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Electromagnetic self-energies of pseudoscalar mesons and Dashen's theorem

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The calculation of the electromagnetic mass shifts of π, K mesons is studied using a model incorporating chiral symmetry and vector dominance. Significant SU(3) breaking to Dashen's theorem is found, allowing a resolution of the long-standing discrepancy between the quark-mass ratio $(m_d-m_u)/m_s$ found from $\eta\to3\pi$ decay and that from the K^+-K^0 mass difference.

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I. INTRODUCTION

The π^+-π^0 mass difference is almost entirely electromagnetic in origin. However, the K^+-K^0 mass difference receives contributions from both electromagnetism and from the u-d mass difference. At lowest order in chiral SU(3) symmetry, the electromagnetic effects are related by Dashen's theorem [1]

$$(m_{K^+}^2-m_{K^0}^2)_{\rm EM}=(m_{\pi^+}^2-m_{\pi^0}^2)_{\rm EM}\ . \tag{1}$$

However, one expects this result to be modified by SU(3) breaking at next order. It is of interest to calculate the electromagnetic splitting of pions and kaons both to increase the understanding of low-energy dynamics and to aid in the extraction of quark masses. As an example of the latter motivation, we note that the quark-mass ratio

$$\frac{m_d-m_u}{m_s-\hat{m}}\,\frac{m_d+m_u}{m_s+\hat{m}}=\frac{m_\pi^2}{m_K^2}\,\frac{[(m_{K^0}^2-m_{K^+}^2)_{\rm QM}]_{\rm DT}}{m_K^2-m_\pi^2}=1.72\times10^{-3} \tag{2}$$

with $\hat{m}=(m_u+m_d)/2$, yields the number quoted when Dashen's theorem is employed. By contrast, an independent measurement of the same quantity [2]

$$\frac{m_d-m_u}{m_s-\hat{m}}\,\frac{m_d+m_u}{m_s+\hat{m}}=\frac{3\sqrt{3}F_\pi^2\,{\rm Re}\,A(\eta\to\pi^+\pi^-\pi^0)}{(m_K^2-m_\pi^2)[1+\Delta_{\eta3\pi}]}\,\frac{m_\pi^2}{m_K^2}=2.35\times10^{-3} \tag{3}$$

where $\Delta_{\eta3\pi}=0.5$ is a factor arising from higher-order chiral effects, yields a different answer. Of course, since the η is rather heavy, one might expect that higher orders in perturbation theory are important. However, it turns out that the conflict can be resolved if Dashen's theorem is appropriately modified at next order in the chiral expansion. We have recently argued that this indeed is the case [3], and we herein describe the general formalism for discussing the electromagnetic mass differences as well as our model for calculating the masses. The method follows that developed in Ref. [4].

The key to our result lies in being able to reliably calculate the electromagnetic self-energy. Thus in Sec. II we present a careful analysis of the low-energy Compton scattering amplitude, which is used to calculate the electromagnetic self-energy. In Sec. III we review previous approaches to the problem, based on the so-called Cottingham rotation [5]. Then in Sec. IV we show how chiral symmetry supplemented by vector dominance can be used to construct a reasonable picture of the electromagnetic self-energy and we study the SU(3)-breaking effects in Dashen's theorem. Our results are summarized in a concluding Sec. V.

II. THE COMPTON AMPLITUDE

The electromagnetic contribution to the pion mass difference can be written as

$$\Delta m_\pi^2=\delta m_+^2-\delta m_0^2=\frac{e^2}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}g^{\mu\nu}[T_{\mu\nu}^+(p,q)-T_{\mu\nu}^0(p,q)] \tag{4}$$

where $T_{\mu\nu}^{+,0}(p,q)$ is the forward Compton scattering amplitude for π^+,π^0 respectively. As we shall see, the contributions to Eq. (4) are dominated by low values of q^2. Consequently it should be sufficient to know the low-energy form of the Compton amplitude in order to obtain a reasonable estimate of the electromagnetic mass splitting. The most reliable way by which this form can be obtained is by use of the effective chiral Lagrangian methods developed by Gasser and Leutwyler [6] since in this fashion the full implications of the chiral symmetry of QCD can be imposed upon the Compton amplitude.

The effective chiral Lagrangian is given in terms of an expansion in powers of the energy-momentum with present applications retaining terms of $O(p^2)$ and $O(p^4)$. The pieces of the general $SU(3)_L\times SU(3)_R$ Lagrangian which are relevant for our work are

$$L=L^{(2)}+L^{(4)}+\cdots ,$$
$$L^{(2)}=\frac{F^2}{4}\,\mathrm{Tr}D_\mu U D^\mu U^\dagger+\frac{F^2}{4}2B_0\,\mathrm{Tr}m\,(U+U^\dagger) ,$$
$$L^{(4)}=\cdots-iL_9\,\mathrm{Tr}(F^L_{\mu\nu}D^\mu U D^\nu U^\dagger+F^R_{\mu\nu}D^\mu U^\dagger D^\nu U)+L_{10}\,\mathrm{Tr}(L_{\mu\nu}UR^{\mu\nu}U^\dagger) . \tag{5}$$

