consistently. An examination of the energy differences as a function of the cutoff radius is shown in Fig. 2, and compared with similar calculations using both a self-consistent determination of $\theta_{H;\omega}(x)$ and the adiabatic result $\theta_H(x)$. For the Kerman-Klein method, the calculated value $E_\Delta - E_N$ increases with cutoff radius since the effective moment of inertia, which leads to most of the energy splitting, can become large. As $r_{\max} \to 0$, the classical hedgehog results are recovered. This should be contrasted with results from the semiclassical approaches, where it is found that $E_\Delta - E_N$ decreases as the cutoff radius increases; this behavior occurs because the cranking frequency needed for self-consistency in the spin J becomes smaller as $r_{\max}$ increases. In Fig. 3, the predictions for the mass of the nucleon states are shown as a function of the cutoff radius. Values computed for observables using $f_\pi = 54.1$ MeV and $g_\pi = 4.842$ are shown as case (I) in Table II.

In the second case, the values of f_π and g_π are altered to permit correct predictions of the nucleon and Δ masses, while the cutoff $r_{\max}$ is fixed at the Compton wavelength of the pion, namely, $\lambda_\pi = 1.43$ fm. Self-

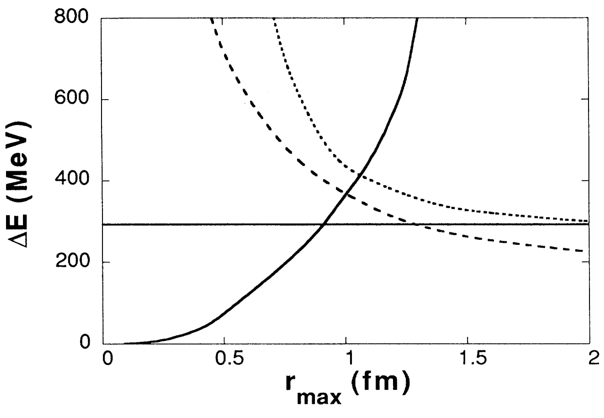

FIG. 2. Energy splitting between the nucleon and Δ states as a function of the cutoff radius $r_{\max}$. The solid line displays the results obtained from the Kerman-Klein method (Sec. IV), the dashed line describes the results found from a self-consistent determination of $\theta_{H;\omega}(r)$ (Sec. II B), and the dotted line shows corresponding results from the adiabatic approximation [2].

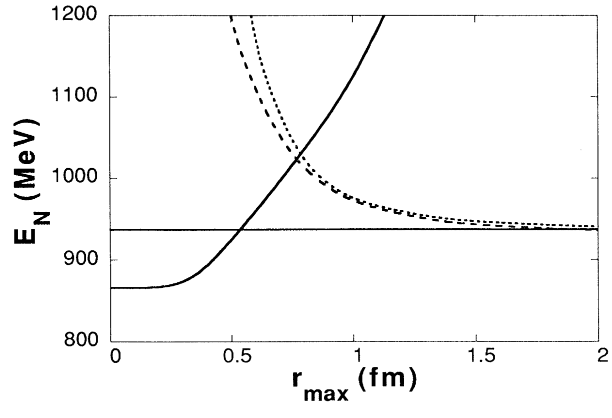

FIG. 3. Energy of the nucleon states as a function of the cutoff radius $r_{\max}$. The solid line displays the results obtained from the Kerman-Klein method (Sec. IV), the dashed line describes the results found from a self-consistent determination of $\theta_{H;\omega}(r)$ (Sec. II B), and the dotted line shows corresponding results from the adiabatic approximation [2].

consistency is found when $f_\pi = 56.0$ MeV and $g_\pi = 6.51$. Values calculated for observables using this method of fixing the input parameters are displayed for the Kerman-Klein method and the semiclassical approaches as case (II) in Table II. Values for the five reduced matrix elements involving the monopole and dipole mode operators are shown in Figs. 4–8. Note that the profiles found using the Kerman-Klein method are less spread out for the reduced matrix elements involving only nucleon states (compared with the semiclassical results), but more spread out when Δ states are involved.

