Analytic solutions of the Yang-Mills field equations with external sources of higher topological indices

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(Received 9 June 1992)

We construct new analytic solutions to the SU(2) Yang-Mills equations with external sources associated with higher topological indices. By manipulating the parameters in the external source, we show that branching can occur in the total energy versus total external charge plot.

PACS number(s): 11.15.Kc, 11.10.Lm, 11.15.Tk

I. INTRODUCTION

The static Coulomb solution in Abelian gauge theory is stable and unique and plays an essential role in classical as well as quantum electrodynamics. However, it does not seem to enjoy the same role in Yang-Mills (YM) theories since it was first found by Mandula [1] that it is unstable when the strength of a spherically symmetric external source exceeds a certain critical magnitude. This has provoked many further efforts into searching for new solutions of the YM equations with external sources; see Ref. [2] for reviews. Notable among the analytic solutions obtained so far are the charge screening solutions and the magnetic dipole solutions found by Sikivie and Weiss [3] and subsequently the type-I and type-II solutions constructed by Jackiw, Jacobs, and Rebbi [4]. The solutions can be roughly classified as follows. Firstly we note that the unit vector $e^a(\mathbf{x})$ from the external charge density $j_0^a(\mathbf{x})=q(\mathbf{x})e^a(\mathbf{x})$ provides a mapping from S^2, defined by $\mathbf{x}^2=R$, $R>0$, to the internal sphere $e^a(\mathbf{x})e^a(\mathbf{x})=1$. The $e^a(\mathbf{x})$ takes over the role of the Higgs field in the topological solution of the YM-Higgs system. This yields the gauge-invariant topological charge M, known as the magnetic charge [5], which can be used to characterize the gauge field solution and the external source. For each value of M, we can have two different types of solutions depending on the asymptotic behavior of the gauge field A_i^a when the charge density $q(\mathbf{x})$ vanishes at infinity. The solution is called type I if the asymptotic A_i^a vanishes identically, and is called type II if the asymptotic A_i^a becomes pure gauge:

$$A_j(\mathbf{x})=-iU^{-1}(\mathbf{x})\partial_j U(\mathbf{x}) , \tag{1}$$

where $U(\mathbf{x})$ is a gauge transformation. The terms type I and type II are generalized from those of Ref. [4] where the case for $M=1$ is discussed. For $M=0$, the Abelian Coulomb solution is of type I while the non-Abelian Coulomb solution [4] is of type II; the magnetic dipole solution [3] also belongs to the $M=0$ class. For $M=1$, the type-I and type-II solutions are as given by [4]. For $M=0$ or 1 one can construct solutions with or without spherical symmetry. For $M>1$, solutions are necessarily nonspherically symmetric and so far only perturbative solutions for weak sources [2] and numerical solutions [6] have been constructed.

An explicit expression for the magnetic charge M of the gauge field and the external source system can be written down [5]:

$$M=\frac{1}{4\pi}\int_{r\to\infty} dS\, n^i\varepsilon_{ijk}(\partial_j a_k+\tfrac{1}{2}\varepsilon^{abc}e^a\partial_j e^b\partial_k e^c) , \tag{2a}$$

where

$$a_k=A_k^a e^a , \quad n^i=x^i/r, \quad r^2=x_1^2+x_2^2+x_3^2 . \tag{2b}$$

When a_k is regular everywhere, the first term of the integrand does not contribute and we have

$$M=\frac{1}{8\pi}\varepsilon_{jk}\varepsilon^{abc}\int_{r\to\infty} dS\, n^i e^a\partial_j e^b\partial_k e^c , \tag{3}$$

which is an element of the second homotopy group $\pi_2[\mathbf{e}]$. Note that the magnetic charge as defined by Eq. (2a) is gauge invariant but the homotopy class as given by expression (3) can change under singular gauge transformation [5]. Expression (3) is also known as the Kronecker index associated with $e^a(\mathbf{x})$. Assuming a_k has no singularity line then M takes integer value n if

$$e^a(\mathbf{x})=\delta_1^a\sin\theta\cos(n\phi)+\delta_2^a\sin\theta\sin(n\phi)+\delta_3^a\cos\theta . \tag{4}$$