Here m represents the quark-mass matrix,

$$B_0=\frac{m_\pi^2}{m_u+m_d}=\frac{m_K^2}{m_u+m_s} \tag{6}$$

is a constant, and

$$U=\exp\left[\frac{i}{F}\sum_{j=1}^{8}\lambda_j\phi_j\right] ,$$
$$D_\mu U=\partial_\mu U-il_\mu U+iUr_\mu ,$$
$$L_{\mu\nu}=\partial_\mu l_\nu-\partial_\nu l_\mu+[l_\mu,l_\nu] ,$$
$$R_{\mu\nu}=\partial_\mu r_\nu-\partial_\nu r_\mu+[r_\mu,r_\nu] . \tag{7}$$

These Lagrangians describe the couplings of the pseudoscalar fields $\phi_j(x)$, $j=1,2,\ldots,8$ to each other and to external left- and right-handed currents l^j_μ, r^j_μ. After renormalization, the constant F becomes the pion decay constant $F_\pi=92.4$ MeV [7].

When this Lagrangian is used to calculate the Compton amplitude, one finds

$$T^+_{\mu\nu}(p_1,q_1,q_2)=\int d^4x\, e^{iq_2\cdot x}\langle\pi^+(p_2)|T[V^{\rm EM}_\nu(x)V^{\rm EM}_\mu(0)]|\pi^+(p_1)\rangle$$
$$=-\frac{T_\mu(p_1,p_1+q_1)T_\nu(p_2+q_2,p_2)}{(p_1+q_1)^2-m_\pi^2}-\frac{T_\nu(p_1,p_1-q_2)T_\mu(p_2-q_1,p_2)}{(p_1-q_2)^2-m_\pi^2}$$
$$+2g_{\mu\nu}+\frac{4}{F_\pi^2}L_9(q_1^2g_{\mu\nu}-q_{1\mu}q_{1\nu}+q_2^2g_{\mu\nu}-q_{2\mu}q_{2\nu})-\frac{8}{F_\pi^2}(L_9+L_{10})(q_1\cdot q_2 g_{\mu\nu}-q_{2\mu}q_{1\nu})+\text{loops} ,$$
$$T^0_{\mu\nu}(p_1,q_1,q_2)=\text{loops} \tag{8}$$

where for simplicity we do not present the explicit forms of the (small) loop contributions, which may be found in Refs. [8]. Here, $T_\mu(p_i,p_f)$ is the off-shell pion electromagnetic vertex and is given by

$$T_\mu(p_i,p_f)=(p_i+p_f)_\mu\left[1+\frac{2L_9}{F_\pi^2}q^2\right]-q_\mu\frac{2L_9}{F_\pi^2}(p_i^2-p_f^2)+\text{loops} \tag{9}$$

where $q=p_i-p_f$ and we again do not present the form of the small loop contributions. Noting that

$$q^\mu T_\mu(p_i,p_f)=p_i^2-p_f^2 \tag{10}$$

we verify gauge invariance of the on-shell Compton amplitude:

$$q_1^\mu T^+_{\mu\nu}(p_1,q_1,q_2)=q_2^\nu T^+_{\mu\nu}(p_1,q_1,q_2)=0 . \tag{11}$$

The empirical constants L_9, L_{10} can be identified in terms of experimental quantities. Indeed, for on-shell pions we have

$$T_\mu(p_i,p_f)=(p_i+p_f)_\mu\left[1+\frac{2L_9}{F_\pi^2}q^2\right] \tag{12}$$

so that L_9 is determined from the pion charge radius as [3]

$$\langle r_\pi^2\rangle=\frac{12}{F_\pi^2}L_9+\text{loops}=(0.44\pm0.01)\ \text{fm}^2,\quad \text{i.e.,}\ L_9\cong(7.1\pm0.3)\times10^{-3} . \tag{13}$$

In the case of L_{10}, things are not quite as simple. In principle the procedure should be straightforward, as the electric, magnetic polarizabilities α_E, β_M of the charged pion are given in terms of the combination L_9+L_{10}:

$$\alpha_E=-\beta_M=\frac{4\alpha}{m_\pi F_\pi^2}(L_9+L_{10}) . \tag{14}$$

Unfortunately, at present there is no experimental agreement on the value of the polarizability, so an alternate tack is required—one can relate the polarizability to a form factor in radiative pion decay $\pi^+\to e^+\nu_e\gamma$ which allows us to measure L_{10} [9]. By taking the soft pion limit $p_2\to0$ and using the PCAC (partial conservation of axial-vector current) condition

$$\partial^\mu A_\mu = F_\pi m_\pi^2 \phi_\pi \tag{15}$$

one finds

$$\lim_{p_2\to 0} T_{\mu\nu}(p_1,q_1,q_2) = \frac{i}{\sqrt{2}F_\pi}[M_{\mu\nu}(p_1,q_1)+M_{\nu\mu}(p_1,q_2)] \tag{16}$$

