PHYSICAL REVIEW D VOLUME 47, NUMBER 4 15 FEBRUARY 1993

Quantum Brownian motion in a general environment. II. Nonlinear coupling and perturbative approach

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(Received 27 April 1992; revised manuscript received 14 September 1992)

We develop a perturbation scheme to treat the dynamics of a quantum Brownian particle (coordinate x) coupled nonlinearly to a bath of oscillators (coordinates q_n) with an interaction action in the form $\lambda f(x) q_n^k$. We derive the influence functionals for the $k=2,3,4$ cases to order λ^2 and derive the master equations for the special cases of local dissipation and white noise, as well as the general cases of nonlocal dissipation and colored noise for $f(x)=x$ and x^2. We show that a generalized fluctuation-dissipation relation exists between the lth-order noise kernels of the kth-order coupling and their corresponding dissipation kernels.

PACS number(s): 05.40.+j, 03.65.Bz, 03.65.Db

I. INTRODUCTION

In a recent paper [1] (called paper I hereafter), we treated the problem of a quantum Brownian particle interacting with a general (non-Ohmic) environment via linear coupling. We used the influence-functional [2] method to derive an exact master equation with time-dependent coefficients valid for both high and low temperatures. This master equation can be used to examine the properties of the most general linear quantum Brownian motion. The linear-coupling models have been analyzed repeatedly, not only because they are technically tractable, but also because they are often viewed as a good approximation to a number of physical situations where the system weakly interacts with the environmental degrees of freedom. In reality, many problems of interest in physics involve some form of nonlinearity in the interaction between the system and environment. The nature of this nonlinear quantum Brownian motion (QBM) has not yet been fully explored, and its analysis is the purpose of the present paper.

We were led to the study of nonlinear QBM by our interest in problems in field theory, gravitation, and cosmology (such as those related to structure formation in inflationary cosmology [3–5], anisotropy dissipation in semiclassical and quantum cosmology [6], the validity of the minisuperspace approximation in quantum cosmology [7,8], and the decoherence and back-reaction processes in the transition from quantum to semiclassical gravity [9]). All these problems, which can be approached using the basic methods of statistical mechanics [10,11], schematically involve, in one way or another, the coarse graining of interacting quantum fields (such as the inflaton in the inflationary universe or some of the gravitational-field modes in semiclassical cosmology). In contrast with problems in laboratory physics where one can prepare one's sample and, to some extent, control its interaction with the environment by conveniently isolating it, in problems involving gravity or other fundamental field theories nonlinearity is often a rule rather than an exception (and this is especially relevant in the strong-field conditions prevailing in the early Universe). There, the interaction is intrinsically fixed. One begins with the whole (closed) system of quantum fields with an infinite number of modes and chooses one's system defining the system-environment separation according to the nature of the physical problem one poses. For example, in the case of the minisuperspace approximation, the system can be regarded as the homogeneous gravitational modes, corresponding to Bianchi universes, and the environment as the inhomogeneous modes, which are the gravitational waves. In invoking the minisuperspace approximation, one ignores these modes whose average effect on the "system" (the minisuperspace sector) can be analyzed using the QBM paradigm. Indeed, they are found to introduce a "dissipative" term in the equation of motion, which turns the Wheeler-DeWitt equation into an effective equation [8]. Another example is the formation of large-scale structures via the gravitational instability mechanism. There, vacuum fluctuations of the scalar field responsible for driving the Universe through inflation are regarded as the primordial seeds of structure. According to a popular scheme, the so-called "stochastic inflation" [3], the scalar field can be separated into two sectors: Those modes with physical wavelengths longer than the horizon are regarded as the system and

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treated as a classical field, while those with wavelengths shorter than the horizon are treated as quantum fluctuations (the bath). The "system" field obeys a Langevin equation driven by a noise associated with the quantum fluctuations. This scheme involves two basic assumptions, both of which need proof: that the long-wavelength modes behave classically and that the short-wavelength quantum modes constitute an effective noise. In the original proposal of Starobinsky [3], it is argued that quantum fluctuations in a free field act as a source of white noise in the Langevin equation for the system. This approach has been widely used, and it was shown that nonlinearities (both in the inflaton potential and gravitationally induced) can produce non-Gaussian features in the distribution of fluctuations (see, for example, [4]). The role of nonlinearities in this context was also emphasized by the present authors [5], who pointed out that the coarse graining of short-wavelength modes of interacting fields will almost always generate a colored-noise source, from which non-Gaussian distributions are naturally expected.

