Nontrivial topology and the chiral Schwinger model

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We analyze the chiral Schwinger model in nontrivial topological sectors, performing its complete bosonization. In order to do this, we propose a prescription for evaluating the fermion determinant in the presence of the zero modes, valid for non-Hermitian Dirac operators, in general. By taking fermionic external sources into account in every step of the calculation, we discover a phase ambiguity which affects the effective action and can be used to render the result invariant with respect to particular choices of the topologically charged background configuration. Consistency requirements on the bosonization procedure fix the phase ambiguity and determine a unique value for the Jackiw-Rajaraman regularization parameter in all sectors with a nonzero topological charge. We thus find that nontrivial sectors have a null contribution to all fermionic correlation functions. Our method is also checked against the analogous results for the Schwinger model.

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I. INTRODUCTION

Topologically charged gauge fields have been considered of physical relevance since the 1970s, in the pioneer works of Rothe and Schroer [1], Crewther [2], Nielsen and Schroer [3], Rothe and Swieca [4], Hortaçsu, Rothe, and Schroer [5], and others [6]. Recently, they arose again in quite different contexts, such as string compactification [7], consistency of two-dimensional SU(2) Weyl fermions [8], two-dimensional gauge theories [9–11], high-T_c superconductivity [12], and QCD strings [13]. Some very interesting phenomena have appeared in these investigations, for instance, correlation functions which would be null in topologically trivial sectors [7,9] or the elimination of instanton contributions by a dynamically generated Chern-Simons term in $2+1$ dimensions [12].

In previous articles [14,15], it has been noted that, in topologically nontrivial sectors, the external fermionic sources play a very important role: they regularize the zero-mode dependence of the generating functional, providing the natural appearance of $\det'D$ (the product of nonzero eigenvalues of the covariant Dirac operator D) instead of $\det D$ (the true fermionic determinant, which is of course zero, due to the inclusion of the zero modes). This fact led us to a different definition of the Jacobian for chiral rotations of the fermionic variables in the path integral, namely,

$$J[\alpha]=\frac{\det'D}{\det'D_\alpha}\mathcal{N}[\alpha]^{-1}\ , \tag{1}$$

with $D_\alpha=e^{\alpha\gamma_5}De^{\alpha\gamma_5}$. The functional $\mathcal{N}[\alpha]$ is a contribution of the fermionic sources to the Jacobian, that exactly cancels the explicit zero-mode dependence of the ratio between $\det'D$ and $\det'D_\alpha$. When D is a normal operator (that is, $DD^\dagger=D^\dagger D$) or a Hermitian one, one can compute $\det'D$ using the ζ-function regularization, as is done, for example, in Refs. [16,17]. However, problems concerning the stability of the null subspace of D preclude the application of this method to non-normal operators, and that is precisely the case of a model of great interest: the chiral Schwinger model (CSM) [18].

Therefore, it is the purpose of this paper to compute the contribution of all topologically nontrivial sectors to the CSM, taking into account the full dependence of the generating functional on the sources. We will base our discussion on the method of bosonization in the presence of nontrivial field configurations as developed by Bardakci and Crescimanno [7] and Manias, Naón, and Trobo [9] for the case of the Schwinger model: we assume that a field configuration with topological charge N can always be decomposed in the following way:

$$A_\mu=A_\mu^{(N)}+a_\mu\ , \tag{2}$$

where $A_\mu^{(N)}$ is a fixed configuration of charge N and a_μ carries zero topological charge, so that, within the N-charge sector, a field change amounts to a change in a_μ

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only. This implies that the functional measure $\mathcal{D}A_\mu$ is restricted to $\mathcal{D}a_\mu$, where the path integral is now to be computed under trivial boundary conditions, which allows us to bosonize the theory completely. In the course of our analysis, we are confronted with the fact that the effective action has an ambiguity due to the process of orthonormalization of the zero modes after chiral rotations. This ambiguity can be fixed with the aid of two criteria: (1) invariance of the generating functional with respect to particular choices of $A_\mu^{(N)}$ and (2) the requirement that the theory is to be bosonized without leaving the topological sector of charge N. As a consequence, the Jackiw-Rajaraman regularization parameter $a_R(N)$ (in principle different for each sector) is shown to be equal to -1 for all $N\neq 0$, only $a_R(0)$ remaining arbitrary. Finally, the computation of arbitrary fermionic correlation functions gives a null contribution from the nontrivial sectors, thus showing that the model can be completely solved by considering only topologically trivial gauge fields.

Our analysis demanded a new definition of $\det' D$, a generalization of one previously proposed for $\det D$ [19]. It reduces quite obviously to the natural one for Hermitian singular operators. We motivate it by working out in some detail the Hermitian case of the Schwinger model.

The article is organized as follows. In Sec. II we consider the definition of $\det' D$ for the Hermitian case and apply it to the Schwinger model. In Sec. III we give an explicit description of the zero modes and compute the Jacobian of chiral rotations for the CSM. Section IV contains the application of our definition of $\det' D$ to the non-normal case, where we compute the full generating functional with the solution of the phase ambiguity. The correlation functions of fermionic fields and currents are the subject of Sec. V and in Sec. VI we present our conclusions.

II. FERMION DETERMINANT AND GENERATING FUNCTIONAL IN THE SCHWINGER MODEL

Before we concentrate on our main goal, let us first consider the Hermitian case of the Schwinger model, defined (in two-dimensional Euclidean spacetime) by the Lagrangian density [20]

$$\mathcal{L}=\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\bar\psi D\psi \ , \tag{3}$$

where $D=i\partial\!\!\!/+eA\!\!\!/$. The gauge field A_μ is assumed to satisfy a vortex quantization condition [21–23]:

$$-\frac{e_R}{2\pi}\oint_\Sigma A_\mu dx_\mu = N \ , \tag{4}$$

where N is an integer and Σ is a closed loop surrounding the vortex. We can decompose A_μ as

$$A_\mu = A_\mu^{(N)} + a_\mu \ , \tag{5}$$

with $A_\mu^{(N)}$ satisfying (4) and a_μ such that

$$-\frac{e_R}{2\pi}\oint_\Sigma a_\mu dx_\mu = 0 \ . \tag{6}$$

We say that the topological charge of $A_\mu^{(N)}$ is N and that of a_μ is zero. We choose $A_\mu^{(N)}$ to be of the form

$$e_R A_\mu^{(N)}(x) = -\tilde\partial_\mu f(x) \ , \tag{7}$$

with the scalar field f satisfying the boundary condition

$$f(x) \underset{|x|\to\infty}{\sim} -N\ln|x| \ . \tag{8}$$

We can rewrite a_μ in terms of scalar fields as well:

$$e_R a_\mu = \partial_\mu\rho - \tilde\partial_\mu\phi \ , \tag{9}$$

where ρ and ϕ satisfy trivial boundary conditions such as

$$\rho(x) \underset{|x|\to\infty}{\sim} \frac{1}{|x|^\gamma} \ , \tag{10}$$

with γ appropriately chosen so that we can perform by parts integrations involving $f(x)$.

It is well known that D is a singular operator with $|N|$ zero modes [1]. We can compute the product of its nonzero eigenvalues using the prescription [24]

$$\det' D = \lim_{\epsilon\to 0^+}\frac{\det(D+\epsilon 1)}{\epsilon^{|N|}} \ . \tag{11}$$

