Updated description of quarkonium by power-law potentials

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The spectra and decay rates of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) levels are well described, for the most part, by a power-law potential $V(r)=\lambda(r^\alpha-1)/\alpha+\text{const}$, where $\alpha\simeq 0$. The results of an up-to-date fit to the data on spin-averaged levels are presented. A fit to levels alone favors $\alpha=-0.045$ and gives very little preference for quark masses, while inclusion of leptonic width data leads to $\alpha=-0.14$ and a b-quark mass in the vicinity of 5 GeV. A fit to the known electric dipole transitions in $b\bar{b}$ systems favors a slightly smaller b-quark mass, but with large errors.

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I. INTRODUCTION

Charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) systems provide a rich source of information on the interquark force at distances ranging from less than 0.1 fm to greater than 1 fm. At short distances, our theoretical prejudices favor a potential which should act like a Coulomb interaction $V(r)\sim\alpha_s(r)/r$. The strong fine-structure constant $\alpha_s(r)$ becomes weaker as $[\ln(1/r)]^{-1}$ at short distances as a result of the asymptotic freedom of quantum chromodynamics (QCD) [1]. At long distances, there are both experimental and theoretical reasons [2] to believe that the interquark force in QCD becomes approximately distance independent, corresponding to a linear potential $V(r)\sim r$. The $c\bar{c}$ and $b\bar{b}$ systems appear to lie in an intermediate range where a power-law potential $V(r)\sim r^\alpha$ ($\alpha\simeq 0$) provides a convenient interpolating form [3–8] between the short-distance Coulomb-like and long-distance linear behavior.

The power-law description is not expected to be of fundamental significance, but it is rather convenient for approximate estimates of $c\bar{c}$ and $b\bar{b}$ properties. It is only necessary to solve the Schrödinger equation once for any power α; the dependence on coupling strength and reduced mass of all quantities is determined by simple scaling laws [4–6]. The laws for the limiting case $\alpha\to 0$, i.e., for the potential $V(r)=C\ln(r/r_0)$ (see Sec. III), are particularly simple because the spacings of energy levels are proportional to C and are independent of reduced mass.

An early fit to quarkonium spectra [7] found a power $\alpha\simeq 0.1$. Since then, data on the P-wave $b\bar{b}$ levels have appeared [9,10], and information on leptonic widths has improved. It is appropriate to update the analysis of Ref. [7] in the light of the new data, for several reasons.

(1) It is so easy to calculate properties of levels in the potential [4] $V(r)=C\ln(r/r_0)$ that it is of interest to see how well such a potential fits present data. The initial motivation for such a potential was the similarity, aside from an overall shift, between $c\bar{c}$ and $b\bar{b}$ spectra.

(2) Power-law potentials can be of use in efforts to interpolate between the $c\bar{c}$ and $b\bar{b}$ systems. One would expect properties of the $b\bar{c}$ states, for example [11–13], to be given rather accurately by a power-law potential fitting $c\bar{c}$ and $b\bar{b}$ spectra and decay widths.

(3) We seek an estimate in phenomenological potentials of the mass difference between b and c quarks, which can be of use (for example) in attempts [14–16] to extract the Cabibbo-Kobayashi-Maskawa matrix element V_{cb} from data on semileptonic B decays.

(4) A recent reevaluation of the partial width for $\Upsilon(1^3D_J)\to\Upsilon\pi\pi$ [17] leads to greater optimism for the possible observation of D-wave $b\bar{b}$ states than in a previous analysis [18]. We recalculate the centers of gravity of $1\,^3D_J$ and $2\,^3D_J$ states in the best-fit power-law potentials.

(5) One would like to see if there is a consistent pattern of data signaling *departure* from a single effective power at short or long distances. There is indeed evidence, through leptonic widths, that the power α probed at shortest distances is more negative. Although this is a far cry from dominantly Coulomb behavior, it is at least a step in the right direction.

In reviewing the status of quarkonium some time ago, Martin [19] pointed out several shortcomings of power-law fits such as that in Ref. [7], including their underestimate of leptonic widths for $b\bar{b}$ states and their overestimate of the $2S$-$1P$ splitting in the $b\bar{b}$ system. It was also stressed [19,20] that power-law descriptions are not adequate for spin-dependent effects such as fine and hyperfine structure. Accordingly, we restrict our attention to spin-averaged levels.

