Flux-tube model, quark-antiquark potential, and Bethe-Salpeter kernel

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(Received 27 March 1992; revised manuscript received 7 October 1992)

We reconsider the problem of the quantization of the relativistic flux-tube model already treated in the literature and show how to make it completely consistent with the effective quark-antiquark Hamiltonian derived from QCD in the context of the Wilson loop approach. We obtain an explicit form for the model Hamiltonian as an expansion in the string tension constant and construct an instantaneous related Bethe-Salpeter kernel including even the spin and the short-range part of the interaction.

PACS number(s): 12.40.Aa, 11.10.St, 12.38.Lg, 12.40.Qq

I. INTRODUCTION

As is well known, an effective semirelativistic quark-antiquark Hamiltonian $H_{q\bar{q}}$ can be derived in the framework of QCD by some appropriate generalizations of the Wilson loop method [1–3]. Such a Hamiltonian has been very useful for an understanding of heavy quarkonium but, since it proceeds in terms of an $1/m^2$ expansion it cannot be significantly extended to the light-quarkonium case.

Among the various attempts to generalize in a more complete relativistic way the above formalism a particularly simple one is provided by the so-called flux-tube model [4,5]. This model rests on the observation that, in presence of a quark-antiquark static pair, the functional integral for the gauge field must be dominated by a purely chromoelectric flux tube connecting the two quarks and the energy σr carried by the tube must be simply proportional to its length r. This is indicated by the Wilson area law [1] and confirmed by numerical lattice simulations [6]. Then the basic area is that if the two quarks move, the flux tube is dragged by them in such a way that in a reference frame comoving with a given segment dr of the tube the energy carried by the segment is again $\sigma\, dr$.

This suggests to assume as the classical Lagrangian for the system *quarks plus flux tube* the expression

$$L=-m_1\sqrt{1-v_1^2}-m_2\sqrt{1-v_2^2}-\sigma\int_0^r dr'\sqrt{1-v_t^{2\prime}} \tag{1.1}$$

that properly defines the model. Here $\mathbf{x}_1,\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2$ are the positions and the velocity of the quark and the antiquark, respectively, $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$ is their relative position, and $\mathbf{v}_t'$ is the purely transverse velocity of the flux-tube segment at the distance r' from the antiquark 2. If the flux tube is treated as a rigid rod with respect to the rotation motion one can write

$$\mathbf{v}_t'=\frac{r'}{r}\mathbf{v}_{1t}+\left[1-\frac{r'}{r}\right]\mathbf{v}_{2t}\ , \tag{1.2}$$

where $\mathbf{v}_{1t}$ and $\mathbf{v}_{2t}$ are the transverse parts of the velocities of the quark and the antiquark, respectively, defined as $\mathbf{v}_{jt}=\mathbf{v}_j-(\mathbf{v}_j\cdot\mathbf{r}/r)(\mathbf{r}/r)$. Notice that the notation used by us is somewhat different from that used in Refs. [4,5] to meet better with our following developments.

In Ref. [4] it is shown how one can quantize the model and the result is applied to the evaluation of the Regge trajectories for a light-quark–antiquark system obtaining straight line trajectories with the correct slope. In Ref. [5] it is also shown that the classical Hamiltonian $\mathcal{H}$ of the flux-tube model coincides at order $1/m^2$ with the confinement part of the effective semirelativistic Hamiltonian $H_{q\bar{q}}$. This is apart from spin related terms, precisely Darwin and purely Thomas precession terms. However, the operator ordering prescription used in the quantization $H_{\rm FT}$ of $\mathcal{H}$ in Ref. [4] is different from that occurring in $H_{q\bar{q}}$ (cf. Sec. II) and so the reordering terms are significantly different in the two cases.

The point is that in Ref. [4] the classical Hamiltonian $\mathcal{H}$ can be obtained only in an implicit way; its quantization is not straightforward and it is performed indirectly in terms of the square radial momentum $q_r^2=-[d^2/dr^2+(2/r)(d/dr)]$ and the square angular momentum $L^2=l(l+1)$; this brings as a consequence a natural order prescription which is not the "correct" one.

In this paper we want to reconsider the problem of the quantization of the model and show how the quantization can be performed even in a Cartesian framework. Then a well defined and very tractable ordering prescription can be given for which $H_{\rm FT}$ becomes identical to the confining spin-independent part of $H_{q\bar{q}}$. We shall also see that the result can be immediately combined with the short-range part of the interaction in an instantaneous Bethe-Salpeter kernel which in the semirelativistic limit reproduces $H_{q\bar{q}}$, spin-dependent terms included, but possibly with some differences in the long-range part of the Darwin term. We give also an explicit expression of $\mathcal{H}(\mathbf{r},\mathbf{q})$ in terms of an expansion in the string tension σ which has exact relativistic kinematics and could be actually already used for the study of light meson spectrum. [1]

[1]Calculations in this line are in progress.