This gives formally the determinant of $D+P_0$, with P_0 being the projector over $\ker D$. Considering a family of operators $\{D_\alpha\}$ given by

$$D_\alpha = i\partial\!\!\!/ + eA\!\!\!/^{(N)} + \alpha a\!\!\!/ \ , \quad 0\le\alpha\le 1 \ , \tag{12}$$

with the eigenvalue equation

$$D_\alpha\varphi_n^\alpha = \lambda_n^\alpha\varphi_n^\alpha \ , \tag{13}$$

we can use (11) and $\det=\exp\operatorname{Tr}\ln$ to establish a differential equation obeyed by D_α, as in Ref. [19],

$$\frac{d}{d\alpha}\det' D_\alpha = \lim_{\epsilon\to 0^+}\frac{\det(D_\alpha+\epsilon 1)}{\epsilon^{|N|}}\operatorname{Tr}\left[(D_\alpha+\epsilon 1)^{-1}\frac{dD_\alpha}{d\alpha}\right] \ , \tag{14}$$

with $A_\mu^{(N)}$ and a_μ given by (7) and (9). The inverse operator $(D_\alpha+\epsilon 1)^{-1}$ exists and can be written in terms of the eigenfunctions of D_α,

$$(D_\alpha+\epsilon 1)^{-1} = \sum_{\lambda_n^\alpha\neq 0}\frac{\varphi_n^\alpha(x)\varphi_n^{\alpha\dagger}(y)}{\lambda_n^\alpha+\epsilon} + \frac{1}{\epsilon}\sum_{i=1}^{|N|}\varphi_{0_i}^\alpha(x)\varphi_{0_i}^{\alpha\dagger}(y)$$
$$\equiv S_\epsilon^\alpha(x,y) + \frac{1}{\epsilon}P_0^\alpha(x,y) \ , \tag{15}$$

so that

$$\frac{d}{d\alpha}\det' D_\alpha = \det' D_\alpha \operatorname{Tr}\left[S^\alpha\frac{dD_\alpha}{d\alpha}\right]$$
$$+\lim_{\epsilon\to 0}\frac{\det(D_\alpha+\epsilon 1)}{\epsilon^{|N|+1}}\operatorname{Tr}\left[P_0^\alpha\frac{dD_\alpha}{d\alpha}\right] \ . \tag{16}$$

There are several ways to see that the second term on the right-hand side (RHS) vanishes. For instance, we can note that $dD_\alpha/d\alpha = a\!\!\!/$ transforms left-handed spinors into right-handed ones and vice versa; thus,

$$\mathrm{Tr}\left[P_0^\alpha \frac{dD_\alpha}{d\alpha}\right] = \sum_i \int d^2x\, \varphi_{0_i}^{\alpha\dagger}(x) \not{a}(x) \varphi_{0_i}^\alpha(x) \tag{17}$$

is in fact the scalar product between a left- and a right-handed spinor (remember that zero modes in the Schwinger model have a definite chirality [1,7,9]). We have then

$$\frac{d}{d\alpha} \ln \det{}' D_\alpha = \mathrm{Tr}\left[S^\alpha \frac{dD_\alpha}{d\alpha}\right], \tag{18}$$

with

$$S^\alpha(x,y) = \sum_{\lambda_n^\alpha \neq 0} \frac{\varphi_n^\alpha(x)\varphi_n^{\alpha\dagger}(y)}{\lambda_n^\alpha}, \tag{19}$$

satisfying

$$D_\alpha S^\alpha(x,y) = \delta(x-y) - P_0^\alpha(x,y) = S^\alpha(x,y) D_\alpha . \tag{20}$$

Although D_α is singular it has a Green's function that can be computed exactly in two dimensions. It does not have a spectral decomposition in terms of the eigenfunctions of D_α and thus cannot be viewed as a distribution over R_c^2 (the compactification of R^2, which must have been made in order to define a discrete set of eigenvalues for D_α) [1]. However, it can be used to construct a function that satisfies (20):

$$S^\alpha(x,y) = G^\alpha(x,y) - \int d^2z\, G^\alpha(x,z) P_0^\alpha(z,y) - \int d^2z\, P_0^\alpha(x,z) G^\alpha(z,y), \tag{21}$$

where

$$D_\alpha G^\alpha(x,y) = \delta(x,y) . \tag{22}$$

Equation (18) is then rewritten as

$$\frac{d}{d\alpha} \ln \det{}' D_\alpha = \mathrm{Tr}\left[G^\alpha \frac{dD_\alpha}{d\alpha}\right] - \mathrm{Tr}\left[G^\alpha P_0^\alpha \frac{dD_\alpha}{d\alpha}\right] - \mathrm{Tr}\left[P_0^\alpha G^\alpha \frac{dD_\alpha}{d\alpha}\right]. \tag{23}$$

The ultraviolet singularities in this expression are all contained in the first term of the RHS. We can regularize them by means of point splitting [25,19] to obtain

$$\begin{aligned}\mathrm{Tr}\left[G^\alpha \frac{dD_\alpha}{d\alpha}\right] &= -\frac{\alpha e^2}{2\pi} \int d^2x \left[a_\mu \left[a_+(N)\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu - \tilde{\partial}_\mu\tilde{\partial}_\nu}{\Box}\right] a_\nu\right] \\ &\quad - \frac{e^2}{2\pi} \int d^2x \left[A_\mu^{(N)} \left[a_+(N)\delta_{\mu\nu} - i a_-(N)\varepsilon_{\mu\nu} - \frac{\tilde{\partial}_\mu\partial_\nu - \partial_\mu\tilde{\partial}_\nu}{\Box}\right] a_\nu\right] \\ &\equiv -2\alpha\Gamma[a_\mu] - \bar{\Gamma}[A_\mu^{(N)}, a_\mu] .\end{aligned} \tag{24}$$

The parameters $a_+(N)$ and $a_-(N)$ in (24) are reminiscent of the regularization freedom of the theory. The values that preserve gauge invariance at the quantum level are $a_+(N)=1$ and $a_-(N)=0$, for all N. The unconventional parameter a_- is a consequence of allowing different interactions between the gauge field and the left- and right-handed fermions (cf. Eq. (11) of Ref. [19]) and appears here only because we have "embedded" the Schwinger model in the generalized Schwinger model [19]. We shall see later how to determine these parameters. In order to compute the other two terms in (23), let us remark that P_0^α can be written as

$$\begin{aligned}P_0^\alpha(x,y) &= \sum_{i=1}^{|N|} \varphi_{0_i}^\alpha(x)\varphi_{0_i}^{\alpha\dagger}(y) \\ &= \sum_{i=1}^{|N|} \varphi_{\alpha_i}^\pm(x)\varphi_{\alpha_i}^{\pm *}(y) \times \begin{cases} \begin{bmatrix}1\\0\end{bmatrix} \otimes (1\,0) \\ \begin{bmatrix}0\\1\end{bmatrix} \otimes (0\,1) \end{cases} \\ &= P_\pm \sum_{i=1}^{|N|} \varphi_{\alpha_i}^\pm(x)\varphi_{\alpha_i}^{\pm *}(y) ,\end{aligned} \tag{25}$$

with $\varphi_{0_i}^\alpha \equiv \varphi_{\alpha_i}^+ \begin{bmatrix}1\\0\end{bmatrix}$, for $N>0$, and $\varphi_{0_i}^\alpha \equiv \varphi_{\alpha_i}^- \begin{bmatrix}0\\1\end{bmatrix}$, for $N<0$, being the orthonormal set of zero modes of D_α. Then we write

$$\mathrm{Tr}\left[G^\alpha P_0^\alpha \frac{dD_\alpha}{d\alpha}\right] = e \int d^2x\, d^2y\, \mathrm{tr}[G^\alpha(x,y) P_\pm \gamma_\mu] \times a_\mu(x) \sum_i \varphi_{\alpha_i}^\pm(x)\varphi_{\alpha_i}^{\pm *}(y) . \tag{26}$$

Using (see [19])

$$G^\alpha(x,y) = (e^{h_+(x) - h_+(y)} P_+ + e^{-[h_-(x) - h_-(y)]} P_-) G_F(x-y) , \tag{27}$$

with

$$h_\pm(x) = f(x) + \alpha[\phi(x) \pm i\rho(x)] \tag{28}$$

and $i\not{\partial} G_F(x-y) = \delta(x-y)$, it can be shown, by making use of the equations of motion for the D_α zero modes,

$$[\partial_z - \partial_z(f + \alpha(\phi - i\rho))]\varphi_{\alpha_i}^{+*} = 0 , \tag{29}$$

$$[\partial_{\bar z}+\partial_{\bar z}(f+\alpha(\phi+i\rho))]\varphi^{-*}_{\alpha_i}=0\ , \tag{30}$$

where $\partial_z=\frac{1}{2}(\partial_1-i\partial_0)$ and $\partial_{\bar z}=\frac{1}{2}(\partial_1+i\partial_0)$, that

$$\mathrm{Tr}\left[G^\alpha P_0^\alpha \frac{dD_\alpha}{d\alpha}\right]=\sum_i \langle \varphi_{0_i}^{\alpha\dagger}(i\rho-\phi\gamma_5)\varphi_{0_i}^\alpha\rangle\ , \tag{31}$$

where angular brackets denote integration over spacetime variables. Because of the property

$$\mathrm{Tr}\left[G^\alpha P_0^\alpha \frac{dD_\alpha}{d\alpha}\right]=\left\{\mathrm{Tr}\left[P_0^\alpha G^\alpha \frac{dD_\alpha}{d\alpha}\right]\right\}^*\ , \tag{32}$$

we come, after integrating over α, to

$$\frac{\det' D}{\det' D^{(N)}}=e^{-\Gamma[a_\mu]-\bar\Gamma[A_\mu^{(N)},a_\mu]}[\det(\langle \varphi_{0_i}^\dagger e^{-2\phi\gamma_5}\varphi_{0_j}\rangle)]^{-1}\ . \tag{33}$$