Generally speaking, our results are of the same quality as or better than those obtained from the projection of the hedgehog results. In view of the very preliminary and

TABLE II. Values of the observables based on a restriction of the range of the time derivatives to $r < r_{\max}$. In case (I) the constants f_π and g_π are fixed to their standard values $f_\pi = 54.1$ MeV and $g_\pi = 4.842$, while the cutoff radius is varied until the $N - \Delta$ energy difference is found self-consistently. In case (II) the cutoff radius is fixed at the Compton wavelength of the pion, while f_π and g_π are varied to fit the masses of the nucleon and Δ states. Three different approaches are used to determine the configurations: the results obtained by using the Kerman-Klein method (Sec. IV) are listed in the columns labeled by (KK), and the results following from a self-consistent determination of $\theta_{H;\omega}(x)$ are denoted by (SC), while the standard results based on assuming that the rotation is adiabatic are listed under (AD).

		(I)			(II)		
	KK	SC	AD	KK	SC	AD	Expt.
f_π (MeV)	54.1	54.1	54.1	56.0	56.7	51.6	93
g_π	4.84	4.84	4.84	6.51	5.13	4.65	6.28
$r_{\max}$ (fm)	0.89	1.23	∞	1.43	1.43	1.43	
M_N (MeV)	1080	950	939	939	939	939	939
$\sqrt{\langle r^2\rangle_{E,T=0;N}}$ (fm)	0.68	0.73	0.68	0.77	0.69	0.73	0.72
$\sqrt{\langle r^2\rangle_{E,T=1;N}}$ (fm)	0.68	1.21	1.05	0.95	1.14	1.17	0.88
$\sqrt{\langle r^2\rangle_{M,T=0;N}}$ (fm)	0.66	0.99	0.96	0.90	0.96	1.02	0.82
$\sqrt{\langle r^2\rangle_{M,T=1;N}}$ (fm)	0.68	1.21	1.05	0.95	1.14	1.17	0.80
M_Δ (MeV)	1373	1245	1232	1232	1232	1232	1232
$\sqrt{\langle r^2\rangle_{E,T=0;\Delta}}$ (fm)	0.69	0.82	0.68	0.84	0.83	0.73	
$\sqrt{\langle r^2\rangle_{E,T=1;\Delta}}$ (fm)	0.71	1.28	1.05	0.96	1.34	1.17	
$\sqrt{\langle r^2\rangle_{M,T=0;\Delta}}$ (fm)	0.67	1.05	0.96	0.90	1.10	1.02	
$\sqrt{\langle r^2\rangle_{M,T=1;\Delta}}$ (fm)	0.70	1.28	1.05	1.04	1.34	1.17	
μ_n	−0.68	−0.98	−1.24	−1.11	−0.95	−1.18	−1.91
μ_p	2.02	1.92	1.97	3.73	1.82	2.02	2.79
μ_Δ^{++}	3.36	4.66	3.99	6.62	4.70	4.15	5–7

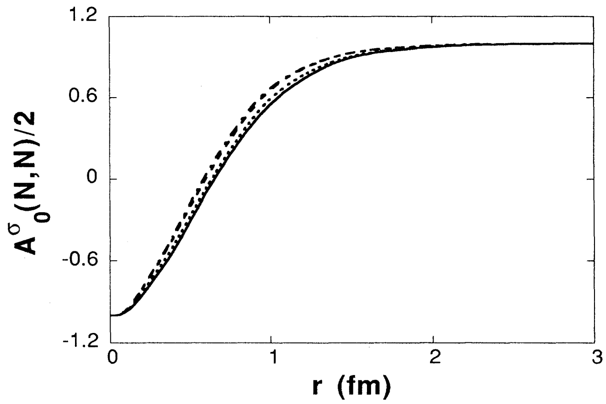

FIG. 4. Reduced matrix element $A_0^\sigma(N,N)$ as a function of the radius r, where the input parameters are determined by fixing the cutoff radius at the pion wavelength. The solid line displays the results obtained from the Kerman-Klein method (Sec. IV), the dashed line describes the results found from a self-consistent determination of $\theta_{H;\omega}(r)$ (Sec. II B), the dotted line shows corresponding results from the adiabatic approximation (with cutoff), and the short-long-dashed curve shows the usual adiabatic results as computed in Ref. [2].