The plan of the paper is the following one. In Sec. II we rewrite $H_{q\bar{q}}$ in a form convenient for us. In Sec. III we simply review the Hamiltonian formulation of the flux-tube model in a Cartesian framework and give an explicit expression for $\mathcal{H}(\mathbf{r},\mathbf{q})$ both as an expansion in $1/m^2$ and in σ. In Sec. IV we discuss the quantization and elaborate the ordering prescription. In Sec. V we deal with the form of the Bethe-Salpeter (BS) kernel.

II. THE $q\bar{q}$ POTENTIAL

We recall that in the center-of-mass system ($\mathbf{p}_1=-\mathbf{p}_2=\mathbf{q}$) the effective semirelativistic quark-antiquark Hamiltonian derived in the context of the Wilson loop formalism can be written [3]

$$H_{q\bar{q}}=m_1+m_2+\frac{q^2}{2m_1}+\frac{q^2}{2m_2}-\frac{q^4}{8m_1^3}-\frac{q^4}{8m_2^3}+V_{\rm stat}+V_{\rm SD}+V_{\rm VD}\ , \tag{2.1}$$

with

$$V_{\rm stat}=-\frac{4}{3}\frac{\alpha_s}{r}+C+\sigma r\ , \tag{2.2a}$$

$$\begin{aligned}V_{\rm SD}=&\left[\frac{1}{m_1^2}\mathbf{L}\cdot\mathbf{S}_1+\frac{1}{m_2^2}\mathbf{L}\cdot\mathbf{S}_2\right]\frac{1}{2}\left[\frac{4}{3}\frac{\alpha_s}{r^3}-\frac{\sigma}{r}\right]\\&+\frac{1}{m_1m_2}(\mathbf{L}\cdot\mathbf{S})\frac{4}{3}\frac{\alpha_s}{r^3}\\&+\frac{4\alpha_s}{m_1m_2r^3}S_1^h\left[\frac{r^hr^k}{r^2}-\frac{1}{3}\delta^{hk}\right]S_2^k\\&+\frac{32\pi\alpha_s}{9m_1m_2}\mathbf{S}_1\cdot\mathbf{S}_2\delta_3(\mathbf{r})\ ,\end{aligned} \tag{2.2b}$$

$$\begin{aligned}V_{\rm VD}=&\frac{1}{8}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]\nabla^2\left[-\frac{4\alpha_s}{3r}+\sigma r\right]\\&-\frac{1}{m_1m_2}\{\hat{q}^h\hat{q}^kS^{hk}(\mathbf{r})\}_{\rm ord}\\&+\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]\{\hat{q}^h\hat{q}^kT^{hk}(\mathbf{r})\}_{\rm ord}\ .\end{aligned} \tag{2.2c}$$

Here the symbol $\{\ \}_{\rm ord}$ denotes the ordering prescription,

$$\begin{aligned}\{q^hq^kX^{hk}(\mathbf{r})\}_{\rm ord}:=&\frac{2}{3}\frac{1}{4}\{q^h,\{q^k,X^{hk}(\mathbf{r})\}\}\\&+\frac{1}{3}\cdot\frac{1}{2}\{q^hq^k,X^{hk}(\mathbf{r})\}\\=&\frac{1}{6}[2q^hq^kX^{hk}(\mathbf{r})+q^hX^{hk}(\mathbf{r})q^k\\&+q^kX^{hk}(\mathbf{r})q^h+2X^{hk}(\mathbf{r})q^kq^k]\ ,\end{aligned} \tag{2.3}$$

and we have set

$$S^{hk}=\delta^{hk}\left[\frac{8\alpha_s}{9r}-\frac{\sigma r}{9}\right]+\left[\frac{1}{3}\delta^{hk}-\frac{r^hr^k}{r^2}\right]\left[\frac{2\alpha_s}{3r}-\frac{\sigma r}{6}\right]\ , \tag{2.4}$$

$$T^{hk}=\delta^{hk}\left[-\frac{C}{4}-\frac{\sigma r}{9}\right]+\left[\frac{1}{3}\delta^{hk}-\frac{r^hr^k}{r^2}\right]\left[-\frac{\sigma r}{6}\right]\ .$$

It should be stressed that the ordering prescription (2.3) is not arbitrary as explained in Ref. [3]. In fact it derives from a discretization rule in the two particle Feynman integral which is essential in order to obtain a *finite* velocity-dependent potential.