The determinant of the zero modes in the above equation comes from the results of Ref. [15], from which we can also write the Jacobian of the transformations

$$\psi=e^{i\rho+\phi\gamma_5}\psi'\ , \tag{34}$$

$$\bar\psi=\bar\psi' e^{-i\rho+\phi\gamma_5}\ , \tag{35}$$

as

$$J[A_\mu^{(N)},a_\mu]=\frac{\det' D}{\det' D^{(N)}}\det(\langle \varphi_{0_i}^\dagger e^{-2\phi\gamma_5}\varphi_{0_j}\rangle)$$
$$=e^{-\Gamma[a_\mu]-\bar\Gamma[A_\mu^{(N)},a_\mu]}\ , \tag{36}$$

which checks with that obtained by Manias, Naón, and Trobo [9] for $a_+(N)=1, a_-(N)=0$. The determinant of zero modes, which appears in the first line of (36), is a consequence of working with the external sources present in every step of the computation.

The fact that the gauge configurations A_μ can be classified according to their topological charge enables us to write

$$Z=\sum_N Z_N\ , \tag{37}$$

such that, for each Z_N, the functional integration over A_μ is restricted over fields of charge N. After making the transformations (34) and (35), we have

$$Z[J_\mu,\eta,\bar\eta]=\sum_N\int \mathcal{D}a_\mu e^{-\bar S_{\mathrm{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle+\langle \bar\eta' S^{(N)}\eta'\rangle}\det' D^{(N)}\prod_{i=1}^{|N|}\langle \bar\eta'\varphi_{0_i}^{(N)}\rangle\langle \varphi_{0_i}^{(N)\dagger}\eta'\rangle\ , \tag{38}$$

with $\eta'=e^{-i\rho+\phi\gamma_5}\eta, \bar\eta'=\bar\eta e^{i\rho+\phi\gamma_5}$ and

$$\bar S_{\mathrm{eff}}[A_\mu^{(N)},a_\mu]=\tfrac{1}{4}\langle F_{\mu\nu}F_{\mu\nu}\rangle+\Gamma[a_\mu]+\bar\Gamma[A_\mu^{(N)},a_\mu]\ . \tag{39}$$

It is convenient to express the generating functional explicitly in terms of the original (nonorthonormal) set of zero modes of $D^{(N)}$ [7,9]:

$$\Phi_{0_i}^{(N)}=e^{f\gamma_5}\begin{cases} z^{i-1}\begin{bmatrix}1\\0\end{bmatrix}\ , & N>0\ ,\\ \bar z^{i-1}\begin{bmatrix}0\\1\end{bmatrix}\ , & N<0\ .\end{cases} \tag{40}$$

To do this, we introduce a rotation matrix between the two sets of functions,

$$\varphi_{0_i}^{(N)}=\sum_j b_{ij}\Phi_{0_j}^{(N)}\ , \tag{41}$$

which yields

$$\prod_{i=1}^{|N|}\langle \bar\eta'\varphi_{0_i}^{(N)}\rangle\langle\varphi_{0_i}^{(N)\dagger}\eta'\rangle=|\det b|^2\prod_{i=1}^{|N|}\langle\bar\eta'\Phi_{0_i}^{(N)}\rangle\langle\Phi_{0_i}^{(N)\dagger}\eta'\rangle\ . \tag{42}$$

Thus the condition $\langle \varphi_{0_i}^{(N)\dagger}\varphi_{0_j}^{(N)}\rangle=\delta_{ij}$ implies

$$|\det b|^2=[\det(\langle\Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)]^{-1}\ . \tag{43}$$

Consequently,

$$Z[J_\mu,\eta,\bar\eta]=\sum_N\int\mathcal{D}a_\mu e^{-\bar S_{\mathrm{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle+\langle\bar\eta' S^{(N)}\eta'\rangle}\det' D[\det(\langle\Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)]^{-1}\prod_{i=1}^{|N|}\langle\bar\eta'\Phi_{0_i}^{(N)}\rangle\langle\Phi_{0_i}^{(N)\dagger}\eta'\rangle\ . \tag{44}$$

With (44) we can check the invariance of the generating functional with respect to the particular choice of the field $A_\mu^{(N)}$. This can be expressed by the equation

$$\frac{\delta}{\delta f(x)} Z[J_\mu,\eta,\bar{\eta}]=0 \ . \tag{45}$$

We comment on two details of the computation. First, we find $[\delta/\delta f(x)]\det' D^{(N)}$ using (33):

$$\det' D^{(N)}[f+\delta f] \equiv \det' D[\delta f, A_\mu^{(N)}=-\tilde{\partial}_\mu f, a_\mu=-\tilde{\partial}_\mu \delta f] \\ =\det' D^{(N)} e^{-\Gamma[-\tilde{\partial}_\mu \delta f]-\bar{\Gamma}[-\tilde{\partial}_\mu f,-\tilde{\partial}_\mu \delta f]} \exp\left[\sum_i \int_0^1 d\alpha \langle \varphi_{0_i}^{\alpha\dagger} 2\delta f \gamma_5 \varphi_{0_i}^{\alpha}\rangle\right] , \tag{46}$$

where $\varphi_{0_i}^{\alpha}$ are the zero modes of $D^{(N)}[f+\alpha\delta f]$. We can express them in terms of the zero modes of $D^{(N)}$ and prove that

$$\sum_i \langle \varphi_{0_i}^{\alpha\dagger} 2\delta f \gamma_5 \varphi_{0_i}^{\alpha}\rangle = \frac{d}{d\alpha} \operatorname{tr} \ln(\langle \Phi_{0_i}^{(N)\dagger} e^{2\alpha\delta f \gamma_5} \Phi_{0_j}^{(N)}\rangle) \ . \tag{47}$$

In the limit $\delta f \to 0$, we obtain

$$\frac{\delta}{\delta f(x)} \det' D^{(N)} = \det' D^{(N)} \left[\frac{1}{2\pi}[a_+(N)+1]\Box f(x) + 2\operatorname{tr}[P_0^{(N)}(x,x)\gamma_5]\right] . \tag{48}$$

Second, we consider

$$\frac{\delta}{\delta f(x)} \det(\langle \Phi_{0_i}^{(N)\dagger} \Phi_{0_j}^{(N)}\rangle) \equiv \frac{\delta}{\delta f(x)} \det A = \frac{\delta}{\delta f(x)} \exp(\operatorname{tr} \ln A) = \det A \ \operatorname{tr}\left[A^{-1} \frac{\delta A}{\delta f(x)}\right] . \tag{49}$$

Thus, we have

$$\frac{\delta}{\delta f(x)} \det A = 2 \det A \ \operatorname{tr}[P_0^{(N)}(x,x)\gamma_5] , \tag{50}$$

as can be easily seen, by expressing $\Phi_{0_i}^{(N)}$ in terms of $\varphi_{0_i}^{(N)}$. After some calculation we are then able to show that

$$\frac{\delta}{\delta f(x)} Z[J_\mu,\eta,\bar{\eta}] = \sum_N \left\{ \int \mathcal{D}\rho\, \mathcal{D}\phi \frac{\delta}{\delta\phi} (e^{-\bar{S}_{\text{eff}} + \langle J_\mu A_\mu\rangle} Z_F^{(N)}[\eta,\bar{\eta}]) \right\} + \frac{ie}{2\pi} a_-(N) \int \mathcal{D}\rho\, \mathcal{D}\phi \Box\rho e^{-\bar{S}_{\text{eff}} + \langle J_\mu A_\mu\rangle} Z_F^{(N)}[\eta,\bar{\eta}] , \tag{51}$$

$Z_F^{(N)}[\eta,\bar{\eta}]$ being the fermion part of the generating functional. The first term on the RHS vanishes, as it is the functional integral of a functional derivative (in fact, it represents the quantum equations of motion for ϕ). Then, the only way to cancel the second term and obtain invariance is to set

$$a_-(N)=0 \quad \text{for all } N\neq 0 \ . \tag{52}$$

This is to be seen as a consistency requirement on the theory, at a mathematical level. Simultaneously, the condition given in (52) is part of the requirement of gauge invariance. Furthermore, the computation of the Green's functions in nontrivial sectors [7] shows that, in order to diagonalize $\bar{S}_{\text{eff}}$ without allowing ϕ and ρ to carry topological charge, it is also necessary that

$$a_+(N)=1 \quad \text{for all } N\neq 0 \ . \tag{53}$$

It then appears that there is a match, in nontrivial sectors, between mathematical consistency and gauge invariance. In the trivial sector, a_+ remains fixed solely on the basis of gauge invariance. However, in the CSM, as no such criterion exists, we expect Eq. (45) to really give extra information.