The two above problems (semiclassical cosmology and stochastic inflation) invoke the paradigm of QBM, upgraded to the field-theoretical level and generalized to include nonlinear couplings. For these and other applications, it is therefore important to devise practical computational methods to find the effective equations for the system including the effect of the environment. As the nonlinear QBM has not been previously analyzed in any detail, the corresponding calculational tools do not exist even for simple quantum-mechanical models. The aim of this paper is to present some useful techniques that can be employed to obtain master equations in the nonlinear QBM problem. Although our motivation comes from field theory and cosmology, we will restrict our attention here to simple quantum-mechanical models. The results will be generalized to field theory and applied to early Universe problems elsewhere (see [12] for the extension to field theory and [5] for some preliminary results in cosmology).

The QBM models we will analyze in this paper have also received increasing attention in recent years, particularly in the context of the quantum-to-classical transition problem. The process of decoherence [13] by which a quantum system loses coherence as a consequence of its interaction with an external environment can be, and has been, naturally modeled and analyzed using the QBM paradigm. The basic idea here is that the interaction with the environment may induce classical behavior by dynamically suppressing the possibility of observing the system in the vast majority of possible states in its Hilbert space. In this way the environment would "choose" a preferred set of states that are relatively immune to the loss of quantum coherence [14]. If decoherence is effective, every initial state for the system would decay into a mixture of the preferred states, called "pointer states" by Zurek, after a time which is the so-called "decoherence time." Finding the pointer states and estimating the decoherence time are two nontrivial issues that, so far, have been analyzed only in simplified models. In the context of quantum-measurement theory [15,16], where the "system" is the measurement apparatus, the usual and natural assumption is to neglect the role of the system's self-Hamiltonian. Consequently, the pointer states are obviously the eigenstates of the interaction Hamiltonian (since they are precisely the states that are undisturbed by the interaction). However, if one tries to explain the existence of classical behavior as a consequence of decoherence in systems for which the self-Hamiltonian is not negligible, the above result is no longer true: Pointer states, defined through a predictability sieve as being the most stable ones, do not generally coincide with the eigenstates of the interaction Hamiltonian [14,17]. The nature of the pointer states as well as the decoherence time scale have been analyzed in various cases using models that reduce, in one way or another, to the linear QBM [15–21]. However, the role of nonlinearities has yet to be explored in this context and this work provides the basic techniques for its pursuit. This paper presents the necessary techniques for deriving the master equation for the nonlinear QBM problem. This equation, which governs the evolution of the reduced density matrix, has proven to be a very useful tool to study the decoherence process. A brief analysis of the application of our results on the calculation of the decoherence rate will be presented in the final section. Further investigations on this and related issues are in progress.

