Universal evolution of Cabibbo-Kobayashi-Maskawa matrix elements

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We derive the two-loop evolution equations for the Cabibbo-Kobayashi-Masakawa (CKM) matrix. We show that to leading order in the mass and CKM hierarchies the scaling of the mixings $|V_{ub}|^2, |V_{cb}|^2, |V_{td}|^2, |V_{ts}|^2$ and of the rephase-invariant CP-violating parameter J is universal to all orders in perturbation theory. In leading order the other CKM elements do not scale. Imposing the constraint $\lambda_b=\lambda_\tau$ at the grand unified theory scale determines the CKM scaling factor to be $\simeq 0.58$ in the minimal supersymmetric standard model.

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The weak interaction quark eigenstates and the quark mass eigenstates differ in the standard model as described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In this paper we show that the scaling of the CKM matrix follows a universal pattern to leading order in the mass and CKM hierarchies: namely, the CKM mixing elements that involve the third generation and *CP*-violation scale together, while the other components of the CKM matrix do not scale to leading order. This makes it much simpler to consider the form of the quark mixings at any other scale, in particular, at the scale of a grand unified theory (GUT). The common scaling is a model-independent feature of the evolution, but the amount of scaling can vary between theories.

The Yukawa matrices **U** and **D** can be diagonalized by biunitary transformations

$$\mathbf{U}^{\rm diag}=V_u^L \mathbf{U} V_u^{R\dagger} , \tag{1}$$

$$\mathbf{D}^{\rm diag}=V_d^L \mathbf{D} V_d^{R\dagger} . \tag{2}$$

The CKM matrix is then given by

$$V \equiv V_u^L V_d^{L\dagger} . \tag{3}$$

The Yukawa matrices evolve with energy scale as determined by renormalization group equations (RGE's). This in turn determines an evolution equation for the "running" CKM matrix $V(\mu)$.

The renormalization group scaling to leading order in the mass and CKM hierarchies can be represented schematically in the following way:

$$\mathbf{U}^{\rm diag}(M_G)=\begin{bmatrix} S_u(\mu)\lambda_u(\mu) & 0 & 0 \\ 0 & S_u(\mu)\lambda_c(\mu) & 0 \\ 0 & 0 & S_t(\mu)\lambda_t(\mu) \end{bmatrix} , \tag{4}$$

$$\mathbf{D}^{\rm diag}(M_G)=\begin{bmatrix} S_d(\mu)\lambda_d(\mu) & 0 & 0 \\ 0 & S_d(\mu)\lambda_s(\mu) & 0 \\ 0 & 0 & S_b(\mu)\lambda_b(\mu) \end{bmatrix} . \tag{5}$$

$$\mathbf{E}^{\rm diag}(M_G)=\begin{bmatrix} S_e(\mu)\lambda_e(\mu) & 0 & 0 \\ 0 & S_e(\mu)\lambda_\mu(\mu) & 0 \\ 0 & 0 & S_\tau(\mu)\lambda_\tau(\mu) \end{bmatrix} , \tag{6}$$

$$|\mathbf{V}|^2(M_G)=\begin{bmatrix} |V_{ud}|^2(\mu) & |V_{us}|^2(\mu) & S(\mu)|V_{ub}|^2(\mu) \\ |V_{cd}|^2(\mu) & |V_{cs}|^2(\mu) & S(\mu)|V_{cb}|^2(\mu) \\ S(\mu)|V_{td}|^2(\mu) & S(\mu)|V_{ts}|^2(\mu) & |V_{tb}|^2(\mu) \end{bmatrix} , \tag{7}$$

where the scale μ is the range $m_t \le \mu \le M_G$ with M_G the GUT scale. The *CP*-violating rephase-invariant parameter J [1] also scales as $J(M_G)=S(\mu)J(\mu)$ to leading order. We have defined our scaling factors to be unity at the GUT scale, but one could equally well choose any convenient scale.

The two light generation quark and lepton Yukawa couplings evolve in a common manner determined by the gauge couplings and traces of the Yukawa matrices, while the third generation Yukawa couplings receive additional Yukawa contributions. This implies that the ratios $\lambda_u/\lambda_c, \lambda_d/\lambda_s, \lambda_e/\lambda_\mu$ do not evolve. The scaling pattern in Eq. (7) violates unitarity of V, but only at subleading order. For example the relation $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2=1$ is violated by terms that are neglected to leading order in the evolution of $|V_{ud}|^2$ and $|V_{us}|^2$. The elements $|V_{ud}|^2$ and $|V_{us}|^2$ must evolve to subleading order to preserve unitarity. A practical strategy is to evolve the small mixings $X=|V_{ub}|^2, Y=|V_{us}|^2, Z=|V_{cb}|^2$, and J which completely determine the other entries in the CKM matrix.