We discuss the input data to our analysis in Sec. II. The scaling properties of energies and wave functions are noted in Sec. III. Fits to experimental masses and leptonic widths are performed in Sec. IV. Constraints on quark masses from electric dipole transition rates appear in Sec. V. We apply the results of our analysis to $b\bar{c}$ states in Sec. VI and to levels involving other quarks (s and t) in Sec. VII. Section VIII concludes.

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II. INPUT DATA

We choose to fit spin-averaged levels. Thus, for S waves, we define

$$M(S)=[M(^1S_0)+3M(^3S_1)]/4 \ , \tag{1}$$

a combination which eliminates the effects of hyperfine splitting. For P waves we take

$$M(P)=\overline{M}(^3P)\equiv[M(^3P_0)+3M(^3P_1)+5M(^3P_2)]/9 \ , \tag{2}$$

which eliminates the spin-orbit and tensor force splittings, and assume that hyperfine splittings are small. This assumption is supported by the recent observation of a candidate for the $1\,^1P_1$ state of charmonium [21] very close in mass to $\overline{M}(^3P)$. The hyperfine splitting in a $1/r$ potential is proportional to $|\Psi(0)|^2$, the square of the wave function at the origin, which is nonzero only for S waves. The small difference between $\overline{M}(^3P)=3525.27\pm0.10$ MeV (see Ref. [21]) and the observed value $M(^1P_1)=3526.2\pm0.15\pm0.2$ MeV thus is evidence for the Coulomb-like nature of the short-distance interquark force.

The masses of states were taken from Ref. [22]. In the absence of information about the hyperfine splittings between $^1S_0$ and $^3S_1$ $b\bar{b}$ levels, we estimated them from the lowest-order QCD expression

$$M(^3S_1)-M(^1S_0)=\frac{32\pi}{9m_b^2}\alpha_s(m_b^2)|\Psi(0)|^2 \ . \tag{3}$$

We took [23] $\alpha_s(m_b^2)=0.19$ and estimated $|\Psi(0)|^2$ from leptonic widths using the formula of Ref. [24], corrected by lowest-order QCD:

$$\Gamma(Q\bar{Q}\to e^+e^-)=\frac{16\pi e_Q^2\alpha^2}{M^2}|\Psi(0)|^2\left[1-\frac{16\alpha_s(m_Q)}{3\pi}\right] . \tag{4}$$

In this manner we found the $1\,^3S_1-1\,^1S_0$ $b\bar{b}$ splitting to be 45 MeV, and to decrease with increasing principal quantum number in accord with the decreasing values of $|\Psi(0)|^2$ as measured by leptonic widths. The actual values of input masses will appear in Sec. IV.

We take as input data the 1S, 2S, and 1P levels of charmonium and the 1S, 2S, 3S, 4S, 1P, and 2P levels of the $b\bar{b}$ system. All but the 4S $b\bar{b}$ level are below flavor threshold. We omit all charmonium levels and all other $b\bar{b}$ levels above flavor threshold from the fit, since their leptonic widths are not well determined and are strongly affected by coupling to open flavor channels.

III. SCALING PROPERTIES OF ENERGIES AND WAVE FUNCTIONS

In this section we briefly consider the dependence of various quantities on the quark mass m and the strength of the potential. We begin from the radial Schrödinger equation for a potential of the form $V(r)=\lambda(r^\alpha-1)/\alpha+C$:

$$-\frac{1}{m}\frac{d^2u(r)}{dr^2}+\frac{1}{m}\frac{l(l+1)}{r^2}u(r)+\lambda\left[\frac{r^\alpha-1}{\alpha}\right]u(r)+Cu(r)=Eu(r) \ , \tag{5}$$

where the term -1 and the factor of $1/\alpha$ are included to permit a smooth limit as $\alpha\to0$. We introduce the dimensionless variables $\rho=(m\lambda)^{1/(\alpha+2)}r$, $w(\rho)=(m\lambda)^{-1/[2(\alpha+2)]}u(r)$. We then find that the radial equation (5) becomes

$$-\frac{d^2w(\rho)}{d\rho^2}+\frac{l(l+1)}{\rho^2}w(\rho)+\left[\frac{\rho^\alpha-1}{\alpha}\right]w(\rho)=\epsilon w(\rho) \ , \tag{6}$$

where the dimensionless eigenvalue ϵ is related to the energy E by

$$E=\begin{cases}\lambda^{2/(\alpha+2)}m^{-\alpha/(\alpha+2)}(\epsilon+1/\alpha)+(C-\lambda/\alpha) & \text{if } \alpha\neq0 \ ,\\ \lambda(\epsilon-\ln\sqrt{m\lambda})+C & \text{if } \alpha=0 \ .\end{cases} \tag{7}$$

Particle masses M are related to the energy eigenvalue E by $M=2m+E$.