As another motivation for our work, let us mention that the techniques we use here could also be used to study the "consistent-histories" approach to quantum mechanics [22], which received renewed attention in recent years partly because of its relevance to the study of the quantum-to-classical transition. In this formulation one analyzes the validity of probability sum rules in the set of possible histories (time-ordered sequences of events, represented by projection operators) of a closed quantum system. The decoherence functional, which gives us information about the validity of the above sum rules, can be computed using a path-integral representation. There are not many realistic computations of the decoherence functional, and one of the canonical examples is, again, given by the linear QBM. There, the environment is coarse grained away and the decoherence functional for the histories of the system is computed. In the case of linear coupling at high temperatures (and simple initial conditions), the decoherence functional for histories of a particle consisting of sequences of projections onto position intervals at finite number of times has recently been computed by Dowker and Halliwell [23]. (The corrections introduced in the low-temperature regime have also been discussed in this case [24].) More recently, Gell-Mann and Hartle [25] and Brun [26] discussed the way in which, in the presence of nonlinear coupling, using the decoherence-functional, effective classical equations can be derived. These equations, containing dissipative terms generated by the environment, are obtained by analyzing the diagonal part of the decoherence functional that, provided the "consistency conditions" (also called "decoherence conditions" by Gell-Mann and Hartle) are satisfied, gives the probability measure in the space of histories. In the nonlinear case, the above authors obtained both the effective equations and the probability distribution (which gives information about the fluctuations around the most probable trajectory) using a perturbative technique that enables them to compute the Feynman-Vernon influence functional for these models. The nonlinear models and the perturbation techniques introduced by Brun, Gell-Mann, and Hartle are structurally very similar to ours (while we aimed at deriving the master equations, their works focus on the computation of the decoherence functional).

The model we study is that of a quantum Brownian oscillator nonlinearly coupled to a bath of harmonic oscillators. The form of the coupling is given by (2.3) (it can be thought of as defining vertices in Feynman-diagram language). The framework we adopt is the Feynman-Vernon influence-functional formalism [2,27,28] and the approach we take for nonlinear coupling is to carry out a perturbation expansion in powers of the coupling constant λ (up to second order). The calculation of the influence functional can be shown to correspond to the computation of an average $\langle \cdots \rangle_0$ that is defined through a path integral and can be expressed in terms of a set of basic two-point functions (which can be thought as defining the propagators in Feynman-diagram language). We introduce a set of Feynman rules to facilitate the perturbation calculations (with lines denoting the propagators and circles denoting the vertices; see Figs. 1–4). The average required to compute the influence functional is diagrammatically obtained by closing the lines upon themselves. The method is exactly identical to that of perturbation theory in the Schwinger-Keldysh (or closed-time-path) [29,30] formalism, which we have applied to field theory before [8,31]. This is not surprising, because, as we mentioned in paper I, the Feynman-Vernon influence-functional approach is formally equivalent to the Schwinger-Keldysh method [30].

There are three aspects of significance arising from these calculations. (1) The derivation of the master equations for the reduced density matrix (or the equations for the reduced Wigner function) from these influence actions for different types of nonlinear coupling. From them, one can carry out various studies of statistical dynamical processes such as decoherence and dissipation. (2) The formulation of a quantum theory of multiplicative noise [32] from the calculated form of the noise kernels. They provide the correlators of the stochastic forces and define the character of the colored noise for different types of nonlinear couplings. These results can be useful for analyzing, say, the growth of fluctuations in terms of classical stochastic dynamics [4,5]. (3) The establishment of a generalized fluctuation-dissipation relation [33,34] for nonlinear couplings of the system and the bath, which involve nonlocal dissipation and colored noise.

After treating different kinds of nonlinear couplings with nonlocal dissipation and colored noise, we discovered that a fluctuation-dissipation (FD) relation exists (3.12) between each lth-order noise kernel of the kth-order coupling and the corresponding dissipation kernel. Except for a different temperature-dependent factor, these relations have the same form for different types of couplings. The forms at high and zero temperatures are identical for both linear and nonlinear couplings. The high-temperature limit gives the famous Callen-Welton-Kubo relation [34]. The zero-temperature FD relation reflects the dissipative effects of quantum fluctuations. The generalization of such a relation to quantum fields in the cosmological context has been anticipated before [35], but the explicit form for the quantum Brownian-motion problem is given for the first time here. The insensitiveness of the FD relation to the different system-bath couplings reflects that it is a categorical relation (back reaction) between the stochastic stimuli (fluctuation noise) of the environment and the averaged response of the system (dissipation relaxation), which has a much deeper and broader meaning than that usually associated with particular cases or under special conditions.