As a by-product of our analysis, we get a functional differential equation for $\det' D^{(N)}$. Using (48) and (50), we obtain

$$\frac{\delta}{\delta f(x)} \ln \det' D^{(N)} = \frac{1}{2\pi}[a_+(N)+1]\Box f(x) \\ + \frac{\delta}{\delta f(x)} \ln \det(\langle \Phi_{0_i}^{(N)\dagger} \Phi_{0_j}^{(N)}\rangle) , \tag{54}$$

or, integrating,

$$\det' D^{(N)} = e^{[a_+(N)+1]\langle f\Box f\rangle/4\pi} \det(\langle \Phi_{0_i}^{(N)\dagger} \Phi_{0_j}^{(N)}\rangle) \ . \tag{55}$$

The result above is important for the complete analysis of the Green's functions of the CSM, which will be the object of Sec. V.

III. ZERO MODES IN THE CHIRAL SCHWINGER MODEL

We now pass to the chiral Schwinger model, i.e., two-dimensional electrodynamics with chiral coupling to fermions, whose covariant Dirac operator is

$$D = i\partial\!\!\!/ + e_R A P_+ \ . \tag{56}$$

Using the gauge field A_μ as in the preceding section, we can now solve the zero-mode problem for D,

$$D\Phi_0=0\ , \tag{57}$$

or, explicitly, in terms of holomorphic coordinates,

$$2\begin{bmatrix} 0 & -\partial_z \\ \partial_{\bar z}-ie_R\bar A & 0\end{bmatrix}\begin{bmatrix}\Phi_R\\ \Phi_L\end{bmatrix}=0\ , \tag{58}$$

where $\bar A=\frac{1}{2}(A_1+iA_0)$. Substituting for all scalar fields, this is equivalent to the set of equations

$$\partial_z\Phi_L=0\ , \tag{59}$$

$$[\partial_{\bar z}-\partial_{\bar z}(f+\phi+i\rho)]\Phi_R=0\ . \tag{60}$$

Although there are no normalizable solutions for (59), there are precisely N for (60), if $N>0$, which are given by

$$\Phi_{0_i}=z^{i-1}e^{f+\phi+i\rho}\begin{bmatrix}1\\0\end{bmatrix}\ ,\quad i=1,\ldots,N\ . \tag{61}$$

If $N<0$, however, there are no normalizable solutions at all for (57). On the other hand, the contrary occurs for the zero-mode equation for the adjoint operator,

$$D^\dagger\chi_0=0\ ; \tag{62}$$

that is, no normalizable solutions exist for $N>0$ and, if $N<0$, the $|N|$ solutions are

$$\chi_{0_i}=\bar z^{i-1}e^{-(f+\phi-i\rho)}\begin{bmatrix}0\\1\end{bmatrix}\ ,\quad i=1,\ldots,|N|\ . \tag{63}$$

The zero modes of $D^\dagger$ are as necessary as those of D, as we can see by considering the generating functional for the present model:

$$Z[J_\mu,\eta,\bar\eta]=\int\mathcal{D}A_\mu\mathcal{D}\bar\psi\mathcal{D}\psi\exp(-\tfrac{1}{4}\langle F_{\mu\nu}F_{\mu\nu}\rangle-\langle\bar\psi D\psi\rangle+\langle J_\mu A_\mu\rangle+\langle\bar\eta\psi\rangle+\langle\bar\psi\eta\rangle)\ , \tag{64}$$

with angular brackets again denoting integration. As D is a non-normal operator, it is convenient to use in each topological sector, as was done by Fujikawa [26], the two sets of orthonormal functions defined by the eigenfunctions of the Laplacian operators related to D:

$$D^\dagger D\varphi_n=\lambda_n^2\varphi_n\ ,\quad D^\dagger D\varphi_{0_i}=0\ , \tag{65}$$

$$DD^\dagger\phi_n=\lambda_n^2\phi_n\ ,\quad DD^\dagger\phi_{0_i}=0\ . \tag{66}$$

The sets $\{\varphi_n\}$ and $\{\phi_n\}$ are orthonormal, as the Laplacians $D^\dagger D$ and $DD^\dagger$ are Hermitian. Moreover,

$$D\varphi_n=\lambda_n\phi_n\ , \tag{67}$$

$$D^\dagger\phi_n=\lambda_n\varphi_n\ , \tag{68}$$

and $\ker D^\dagger D=\ker D$, $\ker DD^\dagger=\ker D^\dagger$. If we decompose ψ and $\bar\psi$ with respect to these bases, we have, for $N>0$,

$$\psi=\sum_{\lambda_n\neq 0}a_n\varphi_n+\sum_{i=1}^{N}a_{0_i}\varphi_{0_i}\ , \tag{69}$$

$$\bar\psi=\sum_n a_n\phi_n^\dagger \tag{70}$$

and, for $N<0$,

$$\psi=\sum_n a_n\varphi_n\ , \tag{71}$$

$$\bar\psi=\sum_{\lambda_n\neq 0}\bar a_n\phi_n^\dagger+\sum_{i=1}^{|N|}\bar a_{0_i}\phi_{0_i}^\dagger\ . \tag{72}$$

The functional fermionic measure may then be written as

$$\begin{aligned}\mathcal{D}\bar\psi\mathcal{D}\psi&\equiv\prod_x d\bar\psi(x)d\psi(x)\\ &=\det[\varphi_n(x)]^{-1}\det[\phi_n^\dagger(x)]^{-1}\\ &\quad\times\prod_x d\bar a_n da_n\left[\prod_i d\bar a_{0_i}\right]^{\theta(-N)}\left[\prod_i da_{0_i}\right]^{\theta(N)}\ ,\end{aligned} \tag{73}$$

where θ is the Heaviside function. The determinants in the formula above do not appear in the Hermitian case, because the transformations (69)–(72) are unitary there (in the present case, they are biunitary) and they are responsible for the phase of $\det' D$. With the aid of the bases (65) and (66), we can construct a distribution $S(x,y)$ as

$$S(x,y)=\sum_{\lambda_n\neq 0}\frac{\varphi_n(x)\phi_n^\dagger(y)}{\lambda_n}\ , \tag{74}$$

which satisfies, for $N>0$,

$$\begin{aligned}DS(x,y)&=\delta(x-y)\ ,\\ S(x,y)D&=\delta(x-y)-P_0(x,y)\ ,\end{aligned} \tag{75}$$

while, for $N<0$,

$$\begin{aligned}DS(x,y)&=\delta(x-y)-\bar P_0(x,y)\ ,\\ S(x,y)D&=\delta(x-y)\ ,\end{aligned} \tag{76}$$

with the two projectors on zero modes given by

$$P_0(x,y)=\sum_{i=1}^{|N|}\varphi_{0_i}(x)\varphi_{0_i}^\dagger(y)\ , \tag{77}$$

$$\bar P_0(x,y)=\sum_{i=1}^{|N|}\phi_{0_i}(x)\phi_{0_i}^\dagger(y)\ . \tag{78}$$

As defined by (74), $S(x,y)$ is the best object that we have at our disposal to try to decouple the sources from the fermion fields, since, as D is noninvertible, we have no Green's function with a spectral decomposition such as that of (74). Performing the translations

$$\psi(x)=\psi'(x)+\int d^2y\, S(x,y)\eta(y)\ , \tag{79}$$

$$\bar\psi(x)=\bar\psi'(x)+\int d^2y\,\bar\eta(y)S(y,x)\ , \tag{80}$$

we obtain

$$Z[J_\mu,\eta,\bar\eta]=\sum_N \int \mathcal{D}a_\mu \mathcal{D}\bar\psi\mathcal{D}\psi \exp(-\tfrac{1}{4}\langle F_{\mu\nu}F_{\mu\nu}\rangle - \langle\bar\psi D\psi\rangle + \langle J_\mu A_\mu\rangle + \langle\bar\eta S\eta\rangle)\exp[\theta(N)\langle\bar\eta P_0\psi\rangle + \theta(-N)\langle\bar\psi \bar P_0 \eta\rangle] . \tag{81}$$