We also need the scaling laws for the radial dipole integrals and for the full wave functions. For the dipole integrals we find

$$\langle u_1|r|u_2\rangle=(m\lambda)^{-1/(\alpha+2)}\langle w_1|\rho|w_2\rangle \ . \tag{8}$$

Similarly, for the wave functions we have

$$\Psi(\mathbf{r})=\frac{u(r)}{r}Y_{lm}=(m\lambda)^{3[2(\alpha+2)]}\frac{w(\rho)}{\rho}Y_{lm} \ . \tag{9}$$

In parametrizing $O(v/c)^2$ corrections to the leptonic widths, we find it necessary to estimate $\langle v^2/c^2\rangle$ for the various states. This can be done by means of the virial theorem (see, e.g., Ref. [6]), which gives for the kinetic energy $T=\langle(r/2)dV/dr\rangle$ and hence $(c=1)$

$$\langle v^2\rangle=\frac{1}{2m}\left\langle r\frac{dV}{dr}\right\rangle=\frac{\lambda}{2m}+\frac{\alpha}{2m}\langle V\rangle \ . \tag{10}$$

Since the best fit to the data is typically given by a potential with $\alpha\simeq0$, we can use $\langle v^2\rangle\simeq\lambda/2m$ in most calculations. Here v is the relative velocity of the quark or antiquark with respect to the center of mass.

A few comments on the m and α dependence of these quantities are in order. We first note that, for $\alpha\simeq0$, the energies depend very weakly on the mass. As a result, we expect that a fit to the experimental masses will give only rather broad limits on the quark masses. Hence we must rely on information about the typical length scales of the system to pin down the mass. This information is provided by the leptonic widths and, to a lesser extent, the radial dipole integrals.

IV. FITS TO EXPERIMENTALLY OBSERVED MASSES AND LEPTONIC WIDTHS

In this section we present results from various fits to the experimental data. We first fit the energy levels of the J/ψ and Υ systems, and then add information on leptonic widths.

We form an estimate of the quality of fit using the variable

$$\chi^2 \equiv \sum_i \left[\frac{x_i^{\text{obs}} - x_i^{\text{pred}}}{\Delta x_i} \right]^2 , \tag{11}$$

where the sum runs over the observables x_i included in the fit, and Δx_i is the error. For masses, we have arbitrarily assigned a uniform error of 10 MeV, in anticipation of the typical error in the fit to masses alone. For leptonic widths, we use the experimental widths and errors quoted in Ref. [22].

Using information on masses alone, we find that the best fit is given by a potential of the form $V(r)=\lambda r^{-0.045}+\text{const}$. We have plotted χ^2 as a function of m_b in Fig. 1.

For positive values of α, the fit favors a value of m_b in the vicinity of 5 GeV. The best fit, however, is provided by negative values of α, and we find that in this case $m_c \to \infty$, $m_b \to \infty$, $m_b - m_c \to 3.19$ GeV. The comparison between the measured masses and those given by the fit in the limit $m_b \to \infty$, $m_c \to \infty$ is given in Table I. Here and elsewhere r^{-1} and λ are understood to be defined in terms of GeV units.

We obtain constraints on the quark masses by including the leptonic decay widths of the $\Upsilon(nS)$ and $\psi(nS)$ in the fit. We use expression (4), with [23]

$$\alpha_s(m_b)=0.189\pm0.008 , \tag{12}$$

$$\alpha_s(m_c)=0.29\pm0.02 . \tag{13}$$

In this case we find that the fit favors a broad range of quark masses around $m_b \simeq 5$ GeV. The parameters of the potential are

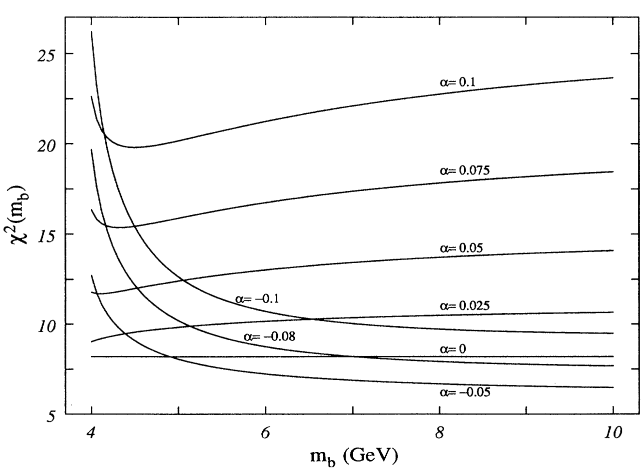

FIG. 1. Plot of χ^2 for the fit to J/ψ and Υ masses as a function of m_b.