In Sec. II we derive the influence action for quadratic, cubic, and quartic couplings. In Sec. III we discuss the general fluctuation-dissipation relations. In Sec. IV we derive the master equation for different cases, first for the Markovian regime of the nonlinear QBM (Sec. IV A) and then for the non-Markovian one (Secs. IV B and IV C). In Sec. V we summarize our results and discuss some qualitative features of the master equations of the nonlinear QBM. After comparing the equations obtained in Sec. IV with the ones obtained in the linear case in paper I, we apply them to a study of some simple aspects of the decoherence process in the Markovian regime. Generalization of this work to field theory will appear in paper III [12].

II. INFLUENCE FUNCTIONAL FOR NONLINEAR COUPLINGS

Consider a Brownian particle interacting with a thermal bath. The classical action of the Brownian particle is given by

$$S[x]=\int_0^t ds\{\tfrac{1}{2}M\dot{x}^2-V(x)\}\ , \tag{2.1}$$

while the environment consists of a set of harmonic oscillators with the classical action

$$S_b[\{q_n\}]=\int_0^t ds\sum_n\{\tfrac{1}{2}m_n\dot{q}_n^2-\tfrac{1}{2}m_n\omega_n^2q_n^2\}\ . \tag{2.2}$$

We will assume that the action for the system-environment interaction has the form

$$\begin{aligned}S_{\text{int}}[x,\{q_n\}]&=\int_0^t ds\sum_n\{-\lambda C_n f(x)q_n^k\}\\&=\int_0^t ds\sum_n v_n(x)q_n^k\ ,\end{aligned} \tag{2.3}$$

where k is an integer and $v_n(x)=-\lambda C_n f(x)$ will play the role of a vertex function in the Feynman rules below. We have added a new dimensionless couple constant λ, which will later be taken as a small parameter to facilitate perturbative expansions.

Following Feynman and Vernon [2], we introduce the evolution operator of the reduced density matrix defined by the relation

$$\rho_r(x_f,x_f',t)=\int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_i' J_r(x_f,x_f',t|x,x_i',0)\rho_r(x_i,x_i',0) . \tag{2.4}$$

This operator has a path-integral representation

$$J_r(x_f,x_f',t|x_i,x_i',0)=\int_{x_i}^{x_f}Dx\int_{x_i'}^{x_f'}Dx' \exp\left[\frac{i}{\hbar}\{S[x]-S[x']\}\right]F[x,x'] , \tag{2.5}$$

where

$$F[x,x']=\prod_n N_n\int_{-\infty}^{+\infty}dq_{nf}\int_{-\infty}^{+\infty}dq_{ni}\int_{-\infty}^{+\infty}dq_{ni}'\rho_b(\{q_{ni}\},\{q_{ni}'\},0) \times\int_{q_{ni}}^{q_{nf}}Dq_n\int_{q_{ni}'}^{q_{nf}}Dq_n'\exp\left[\frac{i}{\hbar}(S_b[\{q_n\}]+S_{\rm int}[x,\{q_n\}] -S_b[\{q_n'\}]-S_{\rm int}[x',\{q_n'\}])\right] \tag{2.6}$$

is the influence functional produced by the environment. In (2.6) we have already assumed that the initial density matrix is factorized as a product of the (reduced) density matrix $\hat{\rho}_r$ of the Brownian particle and that of the bath $\hat{\rho}_b$, i.e.,

$$\hat{\rho}(0)=\hat{\rho}_r(0)\hat{\rho}_b(0) . \tag{2.7}$$

We will also assume that initially the bath is in a thermal-equilibrium state at temperature $T=(k_B\beta)^{-1}$:

$$\rho_b(\{q_{ni}\},\{q_{ni}'\},0)=\prod_n\rho_n(q_{ni},q_{ni}',0)=\prod_n\langle q_{ni}|\exp\{-\beta\hat{H}_n\}|q_{ni}'\rangle . \tag{2.8}$$