We recall that in Eqs. (2.2)–(2.4) the short-range terms in α_s have a perturbative character, the terms in σ correspond to the Wilson area law and to its generalization to the case of a distorted loop for nonstatic quarks, and the terms in C come from an additional perimeter contribution [3]. Such terms are essentially related to the classical dominant configuration of the gauge field in the functional integral expressing the potentials.

Disregarding the terms in α_s and in C, Eq. (2.1) takes the form

$$\begin{aligned}H_{q\bar{q}}=&m_1+m_2+\frac{q^2}{2m_1}+\frac{q^2}{2m_2}-\frac{q^4}{8m_1^3}-\frac{q^4}{8m_2^3}+\sigma r\\&-\frac{1}{2}\left[\frac{1}{m_1^2}\mathbf{L}\cdot\mathbf{S}_1+\frac{1}{m_2^2}\mathbf{L}\cdot\mathbf{S}_2\right]\frac{\sigma}{r}\\&+\frac{1}{8}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}\right]\nabla^2(\sigma r)\\&-\frac{\sigma}{6}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}-\frac{1}{m_1m_2}\right]\{q_t^2r\}_{\rm ord}\ ,\end{aligned} \tag{2.5}$$

with the transverse momentum $\mathbf{q}_t$ defined again as

$$\mathbf{q}_t=\mathbf{q}-\mathbf{q}_r=\mathbf{q}-\left[\mathbf{q}\cdot\frac{\mathbf{r}}{r}\right]\frac{\mathbf{r}}{r}\ . \tag{2.6}$$

III. THE RELATIVISTIC FLUX-TUBE MODEL

Since r is not a relativistic invariant the Lagrangian (1.1) of the model has to be understood in the center-of-mass system. If we treat, however, for the moment, the positions $\mathbf{x}_1,\mathbf{x}_2$ as independent variables, the conjugate momenta are

$$\begin{aligned}\mathbf{p}_1=&\frac{\partial L}{\partial\dot{\mathbf{v}}_1}=\frac{m_1\mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}}+\sigma\int_0^r dr'\frac{(r'/r)\mathbf{v}_t'}{\sqrt{1-\mathbf{v}_t^{2\prime}}}\ ,\\\mathbf{p}_2=&\frac{\partial L}{\partial\dot{\mathbf{v}}_2}=\frac{m_2\mathbf{v}_2}{\sqrt{1-\mathbf{v}_2^2}}+\sigma\int_0^r dr'\frac{[1-(r'/r)]\mathbf{v}_t'}{\sqrt{1-\mathbf{v}_t^{2\prime}}}\ ,\end{aligned} \tag{3.1}$$

while the total linear momentum and the Hamiltonian turn out, respectively, as

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2=\frac{m_1\mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}}+\frac{m_2\mathbf{v}_2}{\sqrt{1-\mathbf{v}_2^2}}+\sigma\int_0^r dr'\frac{\mathbf{v}_t'}{\sqrt{1-\mathbf{v}_t^{2\prime}}} \tag{3.2}$$

and

$$\mathcal{H}=\mathbf{p}_1\cdot\mathbf{v}_1+\mathbf{p}_2\cdot\mathbf{v}_2-L =\frac{m_1}{\sqrt{1-v_1^2}}+\frac{m_2}{\sqrt{1-v_2^2}}+\sigma\int_0^r\frac{dr'}{\sqrt{1-v_t^{2\prime}}} . \tag{3.3}$$

Then, if we restrict ourselves to the center-of-mass system by setting

$$\mathbf{p}_1+\mathbf{p}_2=0$$

and perform explicitly the integrations occurring in (3.3) and (3.1), we obtain

$$\mathbf{q}=\frac{m_1\mathbf{v}_1}{\sqrt{1-\mathbf{v}_1^2}}+\sigma f_{11}(v_{1t},v_{2t})\mathbf{v}_{1t}+\sigma f_{12}(v_{1t},v_{2t})\mathbf{v}_{2t} ,$$
$$-\mathbf{q}=\frac{m_2\mathbf{v}_2}{\sqrt{1-\mathbf{v}_2^2}}+\sigma f_{21}(v_{1t},v_{2t})\mathbf{v}_{1t}+\sigma f_{22}(v_{1t},v_{2t})\mathbf{v}_{2t} \tag{3.4}$$

and

$$\mathcal{H}=\frac{m_1}{\sqrt{1-v_1^2}}+\frac{m_2}{\sqrt{1-v_2^2}}+\frac{\sigma r}{(v_{1t}+v_{2t})}[\arcsin v_{1t}+\arcsin v_{2t}] , \tag{3.5}$$