TABLE I. J/ψ and Υ masses from the fit, not including leptonic widths.

Particle	Experiment	Fit
$J/\psi(1S)$	3.068	3.072
$J/\psi(2S)$	3.663	3.665
$J/\psi(1P)$	3.525	3.519
$\Upsilon(1S)$	9.449	9.444
$\Upsilon(2S)$	10.018	10.037
$\Upsilon(3S)$	10.351	10.355
$\Upsilon(4S)$	10.578	10.571
$\Upsilon(1P)$	9.900	9.891
$\Upsilon(2P)$	10.260	10.258
$\Upsilon(1D)$		10.163
$\Upsilon(2D)$		10.430

$$\alpha=-0.14 , \quad \lambda=0.808 , \quad C=-1.305 \text{ GeV} , \tag{14}$$

while for the quark masses the best fit, yielding a minimum χ^2 of 39.5, is obtained with

$$m_b=5.24 \text{ GeV} , \quad m_c=1.86 \text{ GeV} , \quad m_b-m_c=3.38 \text{ GeV} . \tag{15}$$

The comparison of experiment with theory is given in Table II. We find that the fit to the energies is reasonable, with the worst discrepancies of the order of 20 MeV. The fit to the leptonic widths is somewhat less satisfactory. The widths of charmonium states are overestimated, while those of $b\bar{b}$ states are underestimated.

In Fig. 2 we have plotted χ^2 as a function of m_b. We see that the quality of the fit depends on the b-quark mass, and favors a value in the vicinity of 5 GeV.

To improve the overall fit, we have examined a modification of Eq. (4) by introducing a phenomenological $O(v/c)^2$ correction, in the spirit of a similar approach [23] to gluonic partial widths of quarkonium states. We then have

$$\Gamma(Q\bar{Q}\to e^+e^-)=\frac{16\pi e_Q^2\alpha^2}{M^2}|\Psi(0)|^2\left[1-\frac{16\alpha_s(m_Q)}{3\pi}\right] \times[1+K(v/c)^2] . \tag{16}$$

The inclusion of this correction improves the fit only marginally. In this case the parameters of the potential are

$$\alpha=-0.12 , \quad \lambda=0.801 , \quad C=-0.772 \text{ GeV} , \tag{17}$$

and the quark masses are

$$m_b=4.96 \text{ GeV} , \quad m_c=1.56 \text{ GeV} , \quad m_b-m_c=3.40 \text{ GeV} . \tag{18}$$

For the constant appearing in Eq. (16), we find $K=1.25$. The results are shown in Table II and Fig. 3. The minimum χ^2 value in this case is 35.4. The main change with respect to the fit without the relativistic correction

TABLE II. J/ψ and Υ masses and leptonic widths.

	Mass (GeV)			Width (keV)		
Particle	(Expt)	(NR[a])	(RC[b])	(Expt)	(NR[a])	(RC[b])
$J/\psi(1S)$	3.068	3.077	3.079	5.36 ± 0.29	6.41 ± 0.43	6.29 ± 0.43
$J/\psi(2S)$	3.663	3.654	3.654	2.14 ± 0.21	2.03 ± 0.13	2.04 ± 0.13
$J/\psi(1P)$	3.525	3.524	3.522			
$\Upsilon(1S)$	9.449	9.420	9.423	1.34 ± 0.04	1.21 ± 0.02	1.18 ± 0.02
$\Upsilon(2S)$	10.018	10.044	10.042	0.56 ± 0.14	0.477 ± 0.009	0.475 ± 0.009
$\Upsilon(3S)$	10.351	10.358	10.358	0.44 ± 0.07	0.285 ± 0.006	0.284 ± 0.005
$\Upsilon(4S)$	10.578	10.564	10.567	0.24 ± 0.05	0.197 ± 0.004	0.200 ± 0.004
$\Upsilon(1P)$	9.900	9.903	9.900			
$\Upsilon(2P)$	10.260	10.269	10.267			
$\Upsilon(1D)$		10.181	10.177			
$\Upsilon(2D)$		10.436	10.435			

[a]Nonrelativistic corrections.
[b]With relativistic corrections.

appears to be a shift in the quark masses. Thus, we should not take quark masses obtained in either fit as particularly well determined, although the difference $m_b - m_c$ appears rather stable. In what follows, if we do not specify otherwise, we shall quote results obtained without relativistic corrections.