Under the above assumptions, the influence functional can be written as

$$F[x,x']=\prod_n F_n[x,x'] , \tag{2.9}$$

where the contribution of the nth bath oscillator is

$$F_n[x,x']=N_n\int_{-\infty}^{+\infty}dq_{nf}\int_{-\infty}^{+\infty}dq_{ni}\int_{-\infty}^{+\infty}dq_{ni}'\rho_b(q_{ni},q_{ni}',0)\int_{q_{ni}}^{q_{nf}}Dq_n\int_{q_{ni}'}^{q_{nf}}Dq_n'\exp\left[\frac{i}{\hbar}\{S_b[q_n]+S_{\rm int}[x,q_n] -S_b[q_n']-S_{\rm int}[x',q_n']\}\right]$$

$$=\exp\left[\frac{i}{\hbar}\delta A_n[x,x']\right] , \tag{2.10}$$

and $\delta A_n[x,x']$ is what we have called the influence action. The total influence action is

$$\delta A[x,x']=\sum_n\delta A_n[x,x'] . \tag{2.11}$$

Note that the normalization constant N_n in (2.6) and (2.10) is chosen so that if the interaction term is zero, then the influence functional is equal to unity and the influence action vanishes.

It is clear that for nonlinear couplings the above path integral cannot be computed exactly. However, if the coupling constant λ is small, one can compute (2.10) perturbatively in orders of λ. For this purpose we introduce the influence functional of an environment where the bath oscillators are linearly coupled to the coordinate of the system:

$$F_n^{(1)}[J,J']=\int_{-\infty}^{+\infty}dq_{nf}\int_{-\infty}^{+\infty}dq_{ni}\int_{-\infty}^{+\infty}dq_{ni}'\rho_b(q_{ni},q_{ni}',0)\int_{q_{ni}}^{q_{nf}}Dq_n\int_{q_{ni}'}^{q_{nf}}Dq_n'\exp\left[\frac{i}{\hbar}\left\{S_b[q_n]+\int_0^t ds\, J(s)q_n(s) -S_b[q_n']-\int_0^t ds\, J'(s)q_n'(s)\right\}\right]$$

$$=\left\langle\exp\left[\frac{i}{\hbar}\left[\int_0^t ds\, J(s)q_n(s)-\int_0^t ds\, J'(s)q_n'(s)\right]\right]\right\rangle_0 , \tag{2.12}$$

where the average $\langle Q[q_n,q_n']\rangle_0$ of any function of the bath variables is defined as

$$\langle Q[q_n,q'_n]\rangle_0=N_n\int_{-\infty}^{+\infty}dq_{nf}\int_{-\infty}^{+\infty}dq_{ni}\int_{-\infty}^{+\infty}dq'_{ni}\rho_b(q_{ni},q'_{ni},0)\int_{q_{ni}}^{q_{nf}}Dq_n\int_{q'_{ni}}^{q_{nf}}Dq'_n\exp\left[\frac{i}{\hbar}\{S_b[q_n]-S_b[q'_n]\}\right]Q[q_n,q'_n] . \tag{2.13}$$

In terms of $F_n^{(1)}$, this average can also be obtained as

$$\langle Q[q_n,q'_n]\rangle_0=Q\left[\frac{\hbar}{i}\frac{\delta}{\delta J(s)},-\frac{\hbar}{i}\frac{\delta}{\delta J'(s)}\right]F_n^{(1)}[J,J']\bigg|_{J=J'=0} . \tag{2.14}$$

Using these definitions, the influence functional is related to $F^{(1)}$ by

$$\begin{aligned}F[x,x']&=\left\langle\exp\left[\frac{i}{\hbar}\{S_{\rm int}[x,q_n]-S_{\rm int}[x',q'_n]\}\right]\right\rangle_0\\&=\exp\left\{\frac{i}{\hbar}\left[S_{\rm int}\left[x,\frac{\hbar}{i}\frac{\delta}{\delta J}\right]-S_{\rm int}\left[x',-\frac{\hbar}{i}\frac{\delta}{\delta J'}\right]\right]\right\}F^{(1)}[J,J']\bigg|_{J=J'=0} .\end{aligned} \tag{2.15}$$