Using now the decompositions (69)–(72) and the expression for the measure (73), we see that

$$Z[J_\mu,\eta,\bar\eta]=\sum_N \int \mathcal{D}a_\mu \exp(-\tfrac{1}{4}\langle F_{\mu\nu}F_{\mu\nu}\rangle + \langle J_\mu A_\mu\rangle + \langle\bar\eta S\eta\rangle)\det{}'D \left[\prod_{i=1}^{|N|}\langle\bar\eta\varphi_{0_i}\rangle(-1)^N\right]^{\theta(N)} \left[\prod_{i=1}^{|N|}\langle\phi_{0_i}^\dagger\eta\rangle\right]^{\theta(-N)} , \tag{82}$$

with (see [23,26])

$$\det{}'D=\det[\varphi_n(x)]^{-1}\left|\prod_{\lambda_n\neq 0}\lambda_n\right|\det[\phi_n^\dagger(x)]^{-1} . \tag{83}$$

We can bosonize the theory within each sector, by performing gauge transformations of the fermions according to

$$\psi=e^{(\phi+i\rho)P_+}\psi' , \tag{84}$$

$$\bar\psi=\bar\psi' e^{-(\phi+i\rho)P_-} , \tag{85}$$

with ϕ and ρ defined in (9). As is well known, the fact that these transformations involve γ_5 implies that a Jacobian arises; its computation gives the effective action for the bosonized theory. The rest of this section is devoted to describing the procedure for obtaining this Jacobian. It is to be noted that transformations (84) and (85) take D into $D^{(N)}$, which has the same number of zero modes as D, because both ϕ and ρ obey trivial boundary conditions. This is necessary in order to give meaning to the ratio of their determinants which appears below.

Returning to Eq. (64), and considering, for brevity, only the fermion part of the generating functional in the sector of charge N, we perform transformations (84) and (85) and obtain

$$\begin{aligned} J[A_\mu^{(N)},a_\mu]Z_F^{(N)}[\eta,\bar\eta] &\equiv J[A_\mu^{(N)},a_\mu]\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp[-\langle\bar\psi D^{(N)}\psi\rangle \\ &\qquad + \langle\bar\psi\eta'\rangle + \langle\bar\eta'\psi\rangle] , \end{aligned} \tag{86}$$

with J being the Jacobian, $\eta'=e^{-(\phi+i\rho)P_-}\eta$, $\bar\eta'=\bar\eta e^{(\phi+i\rho)P_+}$ and

$$\begin{aligned} D^{(N)} &= e^{-(\phi+i\rho)P_-}De^{(\phi+i\rho)P_+} \\ &= i\partial\!\!\!/ + e_R A\!\!\!/^{(N)}P_+ . \end{aligned} \tag{87}$$

Therefore,

$$Z_F^{(N)}[\eta,\bar\eta]=e^{\langle\bar\eta' S^{(N)}\eta'\rangle}\det{}'D^{(N)}\left[\prod_{i=1}^{|N|}\langle\bar\eta'\varphi_{0_i}^{(N)}\rangle(-1)^N\right]^{\theta(N)}\left[\prod_{i=1}^{|N|}\langle\phi_{0_i}^{(N)}\eta'\rangle\right]^{\theta(-N)} , \tag{88}$$

where $\varphi_{0_i}^{(N)},\phi_{0_i}^{(N)}$ are the zero modes of $D^{(N)}$ and $D^{(N)^\dagger}$, respectively, and $S^{(N)}$ satisfies

$$\begin{aligned} D^{(N)}S^{(N)}(x,y) &= \delta(x-y) , \\ S^{(N)}(x,y)D^{(N)} &= \delta(x-y) - P_0^{(N)}(x,y) , \end{aligned} \tag{89}$$

for $N>0$ and the analogous equations for $N<0$. It can be proven that $\langle\bar\eta' S^{(N)}\eta'\rangle$ can be replaced by $\langle\bar\eta S\eta\rangle$ in (88) by virtue of the multiplication of the exponential by the products of terms involving the sources [15]. To compare (88) with the fermion part of (81) it is necessary to express the zero modes of D in terms of those of $D^{(N)}$. Thus, given the set $\{\varphi_{0_i}^{(N)}\}$ of zero modes of $D^{(N)}$, we obtain the corresponding set for D as

$$\varphi_{0_i}=e^{(\phi+i\rho)P_+}\sum_j B_{ij}\varphi_{0_i}^{(N)} . \tag{90}$$

The matrix B is introduced to ensure the orthonormality of the set $\{\varphi_{0_i}\}$, when expressed in terms of the $\varphi_{0_i}^{(N)}$ [14,15]. Its determinant is fixed, up to a phase, by

$$\langle\varphi_{0_i}^\dagger\varphi_{0_j}\rangle=\delta_{ij} , \tag{91}$$

which implies

$$|\det B|^2=[\det(\langle\varphi_{0_i}^{(N)^\dagger}e^{2\phi P_+}\varphi_{0_j}^{(N)}\rangle)]^{-1} . \tag{92}$$

However, we need $\det B$ (not just its modulus squared) for the generating functional because

$$\begin{aligned} \prod_{i=1}^{|N|}\langle\bar\eta\varphi_{0_i}\rangle &= \prod_{i=1}^{|N|}\left[\sum_j B_{ij}\langle\bar\eta'\varphi_{0_j}^{(N)}\rangle\right] \\ &= \det B\prod_{i=1}^{|N|}\langle\bar\eta'\varphi_{0_i}^{(N)}\rangle . \end{aligned} \tag{93}$$

A similar computation can be made for the zero modes of $D^\dagger$, giving

$$\prod_{i=1}^{|N|}\langle\phi_{0_i}^\dagger\eta\rangle=\det C^*\prod_{i=1}^{|N|}\langle\phi_{0_i}^{(N)^\dagger}\eta'\rangle , \tag{94}$$

with

$$|\det C|^2=[\det(\langle\phi_{0_i}^{(N)^\dagger}e^{-2\phi P_+}\phi_{0_j}^{(N)}\rangle)]^{-1} . \tag{95}$$

With this we write our Jacobian as

$$J[A_\mu^{(N)},a_\mu]=\frac{\det' D}{\det' D^{(N)}}[e^{i\gamma_+}\det(\langle\varphi_{0_i}^{(N)\dagger}e^{2\phi P_+}\varphi_{0_j}^{(N)}\rangle)^{-1/2}]^{\theta(N)}[e^{i\gamma_-}\det(\langle\phi_{0_i}^{(N)\dagger}e^{-2\phi P_-}\phi_{0_j}^{(N)}\rangle)^{-1/2}]^{\theta(-N)} . \tag{96}$$

The phases γ_+ and γ_- of the determinants of the orthonormalization matrices are in principle arbitrary but we shall see in the following section how this arbitrariness is suitably eliminated.

IV. FERMION DETERMINANT AND PHASE AMBIGUITIES

In the case of the CSM, the analogue to (14),

$$\frac{d}{d\alpha}\det' D_\alpha=\lim_{\epsilon\to 0^+}\frac{\det(D_\alpha+\epsilon 1)}{\epsilon^{|N|}}\mathrm{Tr}\left[(D_\alpha+\epsilon 1)^{-1}\frac{dD_\alpha}{d\alpha}\right] , \tag{97}$$

where now

$$D_\alpha=i\partial\!\!\!/+e_R A\!\!\!/^{(N)}P_+ +\alpha e_R a\!\!\!/ P_+ ,\quad 0\le\alpha\le 1 , \tag{98}$$

simply does not work, because in the present case we do not have a well-defined distribution $(D_\alpha+\epsilon 1)^{-1}$ with a spectral decomposition similar to (15). The reason for this lies in the zero-eigenvalue sector of the spectrum of D_α: as the number of zero modes of D_α and $D_\alpha^\dagger$ is different, we cannot create a "scalar" from $\varphi_{0_i}^\alpha$ and $\phi_{0_j}^{\alpha\dagger}$ that would carry the $1/\epsilon$ singularity of $(D_\alpha+\epsilon 1)^{-1}$. We could also say, in a more physical manner, that putting a mass in the Dirac operator is equivalent to not treating the left-handed fermions as purely free any longer, because a mass term mixes them with the right-handed ones. As the fermions that really matter to us are the right-handed ones ($\psi_R=\gamma_5\psi_R$), we should not use this trick to deal with nonsingular operators here.