where

$$\begin{aligned} M_{\mu\nu}(p,q) &= \int d^4x\, e^{iq\cdot x}\langle 0|T[V_\mu^{\rm EM}(x)A_\nu^-(0)]|\pi^+(p_1)\rangle \\ &= \sqrt{2}F_\pi(p-q)_\nu(2p-q)_\mu \frac{1}{(p-q)^2-m_\pi^2}\left[1+\frac{2L_9}{F_\pi^2}q^2\right] \\ &\quad +\sqrt{2}F_\pi(p-q)_\nu q_\mu \frac{2L_9}{F_\pi^2} - \sqrt{2}F_\pi g_{\mu\nu} + 4\sqrt{2}\frac{L_9+L_{10}}{F_\pi}[(p-q)_\mu q_\nu - g_{\mu\nu}(p-q)\cdot q] \\ &\quad -4\sqrt{2}L_9\frac{1}{F_\pi}(g_{\mu\nu}q^2 - q_\mu q_\nu) \end{aligned} \tag{17}$$

is the axial-vector transition amplitude for radiative pion decay $\pi^+\to e^+\nu_e\gamma$. The coefficient of the kinematic combination $[(p-q)_\mu q_\nu - g_{\mu\nu}(p-q)\cdot q]$ is generally designated by h_A and has been measured as [10]

$$h_A = \frac{4\sqrt{2}(L_9+L_{10})}{F_\pi} = (0.0116\pm 0.0016)m_\pi^{-1} \tag{18}$$

from which it is determined that

$$L_{10} \cong -(5.6\pm 0.3)\times 10^{-3} . \tag{19}$$

It is also possible, by taking a second soft pion limit, to relate both of these amplitudes to an integral over experimentally known spectral functions:

$$\begin{aligned} \lim_{p\to 0} M_{\mu\nu}(p,q) &= \sqrt{2}F_\pi q_\mu q_\nu \frac{1}{q^2-m_\pi^2} - \sqrt{2}F_\pi g_{\mu\nu} - 4\sqrt{2}\frac{L_{10}}{F_\pi}(q_\mu q_\nu - g_{\mu\nu}q^2) + \cdots \\ &= \frac{1}{\sqrt{2}F_\pi}\int d^4x\, e^{iq\cdot x}\langle 0|T[2V_\mu^3(x)V_\nu^3(0) - A_\mu^+(x)A_\nu^-(0)]|0\rangle \\ &= \sqrt{2}F_\pi q_\mu q_\nu \frac{1}{q^2-m_\pi^2} - \sqrt{2}F_\pi g_{\mu\nu} + 2\frac{1}{\sqrt{2}F_\pi}\int ds \frac{1}{s-q^2}[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)](q_\mu q_\nu - g_{\mu\nu}q^2) . \end{aligned} \tag{20}$$

We find then the relations

$$\begin{aligned} -4L_{10} &= \int \frac{ds}{s}[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)] , \\ F_\pi^2 &= \int ds[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)] . \end{aligned} \tag{21}$$

The second of these equations is the first Weinberg sum rule [11] while the first can be written as

$$\begin{aligned} \frac{1}{\sqrt{2}F_\pi}h_A &= \frac{4}{F_\pi^2}(L_9+L_{10}) \\ &= \frac{1}{3}\langle r_\pi^2\rangle - \int \frac{ds}{s}[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)] , \end{aligned} \tag{22}$$

which is the Das-Mathur-Okubo relation [12]. The spectral densities $\rho_V^{(1)}(s),\rho_A^{(1)}(s)$ can be determined empirically from τ decay (for $s<m_\tau^2$), and substitution into the first of Eqs. (21) yields a result [13]

$$h_A = (0.017\pm 0.004)m_\pi^{-1} \tag{23}$$

in reasonable agreement with that obtained in Eq. (18).

Similarly one may analyze the corresponding kaon Compton amplitudes. At the level of the four-derivative chiral expansion the result is identical to Eq. (8) except for the kinematic replacement $p^2=m_\pi^2\to p^2=m_K^2$ [14].

III. ELECTROMAGNETIC MASS DIFFERENCE

We spent considerable time in the previous section attempting to understand the low-energy form of the Compton scattering amplitude for pseudoscalar mesons because of the connection to the electromagnetic mass shifts given in Eq. (4). However, one cannot simply utilize the chiral form of the amplitude given in Eq. (8) as an input to Eq. (4) since the contribution from each term in the chiral expansion will diverge due to the lack of form factors suppressing high-energy effects. The "art" of this calculation lies in using this chiral amplitude in order to generate a believable and convergent result for the mass shift. This is an old problem and has been previously addressed in a number of ways. One of the most interesting approaches is by use of the so-called Cottingham rotation [5]. The idea is that the contracted Compton amplitude

$$T(q^2,\nu)=g^{\mu\nu}T_{\mu\nu}(p,q) \tag{24}$$

is a function of the two kinematic variables q^2 and $\nu=p\cdot q$. The integration in Eq. (4) as it stands is over *all* q_μ and therefore involves both positive and negative values of q^2. It is convenient to rewrite this integration in terms of one involving only negative values of q^2. The singularities in the q_0 plane are located above the real axis for $q^2<0$ and below for $q^2>0$. Thus we can rotate the q_0 integration contour from over the real axis to one from $q_0=-i\infty$ to $q_0=+i\infty$. In this way $q^2<0$ for all values of the integration variables, and we have

$$\Delta m_\pi^2=\frac{e^2}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}T(q^2,\nu)$$
$$=\frac{ie^2}{(2\pi)^3}\int_0^\infty ds^2\int_0^s dq_4\sqrt{s^2-q_4^2}\,\frac{1}{s^2}T(-s^2,imq_4)\ . \tag{25}$$

