BRIEF REPORTS

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Stability of two-dimensional stringy black hole

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We explicitly show that the two-dimensional stringy black hole proposed by Witten is stable under small nonstatic perturbations.

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I. INTRODUCTION

Recently, much attention has been paid to stringy black hole solutions which are obtained from low-energy string effective actions [1–5]. Witten showed that the SL$(2,R)$/U(1) gauged Wess-Zumino-Witten model describes strings in a two-dimensional black hole [1]. His work was generalized to the case of charged black holes, which are obtained by adding a boson to the model and coupling it to the world-sheet gauge field [6]. These stringy black hole solutions would be useful as a toy model for a four-dimensional real black hole [7]. Higher-dimensional analogues of Witten's two-dimensional stringy black holes were also found [8,9].

In all stringy black holes, the dilaton field plays a crucial role. As an illustrative example, the four-dimensional charged black hole solution given in Ref. [10] is unstable due to the dilaton field, although it has a similar form to that of the Reissner-Nordström solution in general relativity [11].

In this Brief Report, we explicitly show that Witten's stringy black hole in two-dimensional target space is stable [12] under small nonstatic perturbations by use of the Chandrasekhar method [13,14]. In Sec. II, we briefly review Witten's two-dimensional stringy black hole. In Sec. III, we show the stability of this solution.

II. TWO-DIMENSIONAL STRINGY BLACK HOLE

After gauging the Wess-Zumino coset model on SL$(2,R)$/U(1), Witten obtained the conformal-invariant representation of the classical action for ρ and θ as

$$S(\rho,\theta)=\frac{\kappa}{4\pi}\int d^2x\sqrt{\gamma}\gamma^{ij}(\partial_i\rho\partial_j\rho+\tanh^2\rho\partial_i\theta\partial_j\theta)-\frac{1}{8\pi}\int d^2x\sqrt{\gamma}\Phi(\rho,\theta)R^{(2)}\ , \tag{1}$$

where $R^{(2)}$ is the curvature on the world sheet, and Φ is the dilaton field. This target space dilaton field is obtained as a finite correction coming from the measure in the integration over an Abelian gauge field which is introduced to the gauge U(1) subgroup [1]. Note that the particular form of the dilaton is not described until later, where it is obtained from the low-energy equation of motion. The central charge [15] of the SL$(2,R)$/U(1) model is given by

$$c=2+\frac{6}{\kappa-2}\ . \tag{2}$$

Defining $r=\rho/\varepsilon$ ($\varepsilon\equiv\sqrt{2/\kappa}$) and absorbing the overall factor $\kappa/2$ in the coordinate ρ, the action can be written as

$$S(r,\theta)=\frac{1}{2\pi}\int d^2x\sqrt{\gamma}\gamma^{ij}\left[\partial_i r\partial_j r+\frac{\tanh^2\varepsilon r}{\varepsilon^2}\partial_i\theta\partial_j\theta\right]-\frac{1}{8\pi}\int d^2x\sqrt{\gamma}\Phi(r,\theta)R^{(2)}\ . \tag{3}$$

This action describes the conformal field theory on the background metric described by the line element

$$ds^2=dr^2+\frac{\tanh^2\varepsilon r}{\varepsilon^2}d\theta^2\ . \tag{4}$$

This background defines a semi-infinite cigar with radius

$$R_0(r)=\frac{\tanh\varepsilon r}{\varepsilon}\ . \tag{5}$$

Witten showed that this target space can be interpreted as a Euclidean black hole asymptotically [1,5]. From the requirement that the one-loop β function, which is obtained from the action in Eq. (3), must vanish, we may get metric equations

$$R_{\mu\nu}-\nabla_\mu\nabla_\nu\Phi=0\ , \tag{6}$$

where $R_{\mu\nu}$ is the Ricci tensor of the target space [16]. These one-loop equations are just the equation of motion obtained from the low-energy string effective action, which is given as

$$L = \int d^2x \sqrt{g}\, e^{\Phi} \left[R + g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + \frac{8}{\kappa - 2} \right] . \tag{7}$$