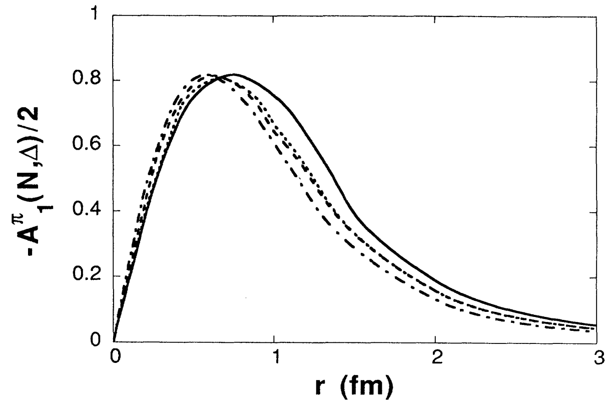

FIG. 6. Reduced matrix element $A_1^\pi(N,\Delta)$ as a function of radius. Refer to Fig. 4 for an explanation of the different curves.

still incomplete character of our theory, the most we can claim is to have made a possibly promising beginning.

V. SUMMARY AND OUTLOOK

Many studies of the Skyrme model have demonstrated that this is a useful approach for gaining a semiquantitative understanding of the properties of baryons, to an accuracy of about 30%. In the present work, we have applied the Kerman-Klein method as an alternative way of analyzing the Skyrme model. Our method of analysis is such that one is guaranteed to reproduce the known classical results in a suitable limit, but the main goal is to describe quantum effects that may be more difficult to obtain by other methods.

From many previous applications to a wide range of problems, it has become clear that the Kerman-Klein approach is a method for implementing the Heisenberg form of quantum mechanics that may be considered to consist of two stages. In the first, the formal stage, the operator equations of motion and the kinematical constraints (commutation relations, constraint equations, etc.) are turned into a set of c-number equations by evaluating matrix elements between exact eigenstates of the Hamiltonian and using the completeness relation for evaluating matrix elements of a product of operators. In the second, the practical stage, the physics of each special application is analyzed to specify a hierarchy of possible truncations of the full set of equations that define, in turn, a convergent sequence of approximations. It is the essence of these approximations that they can be chosen to preserve all the symmetries of the system.

In applying these ideas to the Skyrme model, a number of special problems are encountered.

(i) For the method to be useful, the equations of motion should be polynomial in the basic quantum variables. For the Skyrme model, this dictates the choice of redundant field variables, necessitating, in turn, the application of Dirac's method of quantization.

(ii) The Skyrme model has a field-dependent mass, i.e.,

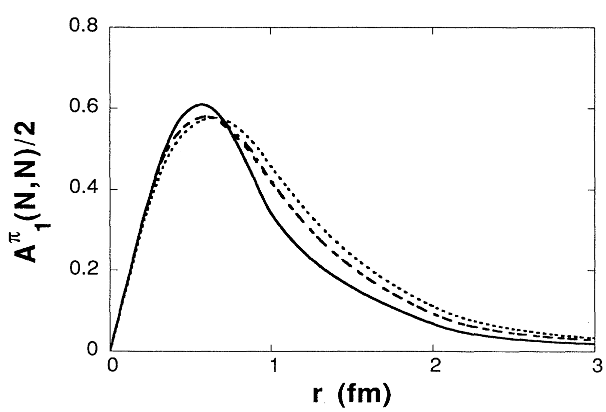

FIG. 5. Reduced matrix element $A_1^\pi(N,N)$ as a function of radius. Refer to Fig. 4 for an explanation of the different curves.

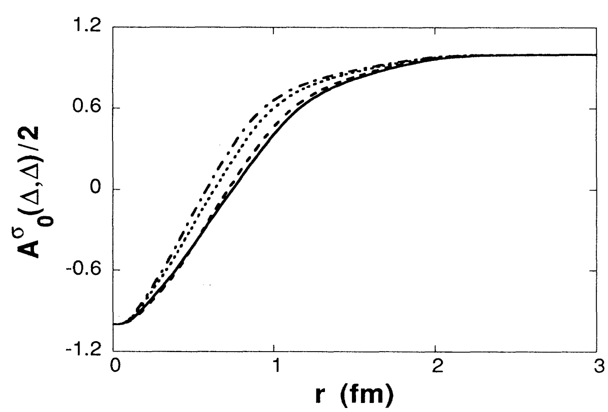

FIG. 7. Reduced matrix element $A_0^\sigma(\Delta,\Delta)$ as a function of radius. Refer to Fig. 4 for an explanation of the different curves.

is defined on a curved field space. For such a model, there are ordering problems involved in the quantization; as a consequence one cannot associate a unique quantum theory with the classical theory. For all previous applications, the equations of motion could be derived from, and therefore be replaced by, a variational principle where the stationary functional is the trace of a Lagrange operator over the eigenstates of interest. For the Skyrme model, we have not succeeded in establishing such a relationship for the equations of motions that emerge for any of the choices of operator ordering that we studied. We have, consequently, taken the unusual step of defining the equations of motion used as those that follow from the variational principle. These are, after all, a set of quantum equations that reduce properly to the classical limit. Furthermore, the variational method becomes the basis for a solution algorithm.