We observe that what is required are values $T(-s^2,imq_4)$ at negative q^2 and imaginary ν, which can in principle be obtained from physical electroproduction data by means of fixed q^2 dispersion relations.

From this perspective, it has been argued that the Born term should make the dominant contribution [15]. The point is that if one compares Born and resonant contributions to $T(q^2,\nu)$ then schematically

$$T_{\rm Born}(q^2,\nu)\sim\frac{1}{q^2+2\nu}f(q^2)\ ,$$
$$T_{\rm res}(q^2,\nu)\sim\frac{1}{q^2+2\nu+m^2-M_R^2}f(q^2) \tag{26}$$

where $f(q^2)$ represents a generic form factor. Since these form factors rapidly damp out such contributions, it is clear that significant contribution to the Cottingham integral should arise only from the region of small q^2 and $\nu\lesssim m\sqrt{-q^2}$. In this kinematic region

$$\frac{T_{\rm res}}{T_{\rm Born}}\sim\frac{\nu}{m^2-M_R^2}\sim\frac{m\sqrt{-q^2}}{m^2-M_R^2}\ll 1\ . \tag{27}$$

An exception to this rule is found in the hard high-energy (where QCD can be treated perturbatively) effects which go into the electromagnetic renormalization of the quark masses. The electromagnetic self-energies of free quarks, of course, produce a divergent mass renormalization

$$\delta m_i\sim\frac{3\alpha}{4\pi}Q_i^2 m_i\ln\Lambda^2\ , \tag{28}$$

which will appear as the short distance, high-energy component of the meson self-energy. Because δm is proportional to the quark mass, it is absent entirely at lowest order in chiral symmetry. However, such dependence is present in general, so that the Cottingham integral must contain a logarithmically divergent high-energy tail. Because this divergence is associated with quark-mass renormalization, it can be entirely absorbed into the renormalized mass parameters. In practice then, there remains a slight ambiguity of how much of the Cottingham integral goes into mass renormalization and what remains as the non-mass-related electromagnetic effect. For a cutoff of order a few GeV, separating the two effects seems to be numerically a rather minor issue. Partially because $\delta m_i\propto m_i$, the cutoff dependence is small compared to the much larger contributions from the low-energy Born term. Thus one general conclusion of the Cottingham approach is that the most important effects are at low energy, in particular from the Born term including form factors.

Of course, there do not exist electroproduction data for pion or kaon targets. Nevertheless, from experience and phenomenology in other areas, we can construct the main ingredients to such reactions. Electron scattering involves the production of resonances at low and moderate energies, with a deep inelastic region at high energy. The pion electromagnetic form factor has been well measured, as have photonic decays of resonances to pions. Combined with the low-energy constraints on the Compton amplitude, we can then present a reasonably complete picture of the ingredients to the Cottingham integral, as we shall show in Sec. IV.

Before doing so, however, it is useful to examine a first approximation to the calculation of electromagnetic masses obtained by including simply the Born plus seagull terms multiplied by appropriate form factors. Indeed this is a model previously proposed by Socolow [16], yielding

$$\Delta m_\pi^2=e^2\int\frac{d^4q}{(2\pi)^4}(3q^2+4q\cdot p-4m_\pi^4)$$
$$\times\frac{1}{q^2(q^2+2p\cdot q)}\left[\frac{m_\rho^2}{m_\rho^2-q^2}\right]^2 \tag{29}$$

as an estimate of the pion mass splitting. The integration is easily performed and the result is

$$\Delta m_\pi^2=\frac{\alpha}{4\pi}\left\{3m_\rho^2+m_\pi^2\left[4+I\left[\frac{m_\rho^2}{m_\pi^2}\right]\right]\right\}\ ,$$

where

$$I(z)=\int_0^1 dy(4-2y)\left[\ln[y^2+z(1-y)]\right.$$
$$\left.+\frac{y^2}{y^2+z(1-y)}\right]\ . \tag{30}$$