However, even though $(D_\alpha+\epsilon 1)^{-1}$ is not well defined, the same is not true for S^α, given by

$$S^\alpha(x,y)=\sum_{\lambda_n\neq 0}\frac{\varphi_n^\alpha(x)\phi_n^{\alpha\dagger}(y)}{\lambda_n} , \tag{99}$$

with S^α constructed from the Laplacians $D_\alpha^\dagger D_\alpha$ and $D_\alpha D_\alpha^\dagger$ in the same way as in (74). So we can still define

$$\frac{d}{d\alpha}\ln\det' D_\alpha=\mathrm{Tr}\left[S^\alpha\frac{dD_\alpha}{d\alpha}\right] . \tag{100}$$

This is a completely well-defined expression (as long as we take care, just as in the case of the Schwinger model, of the ultraviolet divergences), a very natural one, as it has the nonsingular situation as an obvious limit (in which case S^α is a true inverse of D_α) and contains the Hermitian case, where all can be derived from a "reasonable" definition of $\det' D$, Eq. (11).

The equations that S^α has to satisfy are now

$N>0$:

$$\begin{aligned} &D_\alpha S^\alpha(x,y)=\delta(x-y) ,\\ &S^\alpha(x,y)D_\alpha=\delta(x-y)-P_0^\alpha(x,y); \end{aligned} \tag{101}$$

$N<0$:

$$\begin{aligned} &D_\alpha S^\alpha(x,y)=\delta(x-y)-\bar P_0^\alpha(x,y) ,\\ &S^\alpha(x,y)D_\alpha=\delta(x-y) . \end{aligned} \tag{102}$$

This implies

$$\begin{aligned} S^\alpha(x,y)=G^\alpha(x,y)&-\theta(N)\int d^2z\, P_0^\alpha(x,z)G^\alpha(z,y)\\ &-\theta(-N)\int d^2z\, G^\alpha(x,z)\bar P_0^\alpha(z,y) , \end{aligned} \tag{103}$$

with $G^\alpha(x,y)$ being the Green's function (without spectral decomposition) given by

$$G^\alpha(x,y)=(e^{h_+(x)-h_+(y)}P_+ +P_-)G_F(x-y) , \tag{104}$$

and

$$h_+(x)=f(x)+\alpha[\phi(x)+i\rho(x)] . \tag{105}$$

Using the equations of motion for the zero modes of D_α, $\varphi_{0_i}^\alpha\equiv\varphi_i^\alpha\binom{1}{0}$, and for those of $D_\alpha^\dagger$, $\phi_{0_i}^\alpha\equiv\phi_i^\alpha\binom{0}{1}$,

$$[\partial_{\bar z}-\partial_{\bar z}(f+\alpha(\phi+i\rho))]\varphi_i^\alpha=0 , \tag{106}$$

$$[\partial_z+\partial_z(f+\alpha(\phi-i\rho))]\phi_i^\alpha=0 , \tag{107}$$

and the point-splitting method to regularize ultraviolet divergences, we find

$$\begin{aligned} \mathrm{Tr}\left[S^\alpha\frac{dD_\alpha}{d\alpha}\right]=&-2\alpha\Gamma[a_\mu]-\bar\Gamma[A_\mu^{(N)},a_\mu]\\ &+\theta(N)\sum_{i=1}^{|N|}\langle\varphi_{0_i}^{\alpha\dagger}(\phi+i\rho)\varphi_{0_i}^\alpha\rangle\\ &-\theta(-N)\sum_{i=1}^{|N|}\langle\phi_{0_i}^{\alpha\dagger}(\phi+i\rho)\phi_{0_i}^\alpha\rangle , \end{aligned} \tag{108}$$

where (see again [19])

$$\Gamma[a_\mu]=\frac{e_R^2}{8\pi}\int d^2x\, a_\mu\left[a_R(N)\delta_{\mu\nu}-(\partial_\mu+i\tilde\partial_\mu)\frac{1}{\Box}(\partial_\nu+i\tilde\partial_\nu)\right]a_\nu \tag{109}$$

and

$$\bar\Gamma[A_\mu^{(N)},a_\mu]=\frac{e_R^2}{4\pi}\int d^2x\, A_\mu^{(N)}\left[a_R(N)(\delta_{\mu\nu}-i\varepsilon_{\mu\nu})-(\partial_\mu+i\tilde\partial_\mu)\frac{1}{\Box}(\partial_\nu+i\tilde\partial_\nu)\right]a_\nu , \tag{110}$$

$a_R(N)$ being the arbitrary parameter reflecting regularization freedom. Integrating over α we find

$$\frac{\det' D}{\det' D^{(N)}} = e^{-\Gamma[a_\mu]-\bar{\Gamma}[A_\mu^{(N)},a_\mu]} \exp\left[\theta(N)\int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \varphi_{0_i}^{\alpha\dagger}(\phi+i\rho)\varphi_{0_i}^{\alpha}\rangle\right] \exp\left[-\theta(-N)\int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \phi_{0_i}^{\alpha\dagger}(\phi+i\rho)\phi_{0_i}^{\alpha}\rangle\right] . \tag{111}$$

Expressing $\varphi_{0_i}^{\alpha}$ in terms of $\varphi_{0_i}^{(N)}$ and $\phi_{0_i}^{\alpha}$ in terms of $\phi_{0_i}^{(N)}$,

$$\varphi_{0_i}^{\alpha} = e^{\alpha(\phi+i\rho)P_+} \sum_{i=1}^{|N|} B_{ij}\varphi_{0_j}^{(N)} , \tag{112}$$

$$\phi_{0_i}^{\alpha} = e^{-\alpha(\phi-i\rho)P_-} \sum_{i=1}^{|N|} D_{ij}\phi_{0_j}^{(N)} , \tag{113}$$

we get

$$\frac{\det' D}{\det' D^{(N)}} = e^{-\Gamma[a_\mu]-\bar{\Gamma}[A_\mu^{(N)},a_\mu]} [\det(\langle \varphi_{0_i}^{(N)\dagger} e^{2\phi P_+} \varphi_{0_j}^{(N)}\rangle)]^{\theta(N)/2} [\det(\langle \phi_{0_i}^{(N)\dagger} e^{-2\phi P_-} \phi_{0_j}^{(N)}\rangle)]^{\theta(-N)/2}$$
$$\times \exp\left[i\theta(N)\int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \varphi_{0_i}^{\alpha\dagger}\rho\varphi_{0_i}^{\alpha}\rangle\right] \exp\left[-i\theta(-N)\int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \phi_{0_i}^{\alpha\dagger}\rho\phi_{0_i}^{\alpha}\rangle\right] . \tag{114}$$