with $\mathbf{q}=\mathbf{p}_1=-\mathbf{p}_2$ and

$$f_{12}(v_{1t},v_{2t})=f_{21}(v_{1t},v_{2t})=\frac{r}{(v_{1t}+v_{2t})^3}\{[(v_{2t}-\tfrac{1}{2}v_{1t})\sqrt{1-v_{1t}^2}+(v_{1t}-\tfrac{1}{2}v_{2t})\sqrt{1-v_{2t}^2}]$$
$$+(v_{2t}v_{1t}-\tfrac{1}{2})(\arcsin v_{1t}+\arcsin v_{2t})\} ,$$
$$f_{11}(v_{1t},v_{2t})=f_{22}(v_{2t},v_{1t})=\frac{r}{(v_{1t}+v_{2t})^3}\{\tfrac{1}{2}[-(v_{1t}+4v_{2t})\sqrt{1-v_{1t}^2}+3v_{2t}\sqrt{1-v_{2t}^2}]$$
$$+(v_{2t}^2+\tfrac{1}{2})(\arcsin v_{1t}+\arcsin v_{2t})\} . \tag{3.6}$$

In principle Eq. (3.4) should be solved with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$ and the result replaced in (3.5) in order to obtain $\mathcal{H}$ as a function of the canonical variables $\mathbf{r}$ and $\mathbf{q}$. It is obvious that this cannot be done in a closed form. As we said, however, an explicit solution of (3.4) can be obtained only as an expression in $1/m^2$ or in the string tension σ and then even $\mathcal{H}$ is expressed in such a form.

Using an expansion in the inverse of the masses one obtains

$$\mathcal{H}=m_1+m_2+\frac{q^2}{2m_1}+\frac{q^2}{2m_2}-\frac{1}{8}\left[\frac{1}{m_1^3}+\frac{1}{m_2^3}\right]q^4+\sigma r-\frac{\sigma r}{6}\left[\frac{1}{m_1^2}+\frac{1}{m_2^2}-\frac{1}{m_1m_2}\right]\mathbf{q}_t^2 . \tag{3.7}$$

As observed in Ref. [5], Eq. (3.7) is identical to Eq. (2.5) apart from the Darwin term, the spin-dependent term, and possibly terms related to a reordering of noncommuting operators. This shows that a quantization H_{FT} of $\mathcal{H}$, if performed in agreement with the ordering prescription in (2.5), and once the spins are taken into account, provides a full relativistic generalization of $H_{q\bar{q}}$.

Notice that, in order to identify Eq. (3.7) with Eq. (14) of Ref. [5], one has to take into account that $\mathbf{q}_t^2r^2=(\mathbf{r}\times\mathbf{q})^2=L^2$. However, Eq. (3.7) is more appropriate for a comparison with Eq. (2.5).

On the contrary, solving (3.4) by an expansion in σ one obtains

$$\mathcal{H}(\mathbf{r},\mathbf{q})=\sqrt{m_1^2+q^2}+\sqrt{m_2^2+q^2}+\frac{\sigma r}{2}\frac{1}{\sqrt{m_1^2+q^2}+\sqrt{m_2^2+q^2}}\left[\left[\frac{m_2^2+q^2}{m_1^2+q^2}\right]^{1/2}\sqrt{m_1^2+q_r^2}+\left[\frac{m_1^2+q^2}{m_2^2+q^2}\right]^{1/2}\sqrt{m_2^2+q_r^2}\right.$$
$$\left.+\left[\frac{\sqrt{m_1^2+q^2}\sqrt{m_2^2+q^2}}{q_t}\right]\left[\arcsin\frac{q_t}{\sqrt{m_1^2+q^2}}+\arcsin\frac{q_t}{\sqrt{m_2^2+q^2}}\right]\right] , \tag{3.8}$$

at the first order in σ, or, in the equal mass case,

$$\mathcal{H}(\mathbf{r},\mathbf{q})=2\sqrt{m^2+q^2}+\frac{\sigma r}{2}\left[\frac{\sqrt{m^2+q^2}}{q_t}\arcsin\frac{q_t}{\sqrt{m^2+q^2}}+\left[\frac{m^2+q_r^2}{m^2+q^2}\right]^{1/2}\right]$$
$$+\frac{\sigma^2r^2}{16q_t^2}\frac{m^2+q_r^2}{\sqrt{m^2+q^2}}\left[\frac{\sqrt{m^2+q^2}}{q_t}\arcsin\frac{q_t}{\sqrt{m^2+q^2}}-\left[\frac{m^2+q_r^2}{m^2+q^2}\right]^{1/2}\right]^2 . \tag{3.9}$$

at the second order in σ. The worth of Eqs. (3.8) and (3.9) with respect to (3.7) is in the fact that they retain a full relativistic kinematics.