Predictions for centers of gravity of D-wave levels have also been presented in Tables I and II. In the fit to levels and leptonic widths, the centers of gravity of the $1D$ and $2D$ levels are predicted to be 10.18 and 10.43–10.44 GeV, respectively. A logarithmic potential gives 10.16 and 10.43 GeV for these levels, while several QCD-based potentials predict 10.16 and 10.44 GeV [18].

In Ref. [18] it was suggested that the only way to search for Υ (1D) levels was by means of cascades of electric dipole transitions, starting from the $3S$ state. Recently [17] a reevaluation has appeared of the $\Upsilon(1D)\to\Upsilon(1S)\pi\pi$ decay rates, which indicates that this process may be a useful one in which to search for the $1D$ state. While previous estimates [18] implied branching ratios

$$B[\Upsilon(1\,^3D_J)\to\Upsilon(1\,^3S_1)\pi\pi]\simeq 0.25\% ,$$

the results of Ref. [17] indicate branching ratios around 2%.

V. ELECTRIC DIPOLE TRANSITION RATES

In the preceding section we found that the leptonic widths of charmonium and bottomonium $^3S_1$ states can be used to place limits on the quark masses. In a similar way, the rates of electromagnetic transitions in the Υ system are sensitive to the mass of the b quark [27], since the overall sizes of $b\bar{b}$ systems (and hence the magnitudes of dipole matrix elements) depend on m_b. The rate of an electric dipole transition is given by

$$\Gamma=\tfrac{4}{3}e_Q^2\alpha E_\gamma^3 C_f\langle r\rangle^2 , \tag{19}$$

where $\langle r\rangle$ is the radial dipole matrix element, E_γ is the photon energy, and C_f is a statistical factor equal to $(2J_f+1)/9$ for $S\to P$ transitions and $\frac{1}{3}$ for $P\to S$ transitions. The experimental rates for $\Upsilon(3S)\to\chi_{bJ}(2P)$, $\Upsilon(2S)\to\chi_{bJ}(1P)$ for $J=0,1,2$ can be obtained directly from the full widths and branching ratios given in Ref. [22], and are listed in Table III. The dipole matrix ele-

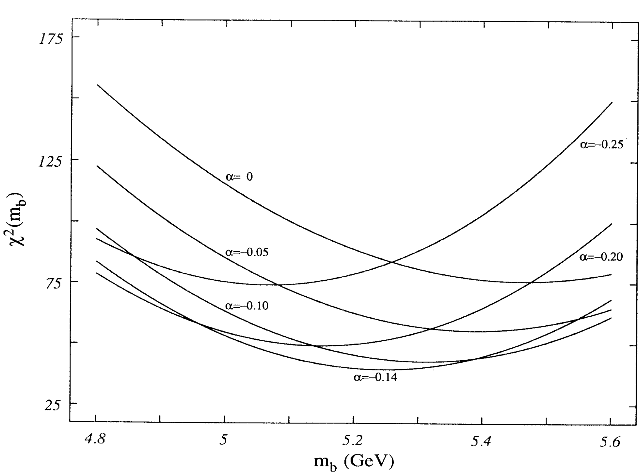

FIG. 2. Plot of χ^2 for the fit to J/ψ and Υ masses and leptonic widths, not including relativistic corrections.

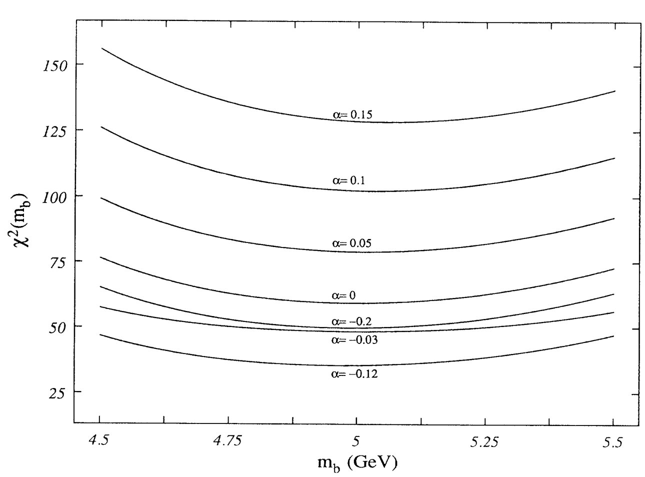

FIG. 3. Plot of χ^2 for the fit to J/ψ and Υ masses and leptonic widths, including relativistic corrections.