(iii) The resulting formulation has been used to study the spectrum and other properties of the one-baryon, zero-strangeness sector. To satisfy in a minimal way all the requirements that have been imposed in our formulation, the minimum choice of a Hilbert space is the spin-isospin tower of states implied by the existence of the hedgehog solution. In practice, we have suggested a means, based on the limiting properties deduced from that solution, to reduce the problem to a coupled-channel calculation for the nucleon and Δ.

(iv) Because the Δ can radiate a pion, we are thwarted, initially, in our goal to treat both nucleon and Δ as stable particles. By the introduction of a cutoff radius in a special way, we are able to to suppress the radiation field, at the same time leaving open the door for its reintroduction at a later stage of the calculation. We end up with a well-determined set of rotationally invariant equations that describe a stable nucleon and a stable Δ.

The solution of these equations for various radial form factors within the space considered provides a basis for the computation of some electromagnetic moments. The results are satisfactory, but we feel that the main contribution of this paper has been to suggest (radically) new methods for the study of models of the Skyrme type. Without attempting to be exhaustive, it is easy to list a number of feasible, though not necessarily trivial, extensions of the calculations presented in this paper.

(1) Inclusion of more states in the coupled-channel equations. For instance, it is not clear that one can neglect the Roper resonance.

(2) Study of the Δ width, by including the radiative decay.

(3) Simultaneous restoration of rotational and translational invariance.

(4) Addition of meson states to reexamine pion-baryon scattering and the estimation of one-loop effects on one-baryon properties.

(5) Application to extended Skyrme models, particularly those in which stabilization of the hedgehog is achieved by the introduction of vector mesons.

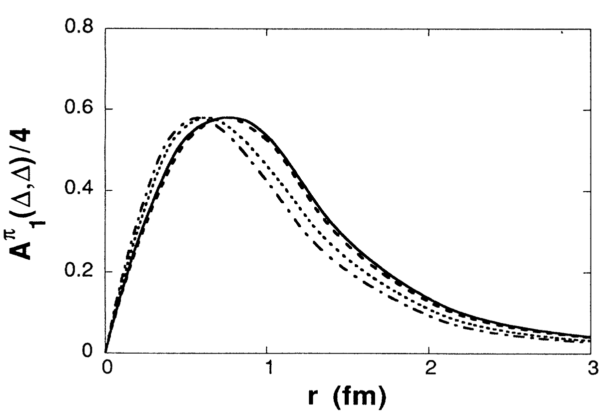

FIG. 8. Reduced matrix element $A_1^{\pi}(\Delta,\Delta)$ as a function of radius. Refer to Fig. 4 for an explanation of the different curves.

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APPENDIX: ELECTROMAGNETIC CURRENT AND ASSOCIATED OBSERVABLES

The electromagnetic current operator is defined by

$$\begin{aligned}\hat{J}^{\mu}_{\rm EM}(\mathbf{x}) &= \frac{1}{2}\hat{\mathcal{B}}^{\mu}(\mathbf{x}) + \hat{J}^{\mu}_{v;3}(\mathbf{x}),\\ \hat{\mathcal{B}}^{\mu}(\mathbf{x}) &= -\frac{1}{12\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\lambda\rho}\\ &\quad\times\mathrm{SYM}\left\{\hat{\phi}_{\alpha},\partial_{\nu}\hat{\phi}_{\beta},\partial_{\lambda}\hat{\phi}_{\gamma},\partial_{\rho}\hat{\phi}_{\delta}\right\},\\ \hat{J}^{\mu}_{v;a}(\mathbf{x}) &= \frac{\varepsilon_{abc}}{g_{\pi}^2}\left(\frac{1}{2}\left\{\hat{\phi}_b,\partial^{\mu}\hat{\phi}_c\right\}\right.\\ &\quad-\frac{1}{8}\left\{\left\{\hat{\phi}_b,\partial^{\mu}\hat{\phi}_c\right\},\left\{\partial_{\nu}\hat{\phi}_{\delta}\partial^{\nu}\hat{\phi}_{\delta}\right\}\right\}\\ &\quad\left.+\frac{1}{8}\left\{\left\{\hat{\phi}_b,\partial^{\nu}\hat{\phi}_c\right\},\left\{\partial_{\nu}\hat{\phi}_{\delta}\partial^{\mu}\hat{\phi}_{\delta}\right\}\right\}\right). \quad \text{(A1)}\end{aligned}$$