On the other hand, we get the dilaton equation of motion from action (7):

$$R - (\nabla\Phi)^2 - 2\nabla^2\Phi + \frac{8}{\kappa-2} = 0 . \tag{8}$$

Using Eq. (6) we get

$$(\nabla\Phi)^2 + \nabla^2\Phi - \frac{8}{\kappa-2} = 0 . \tag{9}$$

By solving Eq. (6), we may get the function form of the background dilaton field:

$$\overline{\Phi}(r) = \ln\cosh^2\varepsilon r + a , \tag{10}$$

where a is a constant, and is related to the mass of black hole. On the other hand, from the space-time effective action given in Eq. (7), one can find the mass of a black hole [1,5]:

$$M = \sqrt{2/(\kappa-2)}\, e^a = \varepsilon e^a + O(\varepsilon^2) . \tag{11}$$

The analytic continuation of the black hole to a Lorentz signature is achieved by setting $\theta = it$:

$$ds^2 = -\frac{\tanh^2\varepsilon r}{\varepsilon^2} dt^2 + dr^2 . \tag{12}$$

A more familiar form of the black hole solution can be obtained by changing the coordinates r,t to Schwarzschild coordinates ϕ,τ in which the dilaton field is a linear function of ϕ. We choose

$$\overline{\Phi}(\phi) = \ln\cosh^2\varepsilon r + a = 2\varepsilon\phi - \ln\varepsilon , \tag{13}$$

$$t = \varepsilon\tau . \tag{14}$$

In these Schwarzschild coordinates, the line element is represented as

$$ds^2 = -(1 - Me^{-2\varepsilon\phi})d\tau^2 + (1 - Me^{-2\varepsilon\phi})^{-1} d\phi^2 , \tag{15}$$

$$M = \varepsilon e^a . \tag{16}$$

Here we used M as the mass of the black hole given only to leading order in ε. This metric describes a Schwarzschild-like black hole which has the event horizon at $\phi_{\mathrm{EH}} \equiv (1/2\varepsilon)\ln M$. In the next section we will show that this black hole solution is stable.

III. STABILITY OF TWO-DIMENSIONAL STRINGY BLACK HOLE

In order to explicitly show the stability of the black hole solution given in Eq. (13), we introduce small nonstatic perturbations $h_{\mu\nu}(\phi,\tau)$ and $n(\phi,\tau)$ around the background solutions $\overline{g}_{\mu\nu}$ and $\overline{\Phi}$, respectively:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} , \tag{17}$$

$$\Phi = \overline{\Phi} + n , \tag{18}$$

where the background dilaton field $\overline{\Phi}$ and the background metric $\overline{g}_{\mu\nu}$ are given in Eqs. (13) and (15). Since two-dimensional space-time is conformally flat, we take small metric perturbations of a form at the conformal gauge such as

$$h_{\mu\nu} = -h(\phi,\tau) \begin{bmatrix} -(1 - Me^{-2\varepsilon\phi}) & 0 \\ 0 & (1 - Me^{-2\varepsilon\phi})^{-1} \end{bmatrix} = -h(\phi,\tau)\overline{g}_{\mu\nu} . \tag{19}$$