IV. QUANTIZATION OF THE MODEL

The model can be quantized in the usual canonical way replacing $\mathbf{r}$ and $\mathbf{q}$ in $\mathcal{H}(\mathbf{r},\mathbf{q})$ with the corresponding operators once an appropriate ordering prescription is chosen. Noting that in principle $\mathcal{H}(\mathbf{r},\mathbf{q})$ can be expanded in powers of the components of $\mathbf{q}$, it becomes apparent that the ordering problem concerns essentially expressions of the form

$$q^{k_1}q^{k_2}\cdots q^{k_n}X^{k_1k_2\cdots k_n}(\mathbf{r}) .$$

We can then easily generalize Eq. (2.3) by setting[2]

$$\{q^{k_1}q^{k^2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_W = \frac{1}{2^n}\sum_{s=0}^{n}\begin{bmatrix} n \\ s \end{bmatrix} q^{k_1}q^{k_2}\cdots q^{k_s}X^{k_1\cdots k_n}(\mathbf{r})q^{k_{s+1}}\cdots q^{k_n} , \tag{4.1a}$$

$$\{q^{k_1}q^{k^2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_S = \frac{1}{2}\{q^{k_1}q^{k_2}\cdots q^{k_n},X^{k_1\cdots k_n}(\mathbf{r})\} ,$$

and

$$\{q^{k_1}q^{k^2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_{\mathrm{ord}} = \tfrac{2}{3}\{q^{k_1}q^{k_2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_W + \tfrac{1}{3}\{q^{k_1}q^{k_2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_S . \tag{4.1b}$$

Indeed, for $n=2$, Eq. (4.1b) becomes identical to Eq. (2.3) and consequently such an order prescription makes Hamiltonian (3.7) identical to (2.5) apart the spin related terms.

We stress that prescription (4.1) does not turn out a purely formal one but, on the contrary, it is very convenient also from the point of view of actual calculations. Precisely we may notice that

$$\langle\mathbf{k}'|\{q^{k_1}q^{k_2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_W|\mathbf{k}\rangle = \left[\frac{k^{h_1\prime}+k^{h_1}}{2}\right]\cdots\left[\frac{k^{h_n\prime}+k^{h_n}}{2}\right]\langle\mathbf{k}'|X^{h_1\cdots h_n}(\mathbf{r})|\mathbf{k}\rangle \tag{4.2}$$

while

$$\langle\mathbf{k}'|\{q^{k_1}q^{k_2}\cdots q^{k_n}X^{k_1\cdots k_n}(\mathbf{r})\}_S|\mathbf{k}\rangle = \tfrac{1}{2}(k^{h_1\prime}\cdots k^{h_n\prime}+k^{h_1}\cdots k^{h_n})\langle\mathbf{k}'|X^{h_1\cdots h_n}(\mathbf{r})|\mathbf{k}\rangle . \tag{4.3}$$

In conclusion for the quantized Hamiltonian H_{FT} we have

$$\langle\mathbf{k}'|H_{\mathrm{FT}}|\mathbf{k}\rangle = \langle\mathbf{k}'|\{\mathcal{H}(\mathbf{r},\mathbf{q})\}_{\mathrm{ord}}|\mathbf{k}\rangle = \int\frac{d\mathbf{r}}{(2\pi)^3}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \times\left[\frac{2}{3}\mathcal{H}\left[\mathbf{r},\frac{\mathbf{k}'+\mathbf{k}}{2}\right] + \frac{1}{3}\frac{\mathcal{H}(\mathbf{r},\mathbf{k}')+\mathcal{H}(\mathbf{r},\mathbf{k})}{2}\right] . \tag{4.4}$$

The problem of the construction of H_{FT} is so reduced to the evaluation of the Fourier transform of the classical Hamiltonian $\mathcal{H}(\mathbf{r},\mathbf{q})$ for $\mathbf{q}$ equal to $\mathbf{k}$, $\mathbf{k}'$, and $(\mathbf{k}+\mathbf{k}')/2$. The expression of the classical Hamiltonian obviously can be obtained by numerical inversion of (3.4) or directly by using Eqs. (3.8) and (3.9).