In terms of $t=\ln(\mu/M_G)$ the two-loop RGE's can be written as

$$\frac{d\mathbf{U}}{dt}=\frac{1}{16\pi^2}\left[(x_1\mathbf{I}+x_2\mathbf{UU}^\dagger+a_u\mathbf{DD}^\dagger)+\frac{1}{16\pi^2}\left[x_3\mathbf{I}+x_4\mathbf{UU}^\dagger+x_5(\mathbf{UU}^\dagger)^2+b_u\mathbf{DD}^\dagger+c_u(\mathbf{DD}^\dagger)^2+d_u\mathbf{UU}^\dagger\mathbf{DD}^\dagger+e_u\mathbf{DD}^\dagger\mathbf{UU}^\dagger\right]\right]\mathbf{U}\ , \tag{8}$$

$$\frac{d\mathbf{D}}{dt}=\frac{1}{16\pi^2}\left[(x_6\mathbf{I}+x_7\mathbf{DD}^\dagger+a_d\mathbf{UU}^\dagger)+\frac{1}{16\pi^2}\left[x_8\mathbf{I}+x_9\mathbf{DD}^\dagger+x_{10}(\mathbf{DD}^\dagger)^2+b_d\mathbf{UU}^\dagger+c_d(\mathbf{UU}^\dagger)^2+d_d\mathbf{DD}^\dagger\mathbf{UU}^\dagger+e_d\mathbf{UU}^\dagger\mathbf{DD}^\dagger\right]\right]\mathbf{D}\ , \tag{9}$$

$$\frac{d\mathbf{E}}{dt}=\frac{1}{16\pi^2}\left[x_{11}\mathbf{I}+x_{12}\mathbf{EE}^\dagger+\frac{1}{16\pi^2}[x_{13}\mathbf{I}+x_{14}\mathbf{EE}^\dagger+x_{15}(\mathbf{EE}^\dagger)^2]\right]\mathbf{E}\ , \tag{10}$$

where the coefficients x_i, a_i, etc. depend upon the particle content of the theory and are functions of the gauge and Yukawa couplings, i.e., $a_i=a_i(g_1^2,g_2^2,g_3^2,\mathrm{Tr}[\mathbf{UU}^\dagger],\mathrm{Tr}[\mathbf{DD}^\dagger],\mathrm{Tr}[\mathbf{EE}^\dagger])$ and Higgs quartic couplings. The coefficients x_i do not enter into the running of the CKM matrix but do influence the diagonal quark Yukawa evolution; only terms involving a factor of $\mathbf{DD}^\dagger$ can rotate the $\mathbf{U}$ matrix, and only terms with a factor of $\mathbf{UU}^\dagger$ can rotate the $\mathbf{D}$ matrix. In the minimal supersymmetric standard model (MSSM) the other coefficients are

$$a_u=a_d=1\ , \tag{11}$$

$$b_u=\tfrac{2}{5}g_1^2-\mathrm{Tr}[3\mathbf{DD}^\dagger+\mathbf{EE}^\dagger]\ , \tag{12}$$

$$b_d=\tfrac{4}{5}g_1^2-\mathrm{Tr}[3\mathbf{UU}^\dagger]\ , \tag{13}$$

$$c_u=c_d=-2\ , \tag{14}$$

$$d_u=d_d=-2\ , \tag{15}$$

$$e_u=e_d=0\ . \tag{16}$$

In the standard model they are given by

$$a_u=a_d=-\tfrac{3}{2}\ , \tag{17}$$

$$b_u==-\tfrac{43}{80}g_1^2+\tfrac{9}{16}g_2^2-16g_3^2-2\lambda+\tfrac{5}{4}Y_2(S)\ , \tag{18}$$

$$b_d==-\tfrac{79}{80}g_1^2+\tfrac{9}{16}g_2^2-16g_3^2-2\lambda+\tfrac{5}{4}Y_2(S)\ , \tag{19}$$

$$c_u=c_d=\tfrac{11}{4}\ , \tag{20}$$

$$d_u=d_d=-\frac{1}{4}\ , \tag{21}$$

$$e_u=e_d=-1\ , \tag{22}$$

where

$$Y_2(S)=\mathrm{Tr}[3\mathbf{UU}^\dagger+3\mathbf{DD}^\dagger+\mathbf{EE}^\dagger]\ . \tag{23}$$

The coefficients x_i can be found in Refs. [2,3].

Following Ma, Pakvasa, Sasaki, and Babu, [4,5] we find the CKM evolution equation

$$\frac{dV_{i\alpha}}{dt}=\frac{1}{16\pi^2}\left[a_u\sum_{\beta,j\neq i}\frac{\lambda_i^2+\lambda_j^2}{\lambda_i^2-\lambda_j^2}\hat\lambda_\beta^2V_{i\beta}V_{j\beta}^*V_{j\alpha}+a_d\sum_{j,\beta\neq\alpha}\frac{\lambda_\alpha^2+\lambda_\beta^2}{\lambda_\alpha^2-\lambda_\beta^2}\hat\lambda_j^2V_{j\beta}^*V_{j\alpha}V_{i\beta}\right]+\frac{1}{(16\pi^2)^2}\left[\sum_{\beta,j\neq i}\frac{2d_u\lambda_i^2\lambda_j^2+e_u(\lambda_i^4+\lambda_j^4)}{\lambda_i^2-\lambda_j^2}\lambda_\beta^2V_{i\beta}V_{j\beta}^*V_{j\alpha}+\sum_{j,\beta\neq\alpha}\frac{2d_d\lambda_\alpha^2\lambda_\beta^2+e_d(\lambda_\alpha^4+\lambda_\beta^4)}{\lambda_\alpha^2-\lambda_\beta^2}\lambda_j^2V_{j\beta}^*V_{j\alpha}V_{i\beta}\right] \tag{24}$$

where

$$\hat{\lambda}_\beta^2=\lambda_\beta^2\left[1+\frac{b_u+c_u\lambda_\beta^2}{16\pi^2 a_u}\right], \tag{25}$$

$$\hat{\lambda}_j^2=\lambda_j^2\left[1+\frac{b_d+c_d\lambda_j^2}{16\pi^2 a_d}\right]. \tag{26}$$