We substitute Eqs. (17) and (18) into Eqs. (6) and (9) and then take terms up to first order of small perturbations. Then we get metric and dilaton equations determining small perturbations,

$$\delta R_{\mu\nu}(h) - \overline{\nabla}_\mu \overline{\nabla}_\nu n(\phi,\tau) + \delta\Gamma^\rho_{\mu\nu}(h)\overline{\nabla}_\rho \overline{\Phi} = 0 , \tag{20}$$

$$\overline{g}^{\mu\nu}\overline{\nabla}_\mu\overline{\nabla}_\nu n - \overline{g}^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}(h)\overline{\nabla}_\rho\overline{\Phi} - h^{\mu\nu}\overline{\nabla}_\mu\overline{\nabla}_\nu\overline{\Phi} + 2\overline{g}^{\mu\nu}\overline{\nabla}_\mu\overline{\Phi}\overline{\nabla}_\nu n - h^{\mu\nu}\overline{\nabla}_\mu\overline{\Phi}\overline{\nabla}_\nu\overline{\Phi} = 0 , \tag{21}$$

where the upper bars indicate background objects and $\delta R_{\mu\nu}$ and $\delta\Gamma^\rho_{\mu\nu}(h)$ are given by

$$\begin{aligned} \delta R_{\mu\nu}(h) &= -\tfrac{1}{2}\overline{\nabla}_\mu\overline{\nabla}_\nu h^\alpha{}_\alpha - \tfrac{1}{2}\overline{\nabla}^\alpha\overline{\nabla}_\alpha h_{\mu\nu} + \tfrac{1}{2}\overline{\nabla}^\alpha\overline{\nabla}_\nu h_{\mu\alpha} + \tfrac{1}{2}\overline{\nabla}^\alpha\overline{\nabla}_\mu h_{\nu\alpha} , \\ \delta\Gamma^\rho_{\mu\nu}(h) &= \tfrac{1}{2}\overline{g}^{\rho\alpha}(\overline{\nabla}_\nu h_{\mu\alpha} + \overline{\nabla}_\mu h_{\nu\alpha} - \overline{\nabla}_\alpha h_{\mu\nu}) . \end{aligned} \tag{22}$$

The nonvanishing components of the background Christoffel symbols are

$$\begin{aligned} \overline{\Gamma}^\phi_{\phi\phi} &= -M\varepsilon e^{-2\varepsilon\phi}(1 - Me^{-2\varepsilon\phi})^{-1} , \\ \overline{\Gamma}^\phi_{\tau\tau} &= M\varepsilon e^{-2\varepsilon\phi}(1 - Me^{-2\varepsilon\phi}) , \\ \overline{\Gamma}^\tau_{\phi\tau} &= M\varepsilon e^{-2\varepsilon\phi}(1 - Me^{-2\varepsilon\phi})^{-1} . \end{aligned} \tag{23}$$

To begin with, we obtain three metric and one dilaton equations for small perturbations from Eqs. (20)–(22) as follows:

$$\overline{\nabla}^2_\phi h_{\tau\tau} + \overline{\nabla}^2_\tau h_{\phi\phi} + 2\overline{g}_{\phi\phi}\overline{\nabla}^2_\tau n - 2\varepsilon\overline{g}_{\tau\tau}\partial_\phi h = 0 , \tag{24}$$

$$\overline{\nabla}^2_\phi h_{\tau\tau} + \overline{\nabla}^2_\tau h_{\phi\phi} + 2\overline{g}_{\tau\tau}\overline{\nabla}^2_\phi n + 2\varepsilon\overline{g}_{\tau\tau}\partial_\phi h = 0 , \tag{25}$$

$$\overline{\nabla}_\phi\overline{\nabla}_\tau n + \varepsilon\partial_\tau h = 0 , \tag{26}$$

$$\overline{g}^{\phi\phi}\overline{\nabla}^2_\phi n + \overline{g}^{\tau\tau}\overline{\nabla}^2_\tau n - 2\varepsilon\overline{g}^{\phi\phi}\delta\Gamma^\phi_{\phi\phi}(h) - 2\varepsilon\overline{g}^{\tau\tau}\delta\Gamma^\phi_{\tau\tau}(h) - h^{\phi\phi}\overline{\nabla}^2_\phi\overline{\Phi} - h^{\tau\tau}\overline{\nabla}^2_\tau\overline{\Phi} + 4\varepsilon\overline{g}^{\phi\phi}\overline{\nabla}_\phi n - 4\varepsilon^2 h^{\phi\phi} = 0 . \tag{27}$$