The main new feature of (114) is the presence of the phases involving traces over the null subspaces of D_α and $D_\alpha^\dagger$ of the longitudinal part of a_μ, the field ρ. These terms are canceled in the Schwinger model due to the occurrence of only

$$\mathrm{Re}\left[\mathrm{Tr}\left[G^\alpha P_0^\alpha \frac{dD_\alpha}{d\alpha}\right]\right]$$

in the expression of the fermion determinant. Considering now Eq. (96) for the Jacobian, we obtain

$$J[A_\mu^{(N)},a_\mu] = e^{-\Gamma[a_\mu]-\bar{\Gamma}[A_\mu^{(N)},a_\mu]}$$
$$\times \exp\left\{i\theta(N)\left[\gamma_+ + \int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \varphi_{0_i}^{\alpha\dagger}\rho\varphi_{0_i}^{\alpha}\rangle\right]\right\} \exp\left\{i\theta(-N)\left[\gamma_- - \int_0^1 d\alpha \sum_{i=1}^{|N|} \langle \phi_{0_i}^{\alpha\dagger}\rho\phi_{0_i}^{\alpha}\rangle\right]\right\} , \tag{115}$$

and, defining $\bar{S}_{\mathrm{eff}}$ as

$$\bar{S}_{\mathrm{eff}} = \tfrac{1}{4}\langle F_{\mu\nu}F_{\mu\nu}\rangle - \ln J[A_\mu^{(N)},a_\mu] , \tag{116}$$

we see that the generating functional is given by

$$Z[J_\mu,\eta,\bar{\eta}] = \sum_N \int \mathcal{D}a_\mu e^{-\bar{S}_{\mathrm{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle+\langle \bar{\eta}' S^{(N)}\eta'\rangle} \det' D^{(N)}$$
$$\times \left[\det(\langle \Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)^{-1/2} \prod_{i=1}^{|N|} \langle \bar{\eta}'\Phi_{0_i}^{(N)}\rangle(-1)^N\right]^{\theta(N)} \left[\det(\langle \chi_{0_i}^{(N)\dagger}\chi_{0_j}^{(N)}\rangle)^{-1/2} \prod_{i=1}^{|N|} \langle \chi_{0_i}^{(N)\dagger}\eta'\rangle\right]^{\theta(-N)}$$
$$\equiv \sum_N \int \mathcal{D}a_\mu e^{-\bar{S}_{\mathrm{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle} Z_F^{(N)}[\eta,\bar{\eta}] . \tag{117}$$

Imposing now invariance under change of the representative of the gauge field homotopy class, $\delta Z[J_\mu,\eta,\bar{\eta}]/\delta f(x)=0$, and noting that

$$\frac{\delta}{\delta f(x)} \det' D^{(N)} = \det' D^{(N)} \left[\frac{1}{4\pi}[a_R(N)+1]\Box f(x) + \theta(N)\mathrm{tr}[P_0^{(N)}(x,x)] - \theta(-N)\mathrm{tr}[\bar{P}_0^{(N)}(x,x)]\right] \tag{118}$$

and

$$\frac{\delta}{\delta f(x)} \det(\langle \Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)^{-1/2} = -\det(\langle \Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)^{-1/2}\mathrm{tr}(P_0^{(N)}(x,x)) , \tag{119}$$

$$\frac{\delta}{\delta f(x)} \det(\langle \chi_{0_i}^{(N)\dagger}\chi_{0_j}^{(N)}\rangle)^{-1/2} = \det(\langle \chi_{0_i}^{(N)\dagger}\chi_{0_j}^{(N)}\rangle)^{-1/2}\mathrm{tr}(\bar{P}_0^{(N)}(x,x)) , \tag{120}$$

we obtain

$$\frac{\delta}{\delta f(x)} Z[J_\mu,\eta,\bar\eta] = \sum_N \Bigg\{ \int \mathcal{D}\rho\,\mathcal{D}\phi \frac{\delta}{\delta\phi(x)} (e^{-\bar S_{\text{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle} Z_F^{(N)}[\eta,\bar\eta]) - \int \mathcal{D}\rho\,\mathcal{D}\phi\, e^{-\bar S_{\text{eff}}[A_\mu^{(n)},a_\mu]+\langle J_\mu A_\mu\rangle} Z_F^{(N)}[\eta,\bar\eta] \times \Bigg[i\theta(N)\left[\frac{\delta}{\delta\phi(x)}-\frac{\delta}{\delta f(x)}\right]\left[\gamma_+ + \int_0^1 d\alpha \sum_{i=1}^{|N|}\langle \varphi_{0_i}^{\alpha\dagger}\rho\varphi_{0_i}^{\alpha}\rangle\right] + i\theta(-N)\left[\frac{\delta}{\delta\phi(x)}-\frac{\delta}{\delta f(x)}\right]\left[\gamma_- - \int_0^1 d\alpha \sum_{i=1}^{|N|}\langle \phi_{0_i}^{\alpha\dagger}\rho\phi_{0_i}^{\alpha}\rangle\right] - \frac{a_R}{4\pi}\Box\rho(x)\Bigg]\Bigg\} . \tag{121}$$

To get a null result to (121), we fix the phases γ_+ and γ_- by choosing them to be

$$\gamma_+ = -\int_0^1 d\alpha \sum_{i=1}^{|N|}\langle \varphi_{0_i}^{\alpha\dagger}\rho\varphi_{0_i}^{\alpha}\rangle + \frac{a_R}{8\pi}\langle(\phi-f)\Box\rho\rangle - \frac{\nu}{8\pi}\langle(\phi+f)\Box\rho\rangle , \tag{122}$$

$$\gamma_- = \int_0^1 d\alpha \sum_{i=1}^{|N|}\langle \phi_{0_i}^{\alpha\dagger}\rho\phi_{0_i}^{\alpha}\rangle + \frac{a_R}{8\pi}\langle(\phi-f)\Box\rho\rangle - \frac{\nu}{8\pi}\langle(\phi+f)\Box\rho\rangle , \tag{123}$$

where the last term in both of the above equations is annihilated by $\delta/\delta\phi-\delta/\delta f$ and represents a residual phase ambiguity, parametrized by ν. It is used in the next section to diagonalize the effective action without allowing ϕ and ρ to carry topological charge. We see then that invariance of the generating functional is achieved only by carefully adjusting the phases of the orthonormal sets of zero modes. The fermionic sources are responsible for uncovering this phase ambiguity.

Finally, using (119) and (120), we can compute explicitly $\det' D^{(N)}$,

$$\det' D^{(N)} = e^{[a_R(N)+1]\langle f\Box f\rangle/8\pi}[\det(\langle \Phi_{0_i}^{(N)\dagger}\Phi_{0_j}^{(N)}\rangle)^{1/2}]^{\theta(N)}[\det(\langle \chi_{0_i}^{(N)\dagger}\chi_{0_j}^{(N)}\rangle)^{1/2}]^{\theta(-N)} , \tag{124}$$

so that we obtain our last expression for the generating functional,

$$Z[J_\mu,\eta,\bar\eta] = \sum_N \int \mathcal{D}\phi\,\mathcal{D}\rho\, e^{-S_{\text{eff}}[A_\mu^{(N)},a_\mu]+\langle J_\mu A_\mu\rangle+\langle\bar\eta' S^{(N)}\eta'\rangle}\left[\prod_{i=1}^{|N|}\langle\bar\eta'\Phi_{0_i}^{(N)}\rangle(-1)^N\right]^{\theta(N)}\left[\prod_{i=1}^{|N|}\langle\chi_{0_i}^{(N)\dagger}\eta'\rangle\right]^{\theta(-N)} , \tag{125}$$

with

$$S_{\text{eff}}[A_\mu^{(N)},a_\mu] = \bar S_{\text{eff}} - \frac{1}{8\pi}[a_R(N)+1]\langle f\Box f\rangle . \tag{126}$$

Expression (125) will be our starting point for the computation of correlation functions in the CSM in the next section.

V. CORRELATION FUNCTIONS IN NONTRIVIAL SECTORS OF THE CHIRAL SCHWINGER MODEL

It is not difficult to see that all correlation functions of the kind

$$\langle \bar\psi(x_1)\Gamma_1\psi(x_1)\cdots\bar\psi(x_N)\Gamma_N\psi(x_N)\rangle , \tag{127}$$

where Γ_n represents any Dirac matrix or product of them, vanish due to the impossibility of pairing the adequate number of functional derivatives of η and $\bar\eta$ over the generating functional to produce a non-null result when $\eta,\bar\eta\to 0$ (we are of course excluding the trivial sector from this analysis). The correlation functions of bosonic fields are also zero. The only objects that remain to be considered are

$$G^{(N)}(x_1,\ldots,x_N) = \langle\psi(x_N)\psi(x_{N-1})\cdots\psi(x_1)\rangle = \frac{1}{Z[0]}\frac{\delta}{\delta\bar\eta(x_N)}\cdots\frac{\delta}{\delta\bar\eta(x_1)}Z[0,\eta,\bar\eta] = \frac{1}{Z[0]}\int\mathcal{D}\rho\,\mathcal{D}\phi\exp\left[-S_{\text{eff}}[A_\mu^{(N)},a_\mu]+\sum_i[\phi(x_i)+i\rho(x_i)]\right]\det\begin{vmatrix}\Phi_{0_1}^{(N)}(x_1) & \cdots & \Phi_{0_N}^{(N)}(x_1)\\ \vdots & & \vdots\\ \Phi_{0_1}^{(N)}(x_N) & \cdots & \Phi_{0_N}^{(N)}(x_N)\end{vmatrix}(-1)^N \tag{128}$$

and

$$\bar{G}^{(N)}(x_1,\ldots,x_N)=\langle\bar{\psi}(x_N)\bar{\psi}(x_{N-1})\cdots\bar{\psi}(x_1)\rangle$$
$$=\frac{1}{Z[0]}(-1)^N\frac{\delta}{\delta\eta(x_N)}\cdots\frac{\delta}{\delta\eta(x_1)}Z[0,\eta,\bar{\eta}]$$
$$=\frac{1}{Z[0]}\int\mathcal{D}\rho\,\mathcal{D}\phi\exp\left[-S_{\text{eff}}[A_\mu^{(N)},a_\mu]-\sum_i[\phi(x_i)+i\rho(x_i)]\right]$$
$$\times\det\begin{vmatrix}\chi_{0_1}^{(N)\dagger}(x_1) & \cdots & \chi_{0_N}^{(N)\dagger}(x_1)\\ \vdots & & \vdots\\ \chi_{0_1}^{(N)\dagger}(x_N) & \cdots & \chi_{0_N}^{(N)\dagger}(x_N)\end{vmatrix}(-1)^N . \tag{129}$$