Here $i,j,k=u,c,t,\ldots$; $\alpha,\beta,\gamma=d,s,b,\ldots$. We henceforth restrict our considerations to the three-generation case. Defining the four independent quantities $X=|V_{ub}|^2$, $Y=|V_{us}|^2$, $Z=|V_{cb}|^2$, and the parameter $J=\mathrm{Im}V_{ud}V_{cs}V_{us}^*V_{cd}^*$ which can completely specify a unitary CKM matrix, the other elements are given by [5]

$$|V_{ud}|^2=1-X-Y, \tag{27}$$

$$|V_{cs}|^2=\frac{[XYZ+(1-X-Y)(1-X-Z)-2K]}{(1-X)^2}, \tag{28}$$

$$|V_{cd}|^2=\frac{[XZ(1-X-Y)+Y(1-X-Z)+2K]}{(1-X)^2}, \tag{29}$$

$$|V_{tb}|^2=1-X-Z, \tag{30}$$

$$|V_{ts}|^2=\frac{[XY(1-X-Z)+(1-X-Y)Z+2K]}{(1-X)^2}, \tag{31}$$

$$|V_{td}|^2=\frac{[X(1-X-Y)(1-X-Z)+YZ-2K]}{(1-X)^2}, \tag{32}$$

where

$$K=[XYZ(1-X-Y)(1-X-Z)-J^2(1-X)^2]^{1/2}. \tag{33}$$

The full evolution equations for X,Y,Z, and J are given in the appendix. Keeping only the leading terms in the mass $(\lambda_c/\lambda_t,\lambda_u/\lambda_c,\lambda_s/\lambda_b,\lambda_d/\lambda_s\ll 1)$ and CKM $(X,Z,J\ll 1)$ hierarchies, these equations simplify considerably [3] and a universal scaling is found

$$\frac{dW_1}{dt}=-\frac{W_1}{8\pi^2}\left[(a_d\hat{\lambda}_t^2+a_u\hat{\lambda}_b^2)+\frac{1}{(16\pi^2)}(e_d+e_u)\lambda_t^2\lambda_b^2\right], \tag{34}$$

where $W_1=|V_{cb}|^2,|V_{ub}|^2,|V_{ts}|^2,|V_{td}|^2,J$ and

$$\frac{dW_2}{dt}=0, \tag{35}$$

where $W_2=|V_{us}|^2,|V_{cd}|^2,|V_{tb}|^2,|V_{cs}|^2,|V_{ud}|^2$. One does not need the mixing between the first two generations to be small $(Y\ll 1)$ which makes the universality an especially good approximation. To leading order it is only necessary to include the third generation Yukawa couplings in $\hat{\lambda}_t^2$ and $\hat{\lambda}_b^2$. Notice that Eqs. (34) and (35) violate unitarity of V, but only at subleading order. The solution of Eq. (34) is

$$W_1(M_G)=W_1(\mu)S(\mu), \tag{36}$$

where S is a scaling factor defined by

$$S(\mu)=\exp\left\{-\frac{1}{8\pi^2}\int_\mu^{M_G}\left[(a_d\hat{\lambda}_t^2+a_u\hat{\lambda}_b^2)+\frac{1}{(16\pi^2)}(e_d+e_u)\lambda_t^2\lambda_b^2\right]d\ln\mu'\right\}. \tag{37}$$

This reduces [3,6] to the scaling factor $y^2(\mu)$ in the one-loop semianalytic treatment (neglecting λ_b and λ_τ), with

$$y(\mu)=\exp\left\{-\frac{1}{16\pi^2}\int_\mu^{M_G}a_d\lambda_t^2(\mu')d\ln\mu'\right\}. \tag{38}$$

The general behavior of $S(\mu)$ is determined by the sign of a_d (and perhaps also a_u in models where $\tan\beta$ is large). In the standard model the scaling factors are greater than one since the one-loop coefficients a_u and a_d are negative.

One might naively have expected there to be contributions to the scaling of the W_1 that are not proportional to W_1; for example, a contribution to the running of $|V_{ub}|^2$ of the form $\lambda_c^2|V_{cb}|^2$ on the right-hand side of Eq. (34) can be of the same order as the contribution $\lambda_t^2|V_{ub}|^2$. We conclude that no such terms arise. We find the following RGE's for the Yukawa couplings:

$$\frac{d\lambda_i}{dt}=\frac{\lambda_i}{16\pi^2}\left[x_1+x_2\lambda_i^2+a_u\sum_\alpha\lambda_\alpha^2|V_{i\alpha}|^2+\frac{1}{16\pi^2}\left\{x_3+x_4\lambda_i^2+x_5\lambda_i^4+\sum_\alpha[b_u\lambda_\alpha^2+c_u\lambda_\alpha^4+(d_u+e_u)\lambda_i^2\lambda_\alpha^2]|V_{i\alpha}|^2\right\}\right], \tag{39}$$