Putting in numbers we find

$$\Delta m_\pi^2(\text{Socolow})=\frac{\alpha}{4\pi}(3m_\rho^2+12.1m_\pi^2)$$
$$=2m_\pi\times 4.2\ \text{MeV}\ , \tag{31}$$

which is in good agreement with the experimental value of $2m_\pi\times 4.6$ MeV. However, it is also clear that one should expect significant violation of Dashen's theorem, since, for the kaon system,

$$\Delta m_K^2(\text{Socolow})=e^2\int\frac{d^4q}{(2\pi)^4}\frac{3q^2+4p\cdot q-4m_K^2}{q^2(q^2+2p\cdot q)}\left[\frac{2}{3}\frac{m_\rho^2}{m_\rho^2-q^2}+\frac{1}{3}\frac{m_\phi^2}{m_\phi^2-q^2}\right]^2$$
$$=\frac{\alpha}{4\pi}\left[\frac{4}{3}m_\rho^2+\frac{1}{3}m_\phi^2+\frac{4}{3}\frac{m_\rho^2 m_\phi^2}{m_\phi^2-m_\rho^2}\ln\frac{m_\phi^2}{m_\rho^2}+6.4m_K^2\right]$$
$$=1.9\Delta m_\pi^2(\text{Socolow})\ . \tag{32}$$

This result certainly captures some of the physics for the electromagnetic mass shift. However, simply modifying the Born term in this way leads to a Compton amplitude which is inconsistent with the constraints of chiral symmetry. In the next section we will show, following Ref. [4], how this result can be improved in order to properly reflect the chiral properties, while remaining sensitive to the important low-energy physics.

However, before leaving this section it is useful to note that the formalism becomes particularly simple in the soft pion limit. In Sec. II we showed how the Compton amplitude is related to the vector and axial-vector spectral functions in the limit $p_\mu\to 0$:

$$\lim_{p\to 0}T_{\mu\nu}(p,q)=\frac{1}{F_\pi^2}\int d^4x\, e^{iq\cdot x}\langle 0|T[2V_\mu^3(x)V_\nu^3(0)-A_\mu^+(x)A_\nu^-(0)]|0\rangle$$
$$=-\frac{2}{F_\pi^2}\left[F_\pi^2\left[\frac{q_\mu q_\nu}{q^2-m_\pi^2}-g_{\mu\nu}\right]+\int ds\frac{1}{s-q^2}[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)](q_\mu q_\nu-g_{\mu\nu}q^2)\right]\ . \tag{33}$$

As we shall see, this relation together with the Weinberg sum rules forms the basis for the result of Das *et al.*, which yields the electromagnetic mass shift in the chiral limit [12]:

$$\Delta m_\pi^2=-\frac{3\alpha}{4\pi F_\pi^2}\int ds\, s\ln s\,[\rho_V^{(1)}(s)-\rho_A^{(1)}(s)]\ . \tag{34}$$

Equation (34) provides the best way to calculate the mass shift to lowest order in m_π^2, and also forms an important constraint on attempts to calculate it for nonzero m_π^2, m_K^2.

IV. ELECTROMAGNETIC MASS DIFFERENCE AND VECTOR DOMINANCE

A form of the pseudoscalar Compton amplitude which provides a convergent result for the electromagnetic mass difference and which obeys the chiral-symmetry strictures at low energy is provided by coupling vector and axial mesons to the chiral formalism. Indeed it is known that such an approach gives a good picture of all low-energy electroweak processes involving the pseudoscalar mesons. Thus, while this technique is only a model, it is very well motivated. In this section we review the successful prediction of the low-energy Compton amplitude via the vector dominance assumption and show how this necessarily implies a set of form factors at higher energy leading to a reasonably complete description of low- and moderate-energy contributions to the electromagnetic mass shift.

As a preliminary motivation for this model, recall from Sec. II that we need to deal with the pion electromagnetic form factor, and the pion polarizability. The former is well known, experimentally and theoretically, to be described in terms of a ρ-meson pole. The latter was shown in Sec. II to be related to the vector and axial-vector spectral functions $\rho_V^{(1)}$ and $\rho_A^{(1)}$. These too are well known experimentally, and the only significant structure in $\rho_V^{(1)}-\rho_A^{(1)}$ is provided by the ρ and $a_1(1270)$ resonances [17]—the other resonances giving only small contributions. Experiment tells us then that the important ingredients are the vector and axial mesons and it is these which will be accounted for in the model.

In order to have the appropriate behavior at high energy it is simplest to utilize an antisymmetric tensor representation for the vector particles instead of the usual four-vector picture [18]. Using $U\equiv u^2$ we write

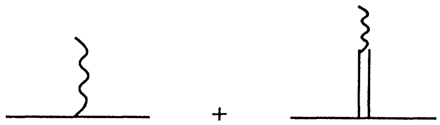
$$L=-\tfrac{1}{2}D^\lambda\rho_{\lambda\mu}^i D_\nu\rho^{i\nu\mu}+\tfrac{1}{4}m_V^2\rho_{\mu\nu}^i\rho^{i\mu\nu}$$
$$+G_V\rho_{\mu\nu}^i\operatorname{Tr}(\lambda^i D^\mu U D^\nu U^\dagger)$$
$$+F_V F^{\mu\nu}\rho_{\mu\nu}^i(\operatorname{Tr}Qu\lambda^i u^\dagger+\operatorname{Tr}Qu^\dagger\lambda^i u) \tag{35}$$

to represent the coupling between pseudoscalar fields and the vector mesons and

$$L=-\tfrac{1}{2}D^\lambda a_{\lambda\mu}^i D_\nu a^{i\nu\mu}+\tfrac{1}{4}m_A^2 a_{\nu\mu}^i a^{i\nu\mu}$$
$$+F_A F^{\mu\nu}a_{\mu\nu}^i(\operatorname{Tr}Qu\lambda^i u^\dagger-\operatorname{Tr}Qu^\dagger\lambda^i u) \tag{36}$$