To construct solutions with higher M, it is convenient to fix the polar angle θ to a constant so that $e^a(\mathbf{x})$ now maps S^1 to S^1. In particular for $\theta=\pi/2$, expression (4) becomes

$$e^a(\mathbf{x})=\delta_1^a\cos(n\phi)+\delta_2^a\sin(n\phi) . \tag{5}$$

The gauge-invariant magnetic charge M can now be expressed as [6]

$$M=\frac{1}{2\pi}\oint_{\rho\to\infty} d\phi\,\rho(C^aC^a)^{1/2} , \tag{6a}$$

$$C^a=\varepsilon^{abc}e^bD_\phi^{cd}e^d =\varepsilon^{abc}e^b\left[\frac{1}{\rho}\frac{d}{d\phi}e^c+\varepsilon^{cfd}A_\phi^f e^d\right] . \tag{6b}$$

In passing we note that there are solutions which do not have a definite M value and hence cannot be classified in the above manner.

The purpose of this paper is to demonstrate that analytic solutions can actually be constructed for $M>1$. Our task is facilitated by the axial-symmetric ansatz found by the authors of Refs. [3,6]. This ansatz reduces the YM equations to equations similar to those for the magnetic dipole solutions [3], from which new solutions can be constructed. Our solutions have finite energy and finite total external non-Abelian charge. The electric field vanishes exponentially fast at large distances while the magnetic field has the magnetic multipole behavior. Our solutions lead to $M=2n$. In Ref. [7], the magnetic dipole solution and the Abelian Coulomb solution, which are supported by the same charge density distribution, are shown to bifurcate from each other. Since for $M\geq 1$ there is no Abelian Coulomb solution, branching of the new solutions from the Abelian Coulomb solutions is out of the question. However, following Ref. [8], we show that branching can occur in the energy versus total external charge plot for the new solutions provided that the parameters in the charge distribution are suitably manipulated.

II. SOLUTIONS

The SU(2) YM equations in the presence of an external static source are

$$(D_\mu F^{\mu\nu})_a=j_a^\nu\ , \tag{7a}$$

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\varepsilon^{abc}A_\mu^b A_c^c\ , \tag{7b}$$

where our metric is $g_{ii}=-g_{00}=1$ and we shall set the gauge field coupling constant $g=1$. Following Sikivie and Weiss [3] and basing on the perturbative analysis [6], an ansatz leading to solutions with the Kronecker index M can be written down:

$$A_0^a=e^a\Phi_n(\rho,z)\ , \tag{8a}$$

$$A^{ai}=\delta_3^a\phi^i A_n(\rho,z)\ , \tag{8b}$$

$$j_\nu^a=e^a q(\rho,z)\delta_\nu^0\ , \tag{8c}$$

$$e^a=\delta_1^a\cos(n\phi)+\delta_2^a\sin(n\phi)\ , \tag{8d}$$

when ϕ^i is the unit vector $\phi^i=\varepsilon^{ij3}x^j/\rho$, $\rho=(x_1^2+x_2^2)^{1/2}$, and ϕ is the azimuthal angle. This ansatz reduces the YM equations into the coupled nonlinear equations [6]

$$-\nabla^2\Phi_n+\Phi_n\left[A_n-\frac{n}{\rho}\right]^2=q(\rho,x_3)\ , \tag{9a}$$

$$\left[\nabla^2-\frac{1}{\rho^2}\right]A_n+\Phi_n^2\left[A_n-\frac{n}{\rho}\right]=0\ . \tag{9b}$$