$$\frac{d\lambda_\alpha}{dt}=\frac{\lambda_\alpha}{16\pi^2}\left[x_6+x_7\lambda_\alpha^2+a_d\sum_i\lambda_i^2|V_{i\alpha}|^2+\frac{1}{16\pi^2}\left\{x_8+x_9\lambda_\alpha^2+x_{10}\lambda_\alpha^4+\sum_i[b_d\lambda_i^2+c_d\lambda_i^4+(d_d+e_d)\lambda_\alpha^2\lambda_i^2]|V_{i\alpha}|^2\right\}\right], \tag{40}$$

$$\frac{d\lambda_a}{dt}=\frac{\lambda_a}{16\pi^2}\left[x_{11}+x_{12}\lambda_a^2+\frac{1}{16\pi^2}[x_{13}+x_{14}\lambda_a^2+x_{15}\lambda_a^4]\right], \tag{41}$$

where $a=e,\mu,\tau$. Including only the third generation in the sums, these equations reduce to the leading-order expressions for $\lambda_t,\lambda_b,\lambda_\tau$ yielding

$$S_t(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_1+x_2\lambda_t^2+a_u\lambda_b^2+\frac{1}{16\pi^2}(x_3+x_4\lambda_t^2+x_5\lambda_t^4+[b_u\lambda_b^2+c_u\lambda_b^4+(d_u+e_u)\lambda_t^2\lambda_b^2])\right]d\ln\mu'\right\}, \tag{42}$$

$$S_b(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_6+x_7\lambda_b^2+a_d\lambda_t^2+\frac{1}{16\pi^2}\{x_8+x_9\lambda_b^2+x_{10}\lambda_b^4+[b_d\lambda_t^2+c_d\lambda_t^4+(d_d+e_d)\lambda_b^2\lambda_t^2]\}\right]d\ln\mu'\right\}, \tag{43}$$

$$S_\tau(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_{11}+x_{12}\lambda_\tau^2+\frac{1}{16\pi^2}[x_{13}+x_{14}\lambda_\tau^2+x_{15}\lambda_\tau^4]\right]d\ln\mu'\right\}, \tag{44}$$

respectively. For the first and second generations the corresponding expressions are

$$S_u(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_1+\frac{1}{16\pi^2}x_3\right]d\ln\mu'\right\}, \tag{45}$$

$$S_d(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_6+\frac{1}{16\pi^2}x_8\right]d\ln\mu'\right\}, \tag{46}$$

$$S_e(\mu)=\exp\left\{\frac{1}{16\pi^2}\int_\mu^{M_G}\left[x_{11}+\frac{1}{16\pi^2}x_{13}\right]d\ln\mu'\right\}. \tag{47}$$

In Fig. 1 we show contours of constant $S(m_t)$ in the MSSM versus the values of the Yukawa couplings λ_t and λ_b at scale m_t and also at the GUT scale. The contribution to the scaling from λ_t can be traded off against the contribution for λ_b as indicated by Eq. (37). These contours are shown versus m_t and $\tan\beta$ in Fig. 2. The contour satisfying the constraints $m_b(m_b)=4.4$ GeV and $\lambda_b(M_G)=\lambda_\tau(M_G)$ is plotted as well. The evolution equation for $R_{b/\tau}\equiv\lambda_b/\lambda_\tau$ at one-loop in the MSSM is given by

$$\frac{dR_{b/\tau}}{dt}=\frac{R_{b/\tau}}{16\pi^2}\left[-\sum d_i g_i^2+\lambda_t^2+3\lambda_b^2-3\lambda_\tau^2\right]. \tag{48}$$

where $d_i=(-4/3,0,16/3)$. For small $\tan\beta$ the bottom-quark and τ Yukawa couplings can be neglected, and the scaling of $R_{b/\tau}$ factorizes into scaling due to the gauge couplings and the scaling factor S due to the top-quark Yukawa coupling. Given a fixed gauge sector scaling, the m_b contours and the contours of constant S coincide for small $\tan\beta$. Note that $m_b\simeq 4.4$ GeV implies $S(m_t)\simeq 0.58$.

The numerical calculations performed here are similar to those described in Ref. [3]. The input values are $\alpha_1(M_Z)^{-1}=58.89$, $\alpha_2(M_Z)^{-1}=29.75$, and $\alpha_3(M_Z)=0.116$ for the gauge couplings and $m_b(m_b)=4.4$ GeV, $m_c(m_c)=1.2$ GeV, $m_s(1\text{ GeV})=0.15$ GeV, $m_d(1\text{ GeV})=0.008$ GeV, $m_u(1\text{ GeV})=0.005$ GeV for the running fermion masses. We take the lepton masses to be $m_\tau=1.7841$ GeV, $m_\mu=0.10566$ GeV, and $m_e=5.1100\times10^{-4}$ GeV. The GUT scale M_G is determined as the scale at which unification of α_1 and α_2 is achieved. Given an input value for $\tan\beta$ the input masses and the gauge couplings determine the Yukawa couplings