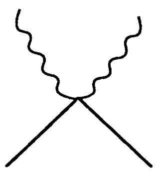
to describe the coupling to the corresponding axial fields. It then becomes a straightforward process to calculate the Compton scattering amplitude. The important diagrams are shown below together with their contributions.

Pion form factor:



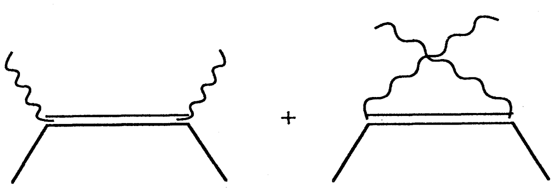
$$T_\mu(p_i,p_f)=(p_i+p_f)_\mu+2F_VG_V\frac{1}{m_V^2-q^2}[(p_i+p_f)_\mu q^2-(p_i-p_f)_\mu(p_i^2-p_f^2)]\ . \tag{37}$$

Seagull:



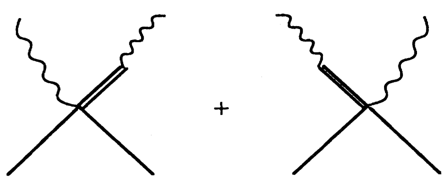
$$2g_{\mu\nu}\ . \tag{38}$$

a_1 pole:



$$-\frac{F_A^2}{F_\pi^2}(g_{\mu\nu}q_1\cdot q_2-q_{1\nu}q_{2\mu})\left[\frac{1-\dfrac{p_1\cdot(p_1+q_1)}{m_A^2}}{m_A^2-(p_1+q_1)^2}+\frac{1-\dfrac{p_1\cdot(p_1-q_2)}{m_A^2}}{m_A^2-(p_1-q_2)^2}\right]\ . \tag{39}$$

Modified seagull:



$$\begin{aligned}&\frac{F_V^2}{F_\pi^2}(g_{\mu\nu}q_1\cdot q_2-q_{1\nu}q_{2\mu})\left[\frac{1}{m_V^2-q_1^2}+\frac{1}{m_V^2-q_2^2}\right]\\&\quad+2\frac{F_VG_V}{F_\pi^2}\left\{g_{\mu\nu}\left[\frac{q_2^2}{m_V^2-q_2^2}+\frac{q_1^2}{m_V^2-q_1^2}-q_1\cdot q_2\left[\frac{1}{m_V^2-q_1^2}+\frac{1}{m_V^2-q_2^2}\right]\right]\right.\\&\qquad\left.-q_{1\mu}q_{1\nu}\frac{1}{m_V^2-q_1^2}-q_{2\mu}q_{2\nu}\frac{1}{m_V^2-q_2^2}+q_{1\nu}q_{2\mu}\left[\frac{1}{m_V^2-q_1^2}+\frac{1}{m_V^2-q_2^2}\right]\right\}\ .\end{aligned} \tag{40}$$

Adding these pieces together and dropping terms in $q^2/m_V^2, q^2/m_A^2$ we find a result in complete agreement with Eq. (8) provided we identify [17]

$$L_9=\frac{F_VG_V}{2m_V^2},\quad L_{10}=-\frac{F_V^2}{4m_V^2}+\frac{F_A^2}{4m_A^2}\ . \tag{41}$$

Numerically, one finds

$$L_9=7.3\times 10^{-3}, \quad L_{10}=-5.5\times 10^{-3} \tag{42a}$$

to be compared with the experimental values (at $\mu=m_\eta$)

$$L_9=(7.1\pm0.3)\times 10^{-3}, \quad L_{10}=-(5.6\pm0.3)\times 10^{-3} \; . \tag{42b}$$

Thus the model offers a quite realistic picture of the low-energy Compton amplitude. Of course, the vector resonances also give a good description of $\rho_V^{(1)}-\rho_A^{(1)}$ and the Weinberg sum rules. Perhaps the only extra ingredient needed to make the model into an excellent description of nature is inclusion of the finite width of the resonances. The model treats ρ and a_1 as narrow states, while the a_1 in particular is relatively broad [10]. However, for quantities which involve integration over the resonance, this approximation does not cause any serious problems.