V. BETHE-SALPETER KERNEL

Let us recall that a relativistic potential theory for two spinless particles with the center-of-mass Hamiltonian

$$H=\sqrt{m_1^2+\mathbf{q}^2}+\sqrt{m_2^2+\mathbf{q}^2}+V \tag{5.1}$$

can be related to an instantaneous BS kernel $I(\mathbf{k}',\mathbf{k})$ by the equation (cf., e.g., Ref. [8])

$$\langle\mathbf{k}'|V|\mathbf{k}\rangle = \frac{1}{(2\pi)^3}\frac{m_1m_2}{\sqrt{w_1(\mathbf{k})w_2(\mathbf{k})w_1(\mathbf{k}')w_2(\mathbf{k}')}}I(\mathbf{k}',\mathbf{k}) \tag{5.2}$$

where $w(\mathbf{k})=\sqrt{\mathbf{k}^2+m^2}$.

Then if we define [cf. (4.4) and (3.8) or (3.9)]

$$\langle\mathbf{k}'|H_{\mathrm{FT}}|\mathbf{k}\rangle = (\sqrt{m_1^2+\mathbf{k}^2}+\sqrt{m_2^2+\mathbf{k}^2})\delta^3(\mathbf{k}-\mathbf{k}') + \mathcal{V}(\mathbf{k}',\mathbf{k}) , \tag{5.3}$$

by inverting Eq. (5.2) we obtain the following instantaneous kernel corresponding to the potential $\mathcal{V}$:

$$I_{\mathrm{conf}}(\mathbf{k}',\mathbf{k}) = (2\pi)^3\frac{\sqrt{w_1(\mathbf{k})w_2(\mathbf{k})w_1(\mathbf{k}')w_2(\mathbf{k}')}}{m_1m_2}\mathcal{V}(\mathbf{k}',\mathbf{k}) . \tag{5.4}$$

Now, taking into account the relation connecting a relativistic potential theory for two spin-$\frac{1}{2}$ particles and the corresponding instantaneous two fermion kernel,

$$\langle\mathbf{k}'\sigma_1'\sigma_2'|V|\mathbf{k}\sigma_1\sigma_2\rangle = \frac{1}{(2\pi)^3}\frac{m_1m_2}{\sqrt{w_1(\mathbf{k})w_2(\mathbf{k})w_1(\mathbf{k}')w_2(\mathbf{k}')}} \times\bar{u}^{(1)}_{\sigma_1'}(\mathbf{k}')\bar{u}^{(2)}_{\sigma_2'}(-\mathbf{k}')I(\mathbf{k}',\mathbf{k}) \times u^{(1)}_{\sigma_1}(\mathbf{k})u^{(2)}_{\sigma_2}(-\mathbf{k}) \tag{5.5}$$

(u stay for the Dirac spinor), the most spontaneous way to introduce the spin in the model, consistently with the Dirac formalism and with the scalar confinement hypothesis, would seem to use (5.4) as the kernel of (5.5). Indeed, when replaced back in Eqs. (5.5) and (5.1), this kernel reproduces (2.5) in the semirelativistic limit with

[2]Here the labels W and S stand for Weyl and symmetric ordering prescriptions (see, e.g., [7]).

the correct velocity-dependent and Thomas precession terms. However, the Darwin-type term turns out to be $-\frac{1}{8}(1/m_1^2+1/m_2^2)\nabla^2(\sigma r)$ and has opposite sign with respect to the correspondent one in (2.5). Such a discrepancy turns out to be of little practical importance for heavy quarks but could become significant in the case of light quarks. It is therefore important to notice that it could be eliminated by including two additional kinematical factors in the definition of $I_{\rm conf}$, at least for nonvanishing quark masses. To this aim we have to set

$$I_{\rm conf}(\mathbf{k}',\mathbf{k})=(2\pi)^3\frac{\sqrt{w_1(\mathbf{k})w_2(\mathbf{k})w_1(\mathbf{k}')w_2(\mathbf{k}')}}{m_1m_2}\times\rho_1\rho_2\mathcal{V}(\mathbf{k}',\mathbf{k}) \ . \tag{5.4'}$$

with

$$\rho_i=\frac{4m_i^2(m_i+w_i(\mathbf{k}'))(m_i+w_i(\mathbf{k}))}{[(m_i+w_i(\mathbf{k}))(m_i+w_i(\mathbf{k}'))-\mathbf{k}'\cdot\mathbf{k}]^2} \ .$$

The factors ρ_i renormalize in an appropriate way the spinor contribution without changing the dominant long-range behavior of the potential and are in this sense essentially unique.