We see that the only contribution to $G^{(N)}$ ($\bar{G}^{(N)}$) is from the Nth sector, $N>0$ ($N<0$). The determinants of zero modes give

$$\det[\Phi_{0_i}^{(N)}(x_j)]=\exp\left[\sum_i f(x_i)\right]\prod_{\substack{i,j=1\\ i>j}}^{|N|}(z_i-z_j)\bigotimes_{i=1}^{|N|}\begin{bmatrix}1\\0\end{bmatrix} , \tag{130}$$

and

$$\det[\chi_{0_i}^{(N)\dagger}(x_j)]=\exp\left[-\sum_i f(x_i)\right]\prod_{\substack{i,j=1\\ i>j}}^{|N|}(\bar{z}_i-\bar{z}_j)\bigotimes_{i=1}^{|N|}(0\;1) . \tag{131}$$

Defining

$$j(x)=\mp\sum_{i=1}^{|N|}\delta(x-x_i) , \tag{132}$$

for $N>0$ and $N<0$, respectively, a final functional integral has to be computed,

$$I[j]=\int\mathcal{D}\rho\,\mathcal{D}\phi\,e^{-S_{\text{eff}}[\rho,\phi,j]} , \tag{133}$$

where

$$S_{\text{eff}}[\rho,\phi,j]=\frac{1}{2e_R^2}\left\langle(f+\phi)\Box\left[\Box-\frac{e_R^2}{4\pi}[a_R(N)+1]\right](f+\phi)\right\rangle$$
$$-\frac{1}{8\pi}[a_R(N)+1]\langle\rho\Box\rho\rangle-\frac{i}{8\pi}[a_R(N)+2-\nu]\langle(f+\phi)\Box\rho\rangle+\langle j(f+\phi+i\rho)\rangle . \tag{134}$$

We can diagonalize S_{eff} by changing variables, first from ρ to σ, through

$$\sigma=\rho+i\frac{a_R(N)+2-\nu}{2[a_R(N)-1]}(f+\phi)-\frac{4\pi i}{a_R(N)-1}\langle\Delta_F j\rangle , \tag{135}$$

where $a_R(N)\neq 1$ and

$$\Delta_F(x-y)=-\frac{1}{2\pi}\ln|x-y| \tag{136}$$

is the Green's function of $\Box$. The effective action then becomes

$$S_{\text{eff}}[\sigma,\phi,j]=\frac{1}{2e_R^2}\langle(f+\phi)\Box[\Box-\mu^2(\nu)](f+\phi)\rangle$$
$$-\frac{1}{8\pi}[a_R(N)-1]\langle\sigma\Box\sigma\rangle+\beta(\nu)\langle j(f+\phi)\rangle+\frac{1}{a_R(N)-1}\sum_{i,j=1}^{|N|}\ln|x_i-x_j| , \tag{137}$$

where we have defined

$$\mu^2(\nu)=\frac{e_R^2}{4\pi}\left[a_R(N)+1-\frac{[a_R(N)+2-\nu]^2}{4[a_R(N)-1]}\right] \tag{138}$$

and

$$\beta(\nu)=\frac{3[a_R(N)-\nu]}{2[a_R(N)-1]} . \tag{139}$$

The asymptotic behavior of σ is

$$\sigma \underset{|x|\to\infty}{\sim} \mp\frac{i|N|}{a_R(N)-1}[a_R(N)+6-\nu]\ln|x| . \tag{140}$$

This fixes ν to be

$$\nu=a_R(N)+6 , \tag{141}$$

because we assume that σ does not carry topological charge. To decouple ϕ from j and f, we use the translation

$$\varphi=f+\phi+\frac{a_R(N)-3}{a_R(N)-1}e_R^2\langle\Delta(\mu)j\rangle , \tag{142}$$

where

$$\Delta(\mu;x-y)=-\frac{1}{2\pi\mu^2}(K_0[\mu|x-y|]+\ln|x-y|) , \tag{143}$$

K_0 being the zeroth-order modified Bessel function. The effective action is then rewritten as

$$S_{\rm eff}=\frac{1}{2e_R^2}\langle\varphi\Box(\Box-\mu^2)\varphi\rangle-\frac{1}{8\pi}[a_R(N)-1]\langle\sigma\Box\sigma\rangle$$
$$+\frac{1}{a_R(N)-1}\sum_{i,j}^{|N|}\ln|x_i-x_j|$$
$$-\frac{[a_R(N)-3]^2}{2[a_R(N)-1]^2}e_R^2\sum_{i,j=1}^{|N|}\Delta(\mu;x_i-x_j) . \tag{144}$$

From the asymptotic behavior of φ,

$$\varphi \underset{|x|\to\infty}{\sim} \mp\frac{|N|\ln|x|}{2e_R[a_R(N)-3]}[a_R(N)+1] , \tag{145}$$

and the trivial homotopy hypothesis, we extract the condition

$$a_R(N)=-1 , \quad \text{for all } N\neq 0 . \tag{146}$$

This then implies

$$I=\exp\left[\frac{1}{2}\sum_{i,j=1}^{|N|}[\ln|x_i-x_j|+2e_R^2\Delta(\mu;x_i-x_j)]\right] . \tag{147}$$

We remark that in the above expression the summation includes the $x_i=x_j$ term; thus, although $\Delta(\mu;0)$ is a regular function, we find $I=0$, showing that nontrivial sectors decouple from the theory.

Finally, we note that if we take $a_R(N)=1$ from the start and perform all computations with that value, it is impossible to diagonalize $S_{\rm eff}$ with a topologically trivial ρ field.

VI. CONCLUSIONS

We have performed a complete analysis of chiral electrodynamics in two dimensions (2D) in nontrivial topological sectors. We have concluded that, unlike the case of the Schwinger model [9], where all topological sectors contribute to the minimal correlation functions (vanishing in the trivial sector), there is no effect of nontrivial topology here. The reason for this relies on two aspects of the theory: the zero-mode structure of chiral gauge theories in 2D, which provides a very peculiar form for the generating functional, and the necessary choice of a regularization ($a_R=-1$), that provides a crucial sign for the term $\sum_{ij}\Delta_F(x_i-x_j)$ which comes from the diagonalization of the effective action. This behavior of the CSM is due to the presence of the longitudinal part ρ, since it does not decouple for any value of $a_R(N)$. Again, this can be contrasted with the Schwinger model, where, for $a_R=1$, ρ disappears from $S_{\rm eff}$.

We have also learned that the effective action has a very important ambiguity in nontrivial sectors, which is decisive for the complete solution of the theory. The mathematical requirement of invariance of the functional integral under changes of background was shown to be a highly nontrivial one, at least when chiral gauge theories are concerned. This requirement and a consistent procedure of bosonization have allowed us to fix completely the form of the effective action and thus solve the theory.

Our analysis has been performed in the so-called "gauge-noninvariant formalism" [27]. It would be interesting to look for the effects, if any, of a Wess-Zumino term on the longitudinal part and in the fixing of $a_R(N)$. Here also the inclusion of the external sources may be important, as one can expect from the results of Ref. [28].

Furthermore, one should try to establish whether these properties occur in the non-Abelian case and in higher dimensions (three and four) as well. Progress in these directions will be reported in the future.

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