Second, we combine Eqs. (24) and (25). Then we get an equation for the combined mode:

$$[(1 - Me^{-2\varepsilon\phi})^2\partial^2_\phi + 2M\varepsilon e^{-2\varepsilon\phi}(1 - Me^{-2\varepsilon\phi})\partial_\phi - \partial^2_\tau] \times (h - n) = 0 . \tag{28}$$

This equation can be reduced to a simple form of equation, which is just a free field equation:

$$\partial^2_{\phi^*} H - \partial^2_\tau H = 0 . \tag{29}$$

Here H is a newly defined field and ϕ^* is a transformed coordinate such that

$$H \equiv h - n , \tag{30}$$

$$\phi^* \equiv \phi + \frac{1}{2\varepsilon}\ln(1 - Me^{-2\varepsilon\phi}) . \tag{31}$$

Note that ϕ^* ranges from $-\infty$ to $+\infty$, while ϕ ranges from the event horizon of the black hole $\phi_{\rm EH}$ to $+\infty$. Now, we take a trial solution such as

$$H(\phi^*,\tau)=K(\phi^*)e^{-ik\tau}\ , \tag{32}$$

and substitute this trial solution into the free field equation in Eq. (29). Then we get an equation such as the time-independent Schrödinger one:

$$\frac{d^2}{d\phi^{*2}}K(\phi^*)+k^2K(\phi^*)=0\ . \tag{33}$$

Its solution is

$$K(\phi^*)=A\ \exp(\pm ik\phi^*)\ . \tag{34}$$

In order to know whether or not there is the exponentially growing mode, we take $k=i\alpha$ (α is positive and real). Since we require that the perturbation falls off to zero for large ϕ, we choose

$$K(\phi^*)=A\ \exp(-\alpha\phi^*)\ . \tag{35}$$

Now in order to cover the event horizon, we are going to transform the solution into a singularity-free region by using the Kruskal transformations which are given by

$$\begin{aligned} u&=\tfrac{1}{2}e^{\varepsilon(\phi^*-\tau)}\ ,\\ v&=-\tfrac{1}{2}e^{\varepsilon(\phi^*+\tau)}\ ,\\ ds^2&=-\frac{1}{\varepsilon^2}\frac{1}{uv-M/4}du\,dv\ . \end{aligned} \tag{36}$$

This metric form of the solution is similar to that of a black hole in Kruskal-Szekeres coordinates [1,17]. The curvature is given by $R=4/\varepsilon^2(uv-M/4)$ which shows that the curvature singularity occurs at $uv=M/4$. The event horizon is given by the lines $uv=0$. Therefore there is not a singularity at the event horizon in the (u,v) coordinate system. Then the perturbation $H(\phi^*,\tau)=Ae^{-\alpha\phi^*}e^{\alpha\tau}$ is given by

$$H(u,v)=2^{-\alpha/\varepsilon}Au^{-\alpha/\varepsilon}\ . \tag{37}$$

Note that α and ε are positive and real. In deciding whether or not the black hole is stable, we start with a perturbation which is regular everywhere in space at the initial time $\tau=0$, and then see whether such a perturbation will grow with time. At $\tau=0$, by choosing a small u, this perturbation can be made as large as we wish. In other words, this perturbation diverges as $u\to 0$, whereas the background solution remains finite. This contradicts the assumption that the perturbation is small compared to the background value. As a result, the perturbation with $k=i\alpha$ is unacceptable and thus cannot exist.