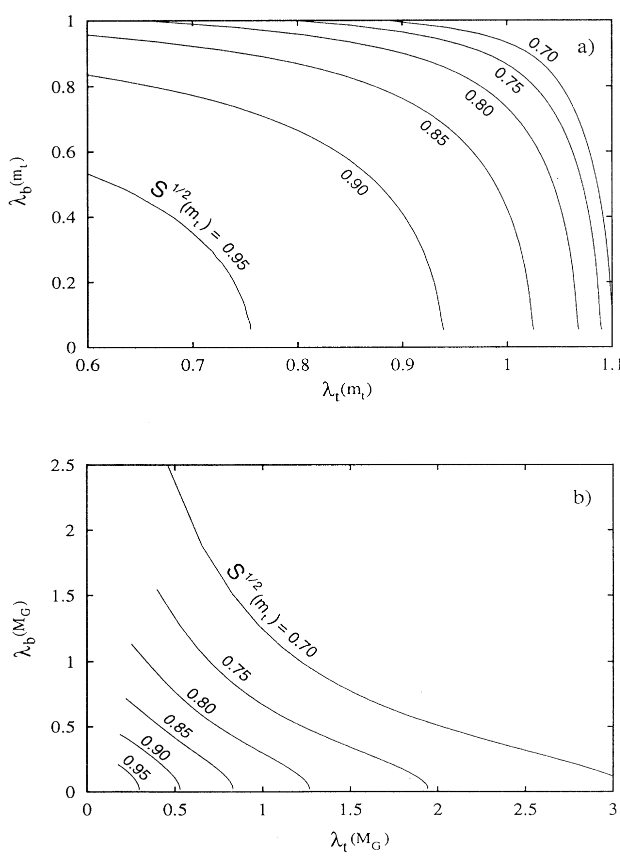

FIG. 1. Contours of constant $S^{1/2}(m_t)$ in the MSSM are shown for values of λ_t and λ_b at (a) $\mu=m_t$ and (b) $\mu=M_G$. We have taken $\alpha_3(M_Z)=0.116$.

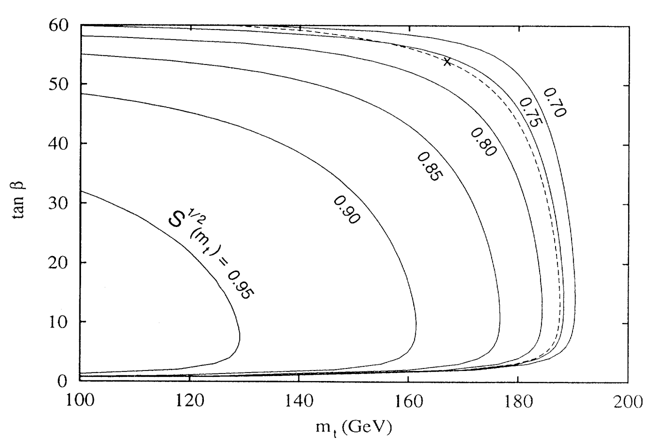

FIG. 2. Contours of constant $S^{1/2}(m_t)$ in the MSSM are shown in the m_t,$\tan\beta$ plane for $\alpha_3(M_Z)=0.116$. The dashed line is the $m_b(m_b)=4.4$ GeV contour obtained from the GUT scale condition $\lambda_b(M_G)=\lambda_\tau(M_G)$. The $\times$ marks the spot at which $\lambda_t(M_G)=\lambda_b(M_G)=\lambda_\tau(M_G)$ for this m_b contour. In the small $\tan\beta$ region a linear relationship exists between m_t and $\sin\beta$ for each contour.

at the scale m_t. We integrate the two-loop RGE's for the gauge and Yukawa couplings as well as the evolution equations for X, Y, Z, and J given in the appendix. Our results are not sensitive to the values of the first and second generation fermion masses or to the input CKM magnitudes $|V_{cb}(m_t)|=0.043$, $|V_{ub}(m_t)|=0.0045$, $|V_{us}(m_t)|=0.221$, $J(m_t)=1.95\times10^{-5}$. For experimentally acceptable values of the quark masses and CKM matrix elements, the exact scaling as given by the equations in the appendix differ from the universal behavior described by Eq. (34) by $\lesssim 0.1\%$.

A good approximation for evolving the CKM matrix is to use Eq. (34) to evolve $|V_{ub}|^2, |V_{cb}|^2$, and J and leave $|V_{us}|^2$ constant as in Eq. (35). Then calculate the remaining magnitudes $|V_{i\alpha}|$ using Eqs. (27)–(32).

One can show that the universal scaling behavior described by Eqs. (34) and (35) is maintained to all orders in perturbation theory. However the quantitative effects of the three-loop contribution are generally smaller than the sub-leading contributions in the mass and CKM hierarchies.

We now give a sketch of a proof of the universal behavior at an arbitrary order in perturbation theory. A higher-order contribution will be of the general form

$$\frac{d\mathbf{U}}{dt}=\frac{1}{(16\pi^2)^q}[f_u^{mn\cdots op}(\mathbf{DD}^\dagger)^m(\mathbf{UU}^\dagger)^n\cdots(\mathbf{UU}^\dagger)^o(\mathbf{DD}^\dagger)^p]\mathbf{U}+\cdots\,, \tag{49}$$

where $q\geq m+n+\cdots+o+p$ represents the loop order. There is an analogous contribution to $d\mathbf{D}/dt$. The exponents m and p could be zero. The coefficient $f_u^{mn\cdots op}$ is calculable in perturbation theory but can be obtained only with tedious effort; it is a function of the gauge couplings g_i and the sum of the eigenvalues of the Yukawa couplings matrices, $\mathrm{Tr}[\mathbf{UU}^\dagger]$, $\mathrm{Tr}[\mathbf{DD}^\dagger]$, $\mathrm{Tr}[\mathbf{EE}^\dagger]$, and possibly other couplings such as the quartic Higgs coupling in the standard model. The term in Eq. (49) generates a new contribution to Eq. (24):