For $n=0$ these equations are just those found in Ref. [3] and apart from the magnetic dipole solution [3], many other solutions, e.g., magnetic multipole solutions, have been constructed [9,10]. Defining $B_n\equiv A_n-n/\rho$ ($n>0$), Eqs. (9a) and (9b) become

$$-\nabla^2\Phi_n+\Phi_n B_n^2=q\ , \tag{10a}$$

$$\left[\nabla^2-\frac{1}{\rho^2}+\Phi_n^2\right]B_n=0\ . \tag{10b}$$

Note that in transforming Eq. (9b) to (10b), no extra singularity is introduced. We can now proceed to solve Eqs. (10a) and (10b) by employing the method of Ref. [3]. Namely, we first choose a suitable functional form for B_n that has the appropriate boundary behavior; the function Φ_n is then derived from Eq. (10b). With the explicit B_n and Φ_n, the source density is evaluated via Eq. (10a). Thus we write

$$B_n=caf_n(y,\theta)P_n^1(\cos\theta)r^{-n-1}\ , \tag{11a}$$

$$y=r/a,\quad r^2=\rho^2+x_3^2\ ,$$

where c denotes the norm of B_n, a indicates the size of the external charge distribution, and $P_n^1(\cos\theta)$ is the associated Legendre polynomial. To ensure finite energy, the function $f_n(y,\theta)$ must tend to one sufficiently fast at large y and must tend to zero as $y\to 0$. For our purpose, an appropriate solution for Eqs. (10a) and (10b) is

$$f_n=\tanh(y^{2n+1})\ , \tag{11b}$$

$$\Phi_n=\frac{\sqrt{2}(2n+1)}{a}y^{2n}\operatorname{sech}(y^{2n+1})\ ; \tag{11c}$$

the source density q is computed from Eq. (10a). At small y, B_n and Φ_n vanish respectively as r^n and r^{2n} while at large y, f_n and Φ_n tend respectively to one and zero exponentially fast. The field strengths are

$$E^{ai}=F^{a0i}$$
$$=-e^a\partial^i\Phi_n-[\cos(n\phi)\delta^{a2}-\sin(n\phi)\delta^{a1}]\phi^i\Phi_n B_n\ , \tag{12a}$$

$$B^{ai}=\tfrac{1}{2}\varepsilon^{ijk}F_{jk}^a=\delta_3^a\varepsilon^{ijk}\partial_j(A_n\phi_k)\ . \tag{12b}$$

At large distances, E^{ai} vanishes exponentially fast whereas B^{ai} is of the order y^{-n-1}. Thus solution (11b) and (11c) is a magnetic multipole solution.

The charge distribution $q(\mathbf{x})$, calculated from Eq. (10a), is regular, vanishes at origin, and decreases exponentially fast at large y. The total charge of the external source projected along the $e^a(\mathbf{x})$ direction is finite:

$$\begin{aligned}Q&=\int d^3x\, j_0^a(\mathbf{x})e^a(\mathbf{x})\\&=\int d^3x\,\Phi_n B_n^2\\&=\frac{4\sqrt{2}\pi(n+1)!}{(2n+1)(n-1)!}I_1\frac{c^2}{a^{2(n-1)}}\\&\equiv k\frac{c^2}{a^{2(n-1)}}\ ,\end{aligned} \tag{13a}$$

where

$$I_1=\int_0^\infty dy[y^{-s}\tanh^2 y\operatorname{sech}y]\ ,\quad s=\frac{2n}{2n+1}\ . \tag{13b}$$

The total energy H can be evaluated from the integral

$$\begin{aligned}H&=\tfrac{1}{2}\int d^3x[(E_i^a)^2+(B_i^a)^2]\\&=\int d^3x[\tfrac{1}{2}(\nabla\Phi_n)^2+\Phi_n^2B_n^2]\ .\end{aligned} \tag{14a}$$

One finds

$$\frac{H}{4\pi}=\frac{k_1}{a}+\frac{k_2c^2}{a^{n-1}} \qquad (14b)$$

with

$$k_1=4n^2(2n+1)I_2-4n(2n+1)^2I_3+(2n+1)^3I_4 ,$$

$$k_2=\frac{2}{3}\frac{(n+1)!}{(n-1)!} ,$$

and

$$I_2=\int_0^\infty dy\, y^s \text{sech}^2 y ,$$

$$I_3=\int_0^\infty dy\, y^{s+1} \tanh y\, \text{sech}^2 y ,$$

$$I_4=\int_0^\infty dy\, y^{s+2} \tanh^2 y\, \text{sech}^2 y .$$

All the above integrals are finite and hence the total energy H has a finite value. Note that Q and H are parametrized by c and a.