TABLE III. Measured partial widths (in keV) for $E1$ transitions between the $\Upsilon(nS)$ and $\chi_{bJ}(nP)$ states.

Transition	Angular momentum of final state $J=2$	$J=1$	$J=0$
$\Upsilon(2S)\rightarrow\chi_{bJ}(1P)$	2.84 ± 0.66	2.88 ± 0.66	1.85 ± 0.55
$\Upsilon(3S)\rightarrow\chi_{bJ}(2P)$	2.77 ± 0.38	2.75 ± 0.36	1.31 ± 0.21

ment for the $\Upsilon(3S)\rightarrow\chi_{bJ}(1P)$ transition can be extracted [25] from the information in Ref. [10], which quotes a product of branching ratios for $3S\rightarrow 1P$ transitions (summed over fine-structure multiplets), followed by transitions to $\Upsilon(1S)$. We interpret this as providing the combination

$$\tfrac{4}{3}e_Q^2\alpha\bar{E}_\gamma^3\langle r\rangle^2\{\tfrac{5}{9}B[\chi_{b2}\rightarrow\Upsilon(1S)\gamma]+\tfrac{1}{3}B[\chi_{b1}\rightarrow\Upsilon(1S)\gamma] + \tfrac{1}{9}B[\chi_{b0}\rightarrow\Upsilon(1S)\gamma]\} =(1.7\pm0.4\pm0.6)\times10^{-3}\Gamma_{\rm tot}[\Upsilon(3S)]\ , \quad (20)$$

where $\bar{E}_\gamma=445$ MeV is the average photon energy for the $\Upsilon(3S)\rightarrow\chi_{bJ}(1P)$ transitions. Using the branching ratios given in Ref. [22], we can obtain the dipole matrix element. Using these data, we find, after averaging over fine-structure multiplets, that the radial dipole integrals are

$$|\langle r\rangle_{3S-1P}|=0.043\pm0.010\ \text{GeV}^{-1}\ , \quad (21)$$

$$|\langle r\rangle_{3S-2P}|=2.66\pm0.10\ \text{GeV}^{-1}\ , \quad (22)$$

$$|\langle r\rangle_{2S-1P}|=1.88\pm0.13\ \text{GeV}^{-1}\ . \quad (23)$$

We now use the scaling of the matrix elements given in Eq. (8) and the potentials found in Sec. IV to estimate the mass of the b quark. Using the potential obtained by a fit to the energies and leptonic widths in Sec. IV ($\alpha=-0.14$ and $\lambda=0.808$), we find by means of a numerical calculation from the bound-state wave functions that the dimensionless matrix elements are

$$|\langle\rho\rangle_{3S-1P}|=0.071\ , \quad (24)$$

$$|\langle\rho\rangle_{3S-2P}|=5.83\ , \quad (25)$$

$$|\langle\rho\rangle_{2S-1P}|=3.26\ , \quad (26)$$

Using Eq. (8), we then use the ratios of the dimensionful and dimensionless matrix elements to estimate m_b:

$$m_b=\begin{cases}3.13\pm1.35 & \text{from } 3S\text{-}1P\ ,\\ 5.33\pm0.37 & \text{from } 3S\text{-}2P\ ,\\ 3.45\pm0.44 & \text{from } 2S\text{-}1P\ .\end{cases} \quad (27)$$

These values are not consistent with one another, but nonetheless they favor a value of m_b in the general range

$$m_b\simeq3.97\pm0.85\ \text{GeV}\ , \quad (28)$$

which is not too far from the expected value $m_b\simeq5$ GeV.

The low value of m_b implied by the $2S$-$1P$ transition is associated with an observed decay rate which is larger than the prediction of many potential models (see, e.g., Ref. [18]). Since the rate is obtained as the product of a branching ratio and the total width of the $\Upsilon(2S)$, it is conceivable that the experimental value of the latter, based essentially on one measurement [26], has been overestimated. However, it is more likely that the present comparison exposes another shortcoming of power-law fits, since the electric dipole transitions of $b\bar{b}$ states (favoring smaller values of m_b) are governed by larger distance scales than the leptonic widths (which favor larger values of m_b).