A successful low-energy representation of the Compton amplitude requires the presence of form factors at higher q^2. The diagrams given above describe resonant intermediate states, and their q^2 variation will modify the size of the Compton amplitude. In particular, if we set $p_1=p_2=p, q_1=q_2=q$ we obtain, for the forward amplitude,

$$\begin{aligned} T_{\mu\nu}(p,q)=&-\frac{T_\mu(p,p+q)T_\nu(p+q,q)}{(p+q)^2-m_\pi^2}-\frac{T_\mu(p-q,p)T_\nu(p,p-q)}{(p-q)^2-m_\pi^2}+2g_{\mu\nu}\\ &+\frac{1}{F_\pi^2}(q^2g_{\mu\nu}-q_\mu q_\nu)\left[\frac{2F_V^2}{m_V^2-q^2}-\frac{F_A^2}{m_A^2-(q+p)^2}-\frac{F_A^2}{m_A^2-(q-p)^2}\right] . \end{aligned} \tag{43}$$

A useful check on this matrix element is provided by studying the soft pion limit. Setting $p_\mu=0$, we find

$$T_{\mu\nu}(0,q)=-2\frac{q_\mu q_\nu}{q^2-m_\pi^2}+2g_{\mu\nu}+\frac{2}{F_\pi^2}(q^2g_{\mu\nu}-q_\mu q_\nu)\left[\frac{F_V^2}{m_V^2-q^2}-\frac{F_A^2}{m_A^2-q^2}\right] . \tag{44}$$

This is exactly the form predicted by the relation between $T_{\mu\nu}$ and the spectral functions $\rho_V^{(1)}-\rho_A^{(1)}$, Eq. (33), provided we identify

$$\rho_V^{(1)}=F_V^2\delta(s-m_V^2), \quad \rho_A^{(1)}=F_A^2\delta(s-m_A^2) \; , \tag{45}$$

which is the narrow width approximation. Moreover, by requiring that Eq. (44) fall with q^2 at high q^2, as required by QCD in the chiral limit, we obtain the Weinberg sum rule constraints on F_V, F_A:

$$F_\pi^2=F_V^2-F_A^2, \quad F_V^2m_V^2=F_A^2m_A^2 \; . \tag{46}$$

Armed with such a model we can now perform the appropriate integration in order to find the electromagnetic mass differences. We have

$$\begin{aligned} \Delta m_\pi^2&=\frac{e^2}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}T(q^2,\nu)\\ &=e^2\int\frac{d^4q}{(2\pi)^4}\left\{\frac{1}{q^2}\frac{1}{q^2+2\nu}\left[3q^2+4\nu-4m^2+4\frac{2m_V^2-q^2}{(m_V^2-q^2)^2}(\nu^2-m^2q^2)\right]+\frac{3}{F_\pi^2}\left[\frac{F_V^2}{m_V^2-q^2}-\frac{F_A^2}{m_A^2-q^2}\right]\right\} \end{aligned} \tag{47}$$

where, for simplicity, we have used the complete vector dominance assumption $G_V=\frac{1}{2}F_V$ [4]. We can rewrite Eq. (47) as

$$\Delta m_\pi^2=e^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\left[\frac{1}{(q^2+2\nu)}(3q^2+4\nu-4m^2)\left[\frac{m_V^2}{m_V^2-q^2}\right]^2+3\frac{m_V^2}{m_V^2-q^2}\left[\frac{m_A^2}{m_A^2-q^2}-\frac{m_V^2}{m_V^2-q^2}\right]\right] \tag{48}$$

where we have used both Weinberg sum rules. Note that this expression is convergent and that the first piece of Eq. (48) is just the expression given by Socolow. However, consistency with chiral-symmetry strictures *requires* the existence of an additional nonzero correction associated with higher mass intermediate states. As we shall see, these corrections are reasonably small, as suggested by Cottingham arguments.

One can also demonstrate the equivalence of this result with the chiral limit calculation of Ref. [4] by setting $\nu\to0, m_\pi^2\to0$. Here we rearrange terms using the Weinberg sum rule relations in order to obtain

$$\begin{aligned}\Delta m_\pi^2 &= 3e^2\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{m_V^2}{m_V^2-q^2}\frac{m_A^2}{m_A^2-q^2}\\ &= \frac{3e^2}{F_\pi^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\left[\frac{F_V^2 m_V^2}{m_V^2-q^2}-\frac{F_A^2 m_A^2}{m_A^2-q^2}\right]\\ &= \frac{3e^2}{F_\pi^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\int ds\frac{s}{s-q^2}[F_V^2\delta(s-m_V^2)-F_A^2\delta(s-m_A^2)]\\ &= -\frac{3\alpha}{4\pi F_\pi^2}\int ds\, s\ln s[F_V^2\delta(s-m_V^2)-F_A^2\delta(s-m_A^2)]\\ &= \frac{3\alpha F_V^2}{4\pi F_\pi^2}\ln\frac{m_A^2}{m_V^2}\end{aligned} \tag{49}$$

which is precisely the formula given by Das *et al.* [12].