The lowest moment of this current is given by the integral over space of the time component:

$$\hat{Q} = \int d^3x\left[\frac{1}{2}\hat{\mathcal{B}}^0(\mathbf{x}) + \hat{J}^0_{v;3}(\mathbf{x})\right]; \quad \text{(A2)}$$

this operator can be identified with the charge of the system in units of e. In the hedgehog limit, the evaluation of this operator between collective (baryon) states can be performed in a straightforward manner since the first term represents the baryon density, and the second term is the density of the z component of isospin (this correspondence is shown in Ref. [3]); therefore the matrix element of this charge operator is given by

$$\langle I,m_I;J,m_J|\,\hat{Q}\,|I,m_I;J,m_J\rangle = \frac{1}{2} + m_I. \quad \text{(A3)}$$

The "electric" mean-square radius for a given configuration is defined by the expression

$$\langle x^2\rangle_E = \langle I, m_I; J, m_J| \int d^3x \hat{J}^0_{\rm EM}(\mathbf{x}) x^2 \,|I, m_I; J, m_J\rangle\,,$$
$$\equiv \frac{1}{2}\langle x^2\rangle_{E,T=0} + m_I \langle x^2\rangle_{E,T=1}\,, \qquad \text{(A4)}$$

where the isoscalar and isovector contributions to this result have been separated.

The differential magnetic density is defined by

$$\hat{\mu}^i(\mathbf{x}) = \varepsilon_{ijk} x_j \hat{J}^k_{\rm EM}(\mathbf{x}). \qquad \text{(A5)}$$

The values of the magnetic moments of specific states are given by

$$\mu_\chi \equiv 2M_N \int d^3x \,\langle \chi\,|\hat{\mu}^3(\mathbf{x})|\,\chi\rangle\,, \qquad \text{(A6)}$$

where M_N is the mass of the nucleon. Weighting these integrands by x^2 and normalizing by the magnetic moment μ_χ allows the magnetic mean-square radius $\langle x^2\rangle_M$ of the states $|\chi\rangle$ to be determined. (Of course, it is also straightforward to calculate the mean-square radius with respect to either the isoscalar or isovector contributions.) Evaluation of these electromagnetic operators in the collective subspace allows the above expressions to be written in terms of the reduced matrix elements.

The expressions for these observables are considerably simplified in the hedgehog limit, as summarized below and discussed in more detail in Ref. [3]. The moments of the charge density are given by

$$\langle x^2\rangle_{E,T=0} = -\frac{2}{\pi}\int_0^\infty dx(x^2)\sin^2\theta_H\theta'_H,$$
$$\langle x^2\rangle_{E,T=1} = \frac{8\pi}{3\Lambda}\int_0^\infty dx(x^2)\sin^2\theta_H[x^2 + x^2(\theta'_H)^2 + \sin^2\theta_H]. \qquad \text{(A7)}$$

It has been argued in Ref. [3] that computations of magnetic moments should utilize the adiabatic approximation to justify the neglect of terms having two time derivatives. (For the results obtained using the Kerman-Klein method, a corresponding omission of these terms leads to decreases of order 1% in the values of the isovector magnetic radius, and reductions of order 0.2 for the magnetic moments.) This approach simplifies the calculations of magnetic moments, and leads to the following results:

$$\langle x^2\rangle_{M,T=0} = \frac{\langle x^4\rangle_{E,T=0}}{\langle x^2\rangle_{E,T=0}},$$
$$\langle x^2\rangle_{M,T=1} = \langle x^2\rangle_{E,T=1}\,, \qquad \text{(A8)}$$

$$\mu_n = 2M_N\left(\frac{1}{12\Lambda}\langle x^2\rangle_{E,T=0} - \frac{\Lambda}{6}\right),$$
$$\mu_p = 2M_N\left(\frac{1}{12\Lambda}\langle x^2\rangle_{E,T=0} + \frac{\Lambda}{6}\right), \qquad \text{(A9)}$$
$$\mu_{\Delta^{++}} = 2M_N\left(\frac{1}{4\Lambda}\langle x^2\rangle_{E,T=0} + \frac{3\Lambda}{10}\right).$$

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