For the time being we want to know the role of dilaton perturbation equation. The dilaton equation (27) combined with Eq. (26) gives the decoupled equation for n as

$$(1-Me^{-2\varepsilon\phi})^2\partial_\phi^2 n-2M\varepsilon e^{-2\varepsilon\phi}(1-Me^{-2\varepsilon\phi})\partial_\phi n+4M\varepsilon^2e^{-2\varepsilon\phi}n-\partial_\tau^2 n=0\ . \tag{38}$$

On the other hand, it is also possible to get the exact same equation by substituting Eq. (26) for the equation which is obtained by subtracting Eq. (25) from Eq. (24). Therefore, Eq. (27) does not give additional information. As de Alwis and Lykken pointed out, this is because the metric and dilaton β functions obey a Bianchi type identity which tells us that if the gravitation equations are satisfied, then the dilaton equation is automatically satisfied up to a constant [4].

Now to complete the analysis we study the other mode $J\equiv h+n$. Combining Eqs. (28), (38), and (26) we get

$$\frac{d^2}{d\phi^{*2}}N(\phi^*)+(k^2-V)N(\phi^*)=0\ , \tag{39}$$

where

$$J(\phi^*,\tau)\equiv h+n=N(\phi^*)e^{-ik\tau}\ , \tag{40}$$

$$V=8M\varepsilon^2e^{-2\varepsilon\phi}(1-Me^{-2\varepsilon\phi})\ . \tag{41}$$

The asymptotic forms of this equation, and the corresponding solutions, are found to be

$$\begin{aligned} &\frac{d^2}{d\phi^{*2}}N_\infty+k^2N_\infty=0,\quad N_\infty\sim\exp(\pm ik\phi)\ \text{ for }\phi\to\infty\ ,\\ &\frac{d^2}{d\phi^{*2}}N_{\rm EH}+k^2N_{\rm EH}=0\ ,\\ &N_{\rm EH}\sim\exp(\pm ik\phi^*)\ \text{ for }\phi\to\phi_{\rm EH}\ . \end{aligned} \tag{42}$$

As in the previous case, set $k=i\alpha$ (α is positive and real). Then the equation and asymptotic solutions will be

$$\frac{d^2}{d\phi^{*2}}N=(\alpha^2+V)N\ , \tag{43}$$

$$N_\infty\sim\exp(\pm\alpha\phi),\quad N_{\rm EH}\sim\exp(\pm\alpha\phi^*)\ . \tag{44}$$

To ensure that the perturbation falls off to zero for large ϕ, we choose $N_\infty\sim\exp(-\alpha\phi)$. In the case of $N_{\rm EH}$, the solution $\exp(\alpha\phi^*)$ goes to zero as $\phi^*\to-\infty$. Then $N_{\rm EH}\sim\exp(\alpha\phi^*)$ cannot be matched to $N_\infty\sim\exp(-\alpha\phi)$ because assuming N to be positive, Eq. (43) shows that $d^2N/d\phi^{*2}$ never becomes negative within the range of ϕ from $\phi_{\rm EH}$ to ∞. Therefore, we conclude that the solution going to zero for large values of ϕ has the asymptotic behavior $N_{\rm EH}\sim\exp(-\alpha\phi^*)$ near the event horizon. This asymptotic solution has exactly the same form as Eq. (35) and gives rise to a divergent perturbation in the Kruskal coordinates as $u\to 0$ at initial time.

Thus we have shown that perturbations with purely imaginary frequencies are physically unacceptable since they are divergent even at the initial moment. This means that the metric and dilaton perturbations should oscillate with time. As a result, a two-dimensional stringy black hole is stable under small perturbations of metric and dilaton fields.

IV. CONCLUSION

We have analyzed the stability of the two-dimensional stringy black hole. We have given the linearized perturbation equations governing the black hole perturbation. Then, we have obtained the solutions for small perturbations $h-n$ and $h+n$, and transformed these solutions to a singularity-free coordinate system in order to cover the event horizon. As a result, we have shown that the regular perturbation solution only exists in the case where k is real. Therefore the perturbation must oscillate with time; i.e., the two-dimensional stringy black hole is stable under small nonstatic perturbation, and thus can be regarded as the final state of gravitational collapse of two-dimensional matter.

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