$$\frac{dV_{i\alpha}}{dt}=\frac{1}{(16\pi^2)^q}\left[f_u^{mn\cdots op}\sum_{j\neq i}\left\{\frac{1}{\lambda_i^2-\lambda_j^2}\sum_{\beta,k,\gamma,\ldots,l,\delta}(\lambda_\beta^{2m}\lambda_k^{2n}\cdots\lambda_l^{2o}\lambda_\delta^{2p}\lambda_j^2+\lambda_i^2\lambda_\beta^{2p}\lambda_k^{2o}\cdots\lambda_l^{2n}\lambda_\delta^{2m})\times V_{i\beta}V_{k\beta}^*V_{k\gamma}V_{l\gamma}^*\cdots V_{j\delta}^*V_{j\alpha}\right\}\right]+\cdots\,. \tag{50}$$

The only terms that contribute to leading order in dX/dt, dY/dt, dZ/dt, dJ/dt are those in which the indices in the second sum above involve the third generation:

$$\frac{dV_{i\alpha}}{dt}=\frac{1}{(16\pi^2)^q}\left[f_u^{mn\cdots op}\sum_{j\neq i}\left\{\frac{\lambda_i^2+\lambda_j^2}{\lambda_i^2-\lambda_j^2}(\lambda_b^{2m}\lambda_t^{2n}\cdots\lambda_t^{2o}\lambda_b^{2p})V_{ib}V_{tb}^*V_{tb}V_{tb}^*\cdots V_{jb}^*V_{j\alpha}\right\}\right]+\cdots\,. \tag{51}$$

Then to leading order, $|V_{tb}|^2\simeq1$, and one has

$$\frac{dV_{i\alpha}}{dt}=\frac{1}{(16\pi^2)^q}\left[f_u^{mn\cdots op}\sum_{j\neq i}\left\{\frac{\lambda_i^2+\lambda_j^2}{\lambda_i^2-\lambda_j^2}(\lambda_t^{2(n+\cdots+o)}\lambda_b^{2(m+\cdots+p)})V_{ib}V_{jb}^*V_{j\alpha}\right\}\right]+\cdots\,,$$

which has the same form as Eq. (24). Consequently

$$\frac{dW_1}{dt}=-\frac{2W_1}{(16\pi^2)^q}[f_u^{mn\cdots op}(\lambda_t^{2(n+\cdots+o)}\lambda_b^{2(m+\cdots+p)})]\,. \tag{52}$$

A similar argument applies to the cases $m=0$ and/or $p=0$.

A summary we have shown that there is a universal scaling pattern in the evolution of the CKM matrix when the only the leading-order terms in the mass and CKM hierarchies are kept. This is a very good approximation given the observed hierarchy of the quark masses and CKM matrix elements. This scaling behavior persists to all orders in perturbation theory. Imposing a GUT scale constraint $\lambda_b(M_G)=\lambda_\tau(M_G)$ constrains the amount of scaling. For $m_b(m_b)=4.4$ GeV the scaling factor of the CKM matrix is $S\simeq0.58$.

After this work was completed we learned that similar conclusions about the scaling of the CKM elements have been obtained at the one-loop level by Babu and Shafi [7].

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APPENDIX

The evolution equations for $|V_{i\alpha}|^2$ can be derived from Eq. (24) using the substitutions in Eqs. (27)–(32), as performed by Babu [5] at the one-loop level. At the two-loop level one obtains

$$\frac{dX}{dt}=\frac{2}{16\pi^2}\left[a_u\frac{\lambda_u^2+\lambda_c^2}{\lambda_u^2-\lambda_c^2}\left\{(\hat\lambda_b^2-\hat\lambda_d^2)XZ+\frac{\hat\lambda_d^2-\hat\lambda_s^2}{1-X}(XYZ-K)\right\}\right.$$
$$+a_u\frac{\lambda_u^2+\lambda_t^2}{\lambda_u^2-\lambda_t^2}\left\{(\hat\lambda_b^2-\hat\lambda_d^2)X(1-X-Z)+\frac{\hat\lambda_d^2-\hat\lambda_s^2}{1-X}[XY(1-X-Z)+K]\right\}$$
$$+a_d\frac{\lambda_b^2+\lambda_s^2}{\lambda_b^2-\lambda_s^2}\left\{(\hat\lambda_u^2-\hat\lambda_t^2)XY+\frac{\hat\lambda_t^2-\hat\lambda_c^2}{1-X}(XYZ-K)\right\}$$
$$\left.+a_d\frac{\lambda_b^2+\lambda_d^2}{\lambda_b^2-\lambda_d^2}\left\{(\hat\lambda_u^2-\hat\lambda_t^2)X(1-X-Y)+\frac{\hat\lambda_t^2-\hat\lambda_c^2}{1-X}[XZ(1-X-Y)+K]\right\}\right]$$
$$+\frac{2}{(16\pi^2)^2}\left[\frac{2d_u\lambda_u^2\lambda_c^2+e_u(\lambda_u^4+\lambda_c^4)}{\lambda_u^2-\lambda_c^2}\left\{(\lambda_b^2-\lambda_d^2)XZ+\frac{\lambda_d^2-\lambda_s^2}{1-X}(XYZ-K)\right\}\right.$$
$$+\frac{2d_u\lambda_u^2\lambda_t^2+e_u(\lambda_u^4+\lambda_t^4)}{\lambda_u^2-\lambda_t^2}\left\{(\lambda_b^2-\lambda_d^2)X(1-X-Z)+\frac{\lambda_d^2-\lambda_s^2}{1-X}[XY(1-X-Z)+K]\right\}$$
$$+\frac{2d_d\lambda_b^2\lambda_s^2+e_d(\lambda_b^4+\lambda_s^4)}{\lambda_b^2-\lambda_s^2}\left\{(\lambda_u^2-\lambda_t^2)XY+\frac{\lambda_t^2-\lambda_c^2}{1-X}(XYZ-K)\right\}$$
$$\left.+\frac{2d_d\lambda_b^2\lambda_d^2+e_d(\lambda_b^4+\lambda_d^4)}{\lambda_b^2-\lambda_d^2}\left\{(\lambda_u^2-\lambda_t^2)X(1-X-Y)+\frac{\lambda_t^2-\lambda_c^2}{1-X}[XZ(1-X-Y)+K]\right\}\right], \quad (53)$$