III. BIFURCATION

We now proceed to show that in the energy H vs total charge Q plot, branching is possible. As a working definition bifurcation is said to occur if there exists at least a common point in the parametric space at which H and Q have their respective stationary values [8]. From expressions (13a) and (14b) and for our purpose, it is sufficient to impose a linear relation between the parameters c and a,

$$c=\alpha a-\beta , \qquad (15)$$

where α and β are constants to be determined so that Q and H can have their respective extrema occurring at the same parametric value. Setting the first derivative of Q with respect to the parameter a equal to zero, we obtain the condition

$$a=\left[\frac{n-1}{n-2}\right]\frac{\beta}{\alpha}, \quad n>2 . \qquad (16a)$$

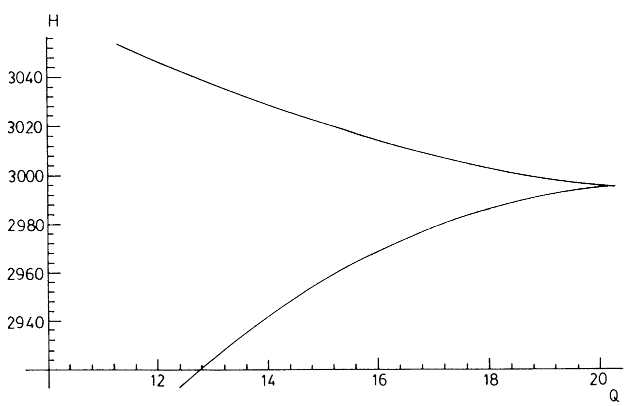

FIG. 1. Plot of H vs Q for the solution (11) when $n=4$, $\alpha=8.4502$, $\beta=10$, $a_c=1.7751$, and the parameter a ranges from 1.4 to 2.8.

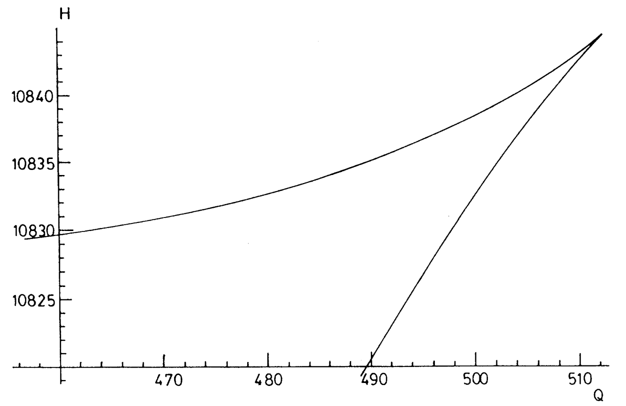

FIG. 2. Plot of H vs Q for the solution (11) when $n=5$, $\alpha=10$, $\beta=6.3827$, $a_c=0.85103$, and the parameter a ranges from 0.78 to 0.91.

The vanishing of the first derivative of H with respect to the parameter a leads to

$$ck_2[(n-1)\beta+\alpha(3-n)a]=k_1a^{n-2} ,$$

and on substituting a from expression (16a), we get

$$k_2\frac{n-1}{(n-2)^2}\beta^2=k_1\left[\frac{n-1}{n-2}\frac{\beta}{\alpha}\right]^{n-2} . \qquad (16b)$$