Once a choice for $I_{\rm conf}$ has been made, one can make the usual additional assumption that in QCD the short-range interaction is correctly described by perturbation theory. Then we are brought to introduce the complete quark-antiquark BS kernel

$$I(\mathbf{k}',\mathbf{k})=I_{\rm conf}(\mathbf{k}',\mathbf{k})+I_{\rm pert}(\mathbf{k}',\mathbf{k}) \ , \tag{5.6}$$

where $I_{\rm conf}$ is defined by Eq. (5.4) or (5.4′) and $I_{\rm pert}$ is given by

$$I_{\rm pert}=(2\pi)^3\left\{\frac{4}{3}\frac{\alpha_s}{2\pi^2}\left[-\frac{\gamma_0^{(1)}\gamma_0^{(2)}}{\mathbf{Q}^2}-\frac{1}{\mathbf{Q}^2}[\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}-(\boldsymbol{\gamma}^{(1)}\cdot\hat{\mathbf{Q}})(\boldsymbol{\gamma}^{(2)}\cdot\hat{\mathbf{Q}})]\right]\right\} \tag{5.7}$$

(with $\mathbf{Q}=\mathbf{k}'-\mathbf{k}$) at the lowest order in α_s and in the Coulomb gauge [9]. The BS equation defined by Eqs. (5.4)–(5.7) provides a full relativistic generalization of the Hamiltonian (2.1) and reproduces completely the potentials (2.2)–(2.4) at the semirelativistic limit but the terms in C and possibly the Darwin term.

The terms in C can be understood as an end effect in the flux tube. In fact in the neighborhood of the quarks the flux-tube field should match with a Coulomb-like field. Then, if we denote by $-C/2$ ($C<0$) the excess of energy of the field in a system comoving with the quark or, respectively, the antiquark, we are brought to subtract from the right-hand side of Eq. (1.1) a term of the form

$$\frac{C}{2}(\sqrt{1-\mathbf{v}_1^2}+\sqrt{1-\mathbf{v}_2^2} \ . \tag{5.8}$$

In the semirelativistic approximation again such a term produces exactly the terms in C in Eqs. (2.2)–(2.4). Obviously in a full relativistic treatment (5.8) can be reabsorbed in a redefinition of the quark masses $m_1\to m_1'=m_1+C/2$, $m_2\to m_2'=m_2+C/2$. At the level of Eq. (2.1), however, this may not be legitimate (cf. Ref. [3]).

Finally, notice that in Eq. (5.2) it has been tacitly assumed that the full one quark propagator can be replaced by its free particle form, $S_F(p)=i(\gamma^\mu p_\mu-m)^{-1}$. If one wanted to take into account self-energy effects one could follow the line of Ref. [10] and write, in the instantaneous approximation,

$$S_F'(p)=i[B(\mathbf{p})\gamma_\mu\hat{p}^\mu-A(\mathbf{p})]^{-1} \ . \tag{5.9}$$

Then, setting for simplicity $m_1=m_2=m$, the kinematic factor in Eq. (5.4) should be replaced by $W(\mathbf{k})W(\mathbf{k}')/A(\mathbf{k})A(\mathbf{k}')$ with $W(\mathbf{k})=\sqrt{A(\mathbf{k})^2+B(\mathbf{k})^2}$.

VI. CONCLUSIONS

In conclusion, we have reformulated the quantization of the relativistic flux model in such a way to make it consistent with the confinement part of the quark-antiquark potential as derived in the context of the Wilson loop approach apart from the terms related to spin.

Then we have constructed an explicit expression for the Hamiltonian of the model in terms of the canonical variables in the form of an expansion in the string tension constant σ [Eqs. (3.8) and (3.9)].

We also have shown how an instantaneous Bethe-Salpeter kernel can be defined as the sum of a long-range confining part which corresponds to the Hamiltonian of the model and a short-range perturbative part directly derived from QCD. This kernel includes the spin and reproduces exactly the $q\bar{q}$ potential in the semirelativistic limit but possibly for the Darwin long-range term.

As we mentioned, the controversial origin of the constant term in the static potential seems to be understandable as a consequence of an end effect in the flux tube. Notice that for static quarks this would be equivalent to subtracting a strip around the contour of the surface delimited by the Wilson loop and so too including a perimeter term in its expectation value as done in Ref. [3]. In this order of ideas, the terms in C can be exactly reabsorbed in effective masses of the quarks as the structure of the corresponding terms in Eqs. (2.1)–(2.4) already suggested.

Concerning the term in σ^2r^2 in Eq. (3.9) we observe that in the two limit situations of large and small m it becomes $\sigma^2r^2q_t^2/64m^3$ and $(\sigma^2r^2/q_t)(q_r^2/q^2)$, respectively. Then it can be seen that the term turns out to be negligible in comparison with σr in the first case ($\sigma \simeq 0.15$ and typically $\langle r \rangle \simeq 1$ GeV^{-1}, $\langle q \rangle \simeq 1$ GeV; on the contrary, it can be more important in the second one. In this context notice that terms proportional to r^2 have already been considered in some phenomenological analysis based on the BS equation [11].