$$\frac{dY}{dt}=\frac{2}{16\pi^2}\left[a_u\frac{\lambda_u^2+\lambda_c^2}{\lambda_u^2-\lambda_c^2}\left\{\frac{\hat\lambda_d^2-\hat\lambda_b^2}{1-X}(XYZ-K)+\frac{\hat\lambda_s^2-\hat\lambda_d^2}{(1-X)^2}Y[XYZ+(1-X-Y)(1-X-Z)-2K]\right\}\right.$$
$$+a_u\frac{\lambda_u^2+\lambda_t^2}{\lambda_u^2-\lambda_t^2}\left\{\frac{\hat\lambda_b^2-\hat\lambda_d^2}{1-X}[XY(1-X-Z)+K]+\frac{\hat\lambda_s^2-\hat\lambda_d^2}{(1-X)^2}Y[XY(1-X-Z)+Z(1-X-Y)+2K]\right\}$$
$$+a_d\frac{\lambda_s^2+\lambda_b^2}{\lambda_s^2-\lambda_b^2}\left\{(\hat\lambda_u^2-\hat\lambda_t^2)XY+\frac{\hat\lambda_t^2-\hat\lambda_c^2}{1-X}(XYZ-K)\right\}$$
$$+a_d\frac{\lambda_s^2+\lambda_d^2}{\lambda_s^2-\lambda_d^2}\left\{(\hat\lambda_u^2-\hat\lambda_t^2)Y(1-X-Y)\right.$$
$$\left.\left.+\frac{\hat\lambda_c^2-\hat\lambda_t^2}{(1-X)^2}[XYZ(1-X-Y)-Y(1-X-Y)(1-X-Z)-K(1-X-2Y)]\right\}\right]$$
$$+\frac{2}{(16\pi^2)^2}\left[\frac{2d_u\lambda_u^2\lambda_c^2+e_u(\lambda_u^4+\lambda_c^4)}{\lambda_u^2-\lambda_c^2}\left\{\frac{\lambda_d^2-\lambda_b^2}{1-X}(XYZ-K)\right.\right.$$
$$\left.+\frac{\lambda_s^2-\lambda_d^2}{(1-X)^2}Y[XYZ+(1-X-Y)(1-X-Z)-2K]\right\}$$
$$+\frac{2d_u\lambda_u^2\lambda_t^2+e_u(\lambda_u^4+\lambda_t^4)}{\lambda_u^2-\lambda_t^2}\left\{\frac{\lambda_b^2-\lambda_d^2}{1-X}[XY(1-X-Z)+K]\right.$$
$$\left.+\frac{\lambda_s^2-\lambda_d^2}{(1-X)^2}Y[XY(1-X-Z)+Z(1-X-Y)+2K]\right\}$$
$$+\frac{2d_d\lambda_s^2\lambda_b^2+e_d(\lambda_s^4+\lambda_b^4)}{\lambda_s^2-\lambda_b^2}\left\{(\lambda_u^2-\lambda_t^2)XY+\frac{\lambda_t^2-\lambda_c^2}{1-X}(XYZ-K)\right\}$$

$$
+\frac{2d_d\lambda_s^2\lambda_d^2+e_d(\lambda_s^4+\lambda_d^4)}{\lambda_s^2-\lambda_d^2}\Bigg\{(\lambda_u^2-\lambda_t^2)Y(1-X-Y)
+\frac{\lambda_c^2-\lambda_t^2}{(1-X)^2}[XYZ(1-X-Y)-Y(1-X-Y)(1-X-Z)
-K(1-X-2Y)]\Bigg\}\Bigg] , \tag{54}
$$

$$
\frac{dZ}{dt}=\frac{2}{16\pi^2}\Bigg[a_u\frac{\lambda_c^2+\lambda_u^2}{\lambda_c^2-\lambda_u^2}\Bigg\{(\hat{\lambda}_b^2-\hat{\lambda}_d^2)XZ+\frac{\hat{\lambda}_d^2-\hat{\lambda}_s^2}{1-X}(XYZ-K)\Bigg\}
$$