Having convinced ourselves that the model amplitude satisfies all chiral constraints, matches the experimental value of the low-energy chiral coefficients, contains the required resonance behavior, and reduces in the soft pion limit to the desired result of Ref. [12], we use it in order to find the pion mass splitting. The Feynman integration is easily performed, yielding

$$\Delta m_\pi^2 = \Delta m_{\text{Socolow}}^2 + \Delta m_{\text{remainder}}^2 \tag{50}$$

with $\Delta m_{\pi\ \text{Socolow}}^2$ given in Eq. (31) while

$$\Delta m_{\text{remainder}}^2 = -3\frac{\alpha}{4\pi}m_V^2\left[1-\frac{1}{1-\dfrac{m_V^2}{m_A^2}}\ln\frac{m_A^2}{m_V^2}\right]. \tag{51}$$

Putting in numbers we find

$$\Delta m_\pi^2 = 2m_\pi\times 5.6\ \text{MeV} \tag{52}$$

which remains close to the experimental value of $2m_\pi\times 4.6$ MeV. This result reduces to that of Ref. [4] in the $m_\pi\to 0$ limit.

In the case of the kaon system we find

$$\begin{aligned}\Delta m_K^2 = e^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2(q^2+2\nu)}\Bigg\{&(3q^2+4\nu-4m_K^2)\\ &-8(m_K^2q^2-\nu^2)\left[\frac{2}{3}\frac{1}{m_\rho^2-q^2}+\frac{1}{3}\frac{1}{m_\phi^2-q^2}\right.\\ &\left.+\frac{q^2}{2}\left[\frac{1}{3}\frac{1}{(m_\rho^2-q^2)^2}+\frac{2}{3}\frac{1}{(m_\rho^2-q^2)(m_\phi^2-q^2)}\right]\right]\\ &+\frac{3}{F_\pi^2}q^2(q^2+2\nu)\left[\frac{2}{3}\frac{F_\rho^2}{m_\rho^2-q^2}+\frac{1}{3}\frac{F_\phi^2}{m_\phi^2-q^2}-\frac{F_{K_A}^2}{m_{K_A}^2-q^2}\right]\Bigg\}\end{aligned} \tag{53}$$

which clearly with $\nu=p\cdot q$ reduces to Eq. (47) in the SU(3) limit, in agreement with Dashen's theorem. However, there are important SU(3)-breaking effects and again we can write

$$\Delta m_K^2(\text{total}) = \Delta m_K^2(\text{Socolow}) + \Delta m_K^2(\text{remainder}) \tag{54}$$

where Δm_K^2(Socolow) is given in Eq. (32). The form of Δm_K^2(remainder) can be read off from Eq. (53), although its form is not as simple as its pion analogue, and we do not display it here. It is straightforward to perform a numerical evaluation of Eq. (53), yielding

$$\Delta m_K^2(\text{total}) = 1.8\Delta m_\pi^2(\text{total})\ . \tag{55}$$

This result is not far from that given by Socolow, Eq. (32), and shows that the correction Δm_K^2(remainder) is not large. We see that Dashen's theorem, which is automatically obeyed in the chiral SU(3) limit, is significantly violated in the full calculation. In searching for the reason, we find that it is largely kinematic, due to the factor of $p^2=m_K^2$ instead of $p^2=m_\pi^2$ in the kaon propagator. Certainly the features of SU(3) breaking in the calculation are well established and not controversial, so that it seems impossible to avoid the conclusion that Dashen's theorem is not a reliable guide to the electromagnetic mass splittings.

V. SUMMARY

Electromagnetic mass differences involve integrals over the forward Compton scattering amplitude. We have described general features which must be satisfied by this amplitude, and have given the resulting form of the mass difference. Analysis of the Cottingham formula reveals that the Born term and low-energy contributions are most important. Experience with the relevant form factors and spectral functions suggests that the main physics ingredients are the couplings of vector and axial-vector mesons. These features are captured in a model calculation including these intermediate states. The result gives a reasonable description of the pion mass difference. It also reveals a significant violation of Dashen's theorem as shown in Eq. (55), which produces a larger electromagnetic effect in kaons than previously realized. Since this breaking is based on known low-energy effects, we conclude that Dashen's theorem is *not* phenomenologically robust. Finally, if we return to the motivations which we used in the Introduction, we now find

$$\frac{m_d-m_u}{m_s-\hat{m}}\frac{m_d+m_u}{m_s+\hat{m}} = \frac{m_\pi^2}{m_K^2}\left[\frac{(m_{K^0}^2-m_{K^+}^2)-1.8(m_{\pi^0}^2-m_{\pi^+}^2)}{m_K^2-m_\pi^2}\right] = 2.11\times10^{-3} \tag{56}$$

which agrees within 10% with the ratio extracted from $\eta\rightarrow3\pi$. Thus an important and remarkable consequence of this violation of Dashen's theorem is to bring the two independent $O(E^4)$ calculations of chiral perturbation theory into agreement.

Note added in proof. Two earlier papers which are relevant for the discussion of the pion electromagnetic mass difference are I. Gerstein, H. J. Schnitzer, T. Wong, and G. S. Guralnik, Phys. Rev. D **1**, 3442 (1970); S. G. Brown and G. West, Phys. Rev. **168**, 1605 (1968).

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