Thus for a given value of β, α can be computed from Eq. (16b); the critical values a_c and c_c are then calculated from Eqs. (16a) and (15), respectively. It is straightforward to show that at $a=a_c$ and $c=c_c$, Q has a local maximum for $n>1$,

$$Q_{\max}=4\sqrt{2}\pi k(\alpha a_c-\beta)^2a_c^{-2(n-1)} , \qquad (17a)$$

whereas H has a local maximum for $n>4$,

$$H_{\max}=4\pi\left[\frac{k_1}{a_c}+\frac{k_2(\alpha a_c-\beta)^2}{a_c^{n-1}}\right] , \qquad (17b)$$

and for $n=3$ and 4, H has a local minimum and inflection point, respectively. In Figs. 1 and 2 we exhibit the cusps in the energy H vs total charge Q plots for $n=4$ and $n=5$, respectively. In Fig. 1, $Q_{\max}=20.287$, $H_{\text{crit}}=2995.5$ and in Fig. 2, $Q_{\max}=512.31$, $H_{\max}=10\,844.3$. Note that H increases with the integer n.

IV. REMARKS

We end with some remarks

(i) Using the expressions (6) it is easy to verify that for our new solution the topological charge M of the source-field system is $2n$. Substituting the ansatz (8) into Eq. (6b) we find, after some straightforward calculation,

$$C^aC^a=\left[A_n+\frac{n}{\rho}\right]^2 . \qquad (18)$$

Using Eq. (6a) and noting $A_n = B_n + n/\rho$, one easily obtains $M = 2n$ since ρB_n vanishes at large ρ. For the choice of $e^a(\mathbf{x})$ as given by Eq. (8d) and with $A_n = B_n + n/\rho$, both terms in Eq. (6b) contribute to M. For the numerical solution of Ref. [6], ρA_n vanishes at large ρ. This means the second term in Eq. (6b) does not contribute to M and thus $M = n$. It is evident that our analytic solution and the numerical solution of Ref. [6] are not related by gauge transformation.

(ii) Gauge transforming Eqs. (8) by using

$$U = \exp\left[\frac{\pi}{4}\hat{\phi}^a\sigma^a\right] , \tag{19a}$$

$$\hat{\phi}^a = -\delta^{a1}\sin(n\phi) + \delta^{a2}\cos(n\phi) , \tag{19b}$$

where σ^a are the Pauli matrices, we obtain

$$A'^a_0 = \delta^a_3\Phi_n(\rho,z) , \tag{20a}$$

$$A'^{ai} = -\left[\left[A_n + \frac{n}{\rho}\right]e^a + \frac{n}{\rho}\delta^a_3\right]\phi^i , \tag{20b}$$

$$j'^a_\nu = \delta^a_3 q(\rho,z)\delta^0_\nu . \tag{20c}$$

Hence the source is now specified by $e'^a = \delta^a_3$, the Abelian gauge frame. One can again compute M by using Eqs. (6). One finds $M = 2n$. In the Abelian gauge frame, the first term in Eq. (6b) gives no contribution and the value of M is contributed solely by the gauge field A'^a_ϕ, namely, the term $(A_n + n/\rho)e^a$ in Eq. (20b), where e^a is given by Eq. (8d) and $A_n = B_n + n/\rho$. Similarly for the numerical solution of Ref. [6] in the Abelian gauge frame, it is the second term in Eq. (6b) that leads to $M = n$.

(iii) The solution (11) is not unique; there may exist many solutions which can also lead to branching in H vs Q plot. For $n = 1$ and 2, the linear relation (15) between the parameters c and a will not generate branching

(iv) That higher M solutions can be derived from the reduced equations (10) which are the same as Eqs. (9) for $n = 0$ is not surprising. In Ref. [10], the type-I solution with topological charge $M = 1$ is derived from the magnetic dipole solution with $M = 0$ by comparing their respective reduced equations. Note that no gauge transformation is involved.

(v) Our solutions are of type II and only lead to $M = 2n$, $n = 1,2,3,\ldots$. We are currently searching for solutions leading to odd integer values of M.

ACKNOWLEDGMENT

We thank Y. Yeo for discussions.

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