$$
+a_u\frac{\lambda_c^2+\lambda_t^2}{\lambda_c^2-\lambda_t^2}\Bigg\{(\hat{\lambda}_b^2-\hat{\lambda}_d^2)Z(1-X-Z)+\frac{\hat{\lambda}_s^2-\hat{\lambda}_d^2}{(1-X)^2}[XYZ(1-X-Z)
-Z(1-X-Y)(1-X-Z)-K(1-X-2Z)]\Bigg\}
$$

$$
+a_d\frac{\lambda_b^2+\lambda_s^2}{\lambda_b^2-\lambda_s^2}\Bigg\{\frac{\hat{\lambda}_u^2-\hat{\lambda}_t^2}{1-X}(K-XYZ)+\frac{\hat{\lambda}_c^2-\hat{\lambda}_t^2}{(1-X)^2}Z[XYZ+(1-X-Y)(1-X-Z)-2K]\Bigg\}
$$

$$
+a_d\frac{\lambda_b^2+\lambda_d^2}{\lambda_b^2-\lambda_d^2}\Bigg\{\frac{\hat{\lambda}_t^2-\hat{\lambda}_u^2}{1-X}[XZ(1-X-Y)+K]
+\frac{\hat{\lambda}_c^2-\hat{\lambda}_t^2}{(1-X)^2}Z[XZ(1-X-Y)+Y(1-X-Z)+2K]\Bigg\}\Bigg]
$$

$$
+\frac{2}{(16\pi^2)^2}\Bigg[\frac{2d_u\lambda_c^2\lambda_u^2+e_u(\lambda_c^4+\lambda_u^4)}{\lambda_c^2-\lambda_u^2}\Bigg\{(\lambda_b^2-\lambda_d^2)XZ+\frac{\lambda_d^2-\lambda_s^2}{1-X}(XYZ-K)\Bigg\}
$$

$$
+\frac{2d_u\lambda_c^2\lambda_t^2+e_u(\lambda_c^4+\lambda_t^4)}{\lambda_c^2-\lambda_t^2}\Bigg\{(\lambda_b^2-\lambda_d^2)Z(1-X-Z)
+\frac{\lambda_s^2-\lambda_d^2}{(1-X)^2}[XYZ(1-X-Z)-Z(1-X-Y)(1-X-Z)-K(1-X-2Z)]\Bigg\}
$$

$$
+\frac{2d_d\lambda_b^2\lambda_s^2+e_d(\lambda_b^4+\lambda_s^4)}{\lambda_b^2-\lambda_s^2}\Bigg\{\frac{\lambda_u^2-\lambda_t^2}{1-X}(K-XYZ)+\frac{\lambda_c^2-\lambda_t^2}{(1-X)^2}Z[XYZ+(1-X-Y)(1-X-Z)-2K]\Bigg\}
$$

$$
+\frac{2d_d\lambda_b^2\lambda_d^2+e_d(\lambda_b^4+\lambda_d^4)}{\lambda_b^2-\lambda_d^2}\Bigg\{\frac{\lambda_t^2-\lambda_u^2}{1-X}[XZ(1-X-Y)+K]
+\frac{\lambda_c^2-\lambda_t^2}{(1-X)^2}Z[XZ(1-X-Y)+Y(1-X-Z)+2K]\Bigg\}\Bigg] , \tag{55}
$$

$$
\frac{dJ}{dt}=\frac{-(J/2)}{16\pi^2}\Bigg[a_u\sum_{\beta,j\neq i}\frac{\lambda_i^2+\lambda_j^2}{\lambda_i^2-\lambda_j^2}\hat{\lambda}_\beta^2(|V_{i\beta}|^2-|V_{j\beta}|^2)+a_d\sum_{j,\beta\neq\alpha}\frac{\lambda_\alpha^2+\lambda_\beta^2}{\lambda_\alpha^2-\lambda_\beta^2}\hat{\lambda}_j^2(|V_{j\alpha}|^2-|V_{j\beta}|^2)\Bigg]
$$

$$
+\frac{-(J/2)}{(16\pi^2)^2}\Bigg[\sum_{\beta,j\neq i}\frac{2d_u\lambda_i^2\lambda_j^2+e_u(\lambda_i^4+\lambda_j^4)}{\lambda_i^2-\lambda_j^2}\lambda_\beta^2(|V_{i\beta}|^2-|V_{j\beta}|^2)
+\sum_{j,\beta\neq\alpha}\frac{2d_d\lambda_\alpha^2\lambda_\beta^2+e_d(\lambda_\alpha^4+\lambda_\beta^4)}{\lambda_\alpha^2-\lambda_\beta^2}\lambda_j^2(|V_{j\alpha}|^2-|V_{j\beta}|^2)\Bigg] . \tag{56}
$$

Replacing $\hat{\lambda}_i\rightarrow\lambda_i,\hat{\lambda}_\alpha\rightarrow\lambda_\alpha$ and omitting the second contributions proportional to $(16\pi^2)^{-2}$, one recovers the one-loop results of Babu [5] (our definitions of X and Z differ from Ref. [5]). The two-loop equations have the same overall structure as the one-loop equations because both contain the same factor $V_{i\beta}V^*_{j\beta}V_{j\alpha}$ in Eq. (34). Eqs. (53)–(56), together with the evolution equations for the gauge couplings g_i and the Yukawa couplings λ_i, form a coupled set of differential equations that can be solved numerically. In their full form these equations together with Eqs. (27)–(32) preserve the unitarity of the CKM matrix to all orders in the hierarchy.

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