VI. $b\bar{c}$ LEVELS

We can use the potentials found in Sec. IV to estimate the masses of $b\bar{c}$ states. In doing so, we need only replace the mass m in Eq. (7) with 2μ, where $\mu=m_bm_c/(m_b+m_c)$ is the reduced mass. The results, for each of the three fits discussed in Sec. IV, are given in Table IV. The resulting masses are more or less consistent with one another, indicating that the potentials found in Sec. IV can, in fact, be used for purposes of interpolating between the $b\bar{b}$ and $c\bar{c}$ systems. The mass of the $b\bar{c}(1S)$ state has been obtained elsewhere [11] using a different method, and is consistent with the estimate given in Table IV.

VII. LEVELS WITH OTHER QUARKS

As a final application of the potentials found in Sec. IV, we estimate the masses of states involving lighter (strange) and heavier (top) quarks. The application to strange quarks, while questionable from the standpoint of a nonrelativistic model, was reasonably successful on the basis of a fit with a small positive power [7]. As for the top quark, we expect more realistic treatments based on a short-distance interquark force derived from QCD [28,29] to be superior to a power-law description, but it is of interest to see if the predictions of a power-law scheme and a QCD-based one can be distinguished from one another.

To estimate the masses of particles containing a strange quark, we use the method of Ref. [30] to extract a value for the strange quark mass from the measured mass of the ϕ. We compute the center of gravity of the $s\bar{s}(1S)$ state using Eq. (7), and add to this a hyperfine splitting which is estimated by rescaling the $J/\psi(1S)$-η_c splitting by an appropriate power of the strange quark mass. Setting this equal to the mass of ϕ gives $m_s\simeq0.604$ GeV.

TABLE IV. Predicted masses (in GeV) of $b\bar{c}$ states using different fits.

Particle	Fit to levels	Fit to levels and leptonic widths: No relativistic correction	With relativistic correction
$b\bar{c}(1S)$	6.319	6.304	6.318
$b\bar{c}(2S)$	6.905	6.898	6.908
$b\bar{c}(1P)$	6.760	6.764	6.772

Using the potential of Sec. IV, ($\alpha=-0.14$, $\lambda=0.808$), we find that the masses of the $c\bar{s}(1S)$, $(1P)$ states are

$$M_{c\bar{s}}(1S)=2.085 \text{ GeV} , \quad (29)$$

$$M_{c\bar{s}}(1P)=2.509 \text{ GeV} . \quad (30)$$

For the mass of the $b\bar{s}(1S)$ state, we find

$$M_{b\bar{s}}(1S)=5.401 \text{ GeV} . \quad (31)$$

The first and last values agree closely with previous estimates based on a potential with a small positive power [7,31], the changes being of the order of a few MeV. The prediction for the P-wave $c\bar{s}$ level is somewhat below the one for the earlier set of parameters in Ref. [7], which gave [33] a value of 2.532 GeV.

The prediction for $M_{c\bar{s}}(1S)$ is close to the experimental spin-averaged value [22] of 2.075 GeV, while there are several candidates [32] for the $c\bar{s}(1P)$ levels, whose average mass [22] is about 2.536 GeV. (It is not clear that the level reported in the first of Refs. [32] is the same as that in the other two.) The fit with a small negative power is slightly poorer, therefore, for extrapolating to light-quark systems than that [7] with $\alpha\simeq 0.1$. This might be as expected if the quark-confining force at large distances behaves [19,20] as a linear potential.

For toponium, the highest top-quark mass for which one may expect to be able to see the $1S$-$2S$ splitting is about $m_t=130$ GeV [28]. For this value, we find that the $1S$-$2S$ splitting is roughly 0.8 GeV. We anticipate that the observed splitting in toponium will be significantly larger than this, since the value of α in a more realistic $t\bar{t}$ potential should be more negative than that found in the lighter $b\bar{b}$ and $c\bar{c}$ systems. (The small Compton wavelength of the top quark allows it to probe much smaller distances than those to which the c and b quarks are sensitive.) As a result, the conversion from dimensionless eigenvalues to energies in the $t\bar{t}$ system will have a stronger mass dependence than that which occurs in the lighter systems, giving rise to larger splittings. A $1S$-$2S$ splitting of 0.8 GeV (for $m_t=130$ GeV) thus represents a very conservative lower bound based purely on the phenomenology of the $c\bar{c}$ and $b\bar{b}$ systems.