Finally we remark that, as already stressed in the Introduction, the flux tube is treated in the model somehow like a rigid rod. Obviously the quantum fluctuation of the gauge field should introduce correction to this picture. In this order of ideas one should compare the present treatment with the somewhat complementary point of view of Refs. [12,13] in the static limit.

ACKNOWLEDGMENTS

We would like to thank our friend J. F. Lagaë for interesting discussions.

[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974); L. S. Brown and W. I. Weisberger, *ibid.* **20**, 3229 (1979); T. Appelquist, M. Dine, and I. J. Muzinich, *ibid.* **17**, 2074 (1978); E. Eichten and F. Feinberg, *ibid.* **23**, 2724 (1981); M. A. Peskin, in *Dynamics and Spectroscopy at High Energy,* Proceedings of the 11th SLAC Summer Institute on Particle Physics, Stanford, California, 1983, edited by P. M. McDonough (SLAC Report No. 267, Stanford, 1984), p. 151; D. Gromes, Z. Phys. C **22**, 265 (1984); A. Barchielli, E. Montaldi, and G. M. Prosperi, Nucl. Phys. **B296**, 625 (1988); **B303**, 752(E) (1988).

[2] For a recent comprehensive review, see W. Lucha, F. F. Schöberl, and D. Gromes, Phys. Rep. **200**, 127 (1990).

[3] A. Barchielli, N. Brambilla, and G. M. Prosperi, Nuovo Cimento **103A**, 59 (1990); A. Barchielli, E. Montaldi, and G. M. Prosperi, Nucl. Phys. **B296**, 625 (1988); cf. also N. Brambilla and G. M. Prosperi, Phys. Lett. B **236**, 69 (1990).

[4] D. LaCourse and M. G. Olsson, Phys. Rev. D **39**, 2751 (1989); see also W. Buchmüller, Phys. Lett. **112B**, 479 (1982); M. Ida, Prog. Theor. Phys. **59**, 1661 (1978); A. Chodos and C. B. Thorn, Nucl. Phys. **B72**, 509 (1974); I. Bars, *ibid.* **B111**, 413 (1976). A selection of general references on the flux-tube model and its stringlike idealization is H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973); G. 't Hooft, *ibid.* **B72**, 461 (1974); C. B. Thorn, Phys. Rev. D **19**, 639 (1979); G. Parisi, Phys. Lett **81B**, 357 (1979); A. M. Polyakov, *ibid.* **82B**, 247 (1979). Among recent contributions see also M. Baker, J. S. Ball, and F. Zachariasen, Phys. Rev. D **41**, 2612 (1990), and references therein.

[5] C. Olson, M. G. Olsson, and K. Williams, Phys. Rev. D **45**, 4307 (1992).

[6] See, e.g., A. Di Giacomo, M. Maggiore, and S. Olejnik, Phys. Lett. B **236**, 199 (1990); for a general review on lattice QCD, cf. M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, England, 1983).

[7] F. Langouche, D. Roekaerts, and E. Tirapegui, *Functional Integration and Semiclassical Expansions* (Reidel, Dordrecht, 1982).

[8] N. Brambilla and G. M. Prosperi, Phys. Rev. D **46**, 1096 (1992); **46**, 4105(E) (1992); N. Brambilla, Nuovo Cimento **105A**, 949 (1992); see also Ref. [2]. For a general bibliography on the BS equation, cf. T. Murota, Prog. Theor. Phys. Suppl. **95**, 45 (1988), and references therein.

[9] W. Celmaster and F. S. Henvey, Phys. Rev. D **17**, 3268 (1978).

[10] J. F. Lagaë, Phys. Rev. D **45**, 305 (1992); **45**, 317 (1992); see also K. Aoki, M. Bando, T. Kugo, M. Mitchard, and H. Nakatami, Prog. Theor. Phys. **84**, 683 (1990).

[11] A. Mittal and A. N. Mitra, Phys. Rev. Lett. **57**, 290 (1986), and references therein.

[12] M. Lüscher, G. Münster, and P. Weisz, Nucl. Phys. **B180**, 1 (1981); M. Lüscher, *ibid.* **B180**, 317 (1981); M. Luscher, K. Symanzik, and P. Weisz, *ibid.* **B173**, 365 (1980).

[13] M. Caselle, R. Fiore, F. Gliozzi, P. Provero, and S. Vinti, Phys. Lett. B **272**, 272 (1991).