VIII. CONCLUSIONS

We have performed fits to charmonium and bottomonium spectra and leptonic widths using potentials of the power-law variety, $V(r)\sim r^{\alpha}+\text{const}$. The best fit is obtained with $\alpha=-0.14$, a slightly negative power. The corresponding quark masses are $m_c=1.86$ GeV, $m_b=5.24$ GeV. The mass difference between the b and c quarks in this fit is 3.38 GeV, very close to what was found some time ago [27] from a fit based on inverse scattering techniques.

When a relativistic correction, amounting to a rescaling of the connection between the wave function at the origin and the leptonic width by a phenomenological correction factor $1+K(v^2/c^2)$, is taken into account, the power in the potential changes only slightly, to $\alpha=-0.12$, but the quark masses shift appreciably, with $m_c=1.56$ GeV, $m_b=4.96$ GeV, $m_b-m_c=3.40$ GeV corresponding to the best fit. Thus, the exact values of quark masses cannot be regarded as well determined, though their difference is reasonably stable.

The pattern of deviations from the predictions of a power-law potential indicates that the leptonic widths of the $c\bar{c}$ and $b\bar{b}$ states are consistently lower and higher, respectively, than the predicted values. This is to be expected if the effective power α of the potential becomes more negative at shorter distances, as would happen when a short-distance Coulomb-like behavior is joined onto linear behavior at large interquark separations.

The centers of gravity of the $\Upsilon(1D)$ and $\Upsilon(2D)$ states are found to lie near 10.18 and 10.43–10.44 GeV. It may be worth searching for these levels not only in electromagnetic cascades [18], but also in $\Upsilon\pi\pi$ final states.

The magnitudes of the matrix elements for the electric dipole transitions from $3S$ to $1P$ and $2P$ levels in $b\bar{b}$ systems are compatible with the above parameters. The $2S$-$1P$ transition strength appears somewhat too strong for consistency with the other transitions. Slightly smaller values of m_b are favored by fits to the dipole matrix elements than by fits to spectra and leptonic widths.

The spin-averaged masses of some low-lying $b\bar{c}$ states have been estimated. For the $1S$ level, we find a value around 6.32 GeV from our fit to levels and leptonic widths when a relativistic correction which rescales the connection between $c\bar{c}$ and $b\bar{b}$ levels is included. A similar value is found when fitting levels alone. A slightly lower value (by about 15 MeV) is found when fitting levels and leptonic widths, but omitting the relativistic correction. Reviews of other predictions for this level may be found in Refs. [11,13].

Estimates of the masses of various $b\bar{s}$, $c\bar{s}$ states have been given. We find $M_{c\bar{s}}(1S)=2.085$ GeV, $M_{c\bar{s}}(1P)=2.509$ GeV, and $M_{b\bar{s}}(1S)=5.401$ GeV for the centers of gravity of these states, in rough agreement with previous potential model estimates [7,31].

A prediction for toponium levels has been presented for the sake of contrast with more realistic potentials. For $m_t=130$ GeV, the $2S$-$1S$ splitting is predicted in a power-law potential with $\alpha=-0.14$ to be about 0.8 GeV, while a QCD-based potential [28] gives a $2S$-$1S$ spacing in excess of 1 GeV. A logarithmic potential would have given a $2S$-$1S$ spacing slightly below 0.6 GeV. For a top-quark mass of 130 GeV, the large width of the top quark (nearly $\frac{1}{2}$ GeV) broadens the levels to the point that their spacing is nearly imperceptible, but detailed analysis of the threshold behavior of $t\bar{t}$ production [28,29] still can provide some information on properties of the levels.

Despite our prejudices in favor of a QCD-based description of the interquark force, fits to quarkonium spectra based on power-law potentials are still largely adequate. Exceptions appear to be a noticeable trend in leptonic widths when comparing charmonium and bottomonium levels, a slight discrepancy in the predicted

$2S \rightarrow 1P$ electric dipole rate for the $b\bar{b}$ system, and the failure to reproduce spin-dependent effects [19,20]. If the top quark is light enough, the spectroscopy of $t\bar{t}$ levels may provide some additional distinction between potential which are merely phenomenological, and those which have some fundamental basis.

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