If we expand the exponential in this equation, we obtain a perturbative expansion for the influence functional. The result, up to the second order of λ, is given by

$$\begin{aligned}\delta A_n[x,x']=&\{\langle S_{\rm int}[x,q_n]\rangle_0-\langle S_{\rm int}[x',q'_n]\rangle_0\}+\frac{i}{2\hbar}\{\langle(S_{\rm int}[x,q_n])^2\rangle_0-(\langle S_{\rm int}[x,q_n]\rangle_0)^2\}\\&-\frac{i}{\hbar}\{\langle S_{\rm int}[x,q_n]S_{\rm int}[x',q'_n]\rangle_0-\langle S_{\rm int}[x,q_n]\rangle_0\langle S_{\rm int}[x',q'_n]\rangle_0\}\\&+\frac{i}{2\hbar}\{\langle(S_{\rm int}[x',q'_n])^2\rangle_0-(\langle S_{\rm int}[x',q'_n]\rangle_0)^2\} .\end{aligned} \tag{2.16}$$

Each term in (2.16) can be computed in terms of the unperturbed influence functional $F_n^{(1)}[J,J']$, which has the exact form [29,30]

$$\begin{aligned}F_n^{(1)}[J,J']=\exp\Bigg\{&-\frac{i}{\hbar}\int_0^t ds_1\int_0^{s_1}ds_2[J(s_1)-J'(s_1)]\eta_n^{(1)}(s_1-s_2)[J(s_2)+J'(s_2)]\\&-\frac{1}{\hbar}\int_0^t ds_1\int_0^{s_1}ds_2[J(s_1)-J'(s_1)]\nu_n^{(1)}(s_1-s_2)[J(s_2)-J'(s_2)]\Bigg\} .\end{aligned} \tag{2.17}$$

Here $\eta_n^{(1)}(s)$ and $\nu_n^{(1)}(s)$ are the dissipation and noise kernels given, respectively, by

$$\eta_n^{(1)}(s)=-\frac{1}{2m_n\omega_n}\sin\omega_n s , \tag{2.18a}$$

$$\nu_n^{(1)}(s)=\frac{1}{2m_n\omega_n}z\cos\omega_n s , \tag{2.18b}$$

and the parameter z denotes

$$z=\coth\tfrac{1}{2}\beta\hbar\omega . \tag{2.18c}$$

We are using here a superscript (1) to indicate that the influence functional, and the kernels are those corresponding to the linear-coupling case we have already encountered in paper I. They act here as the unperturbed quantities from which we construct the higher-order nonlinear-dissipation and noise kernels.

From the known form of the unperturbed influence functional $F^{(1)}[J_1,J_2]$ given by (2.17), one can easily show that

$$\langle q_n(s)\rangle_0=\langle q'_n(s)\rangle_0=0 \tag{2.19}$$

and compute the two-point functions

$$\langle q_n(s_1)q_n(s_2)\rangle_0=-i\hbar\{-\eta_n^{(1)}(s_1-s_2)\mathrm{sgn}(s_1-s_2)+i\nu_n^{(1)}(s_1-s_2)\} , \tag{2.20a}$$

$$\langle q'_n(s_1)q'_n(s_2)\rangle_0=-i\hbar\{+\eta_n^{(1)}(s_1-s_2)\mathrm{sgn}(s_1-s_2)+i\nu_n^{(1)}(s_1-s_2)\} , \tag{2.20b}$$

$$\langle q_n(s_1)q'_n(s_2)\rangle_0=-i\hbar\{\eta_n^{(1)}(s_1-s_2)+i\nu_n^{(1)}(s_1-s_2)\} , \tag{2.20c}$$

where the sign function is $\mathrm{sgn}(s)=s/|s|$.

One can also introduce Feynman diagrams to facilitate the perturbative calculation: Equations (2.20a)–(2.20c) define the bath propagators, which in Fig. 1 are denoted by a wavy line, a dashed line, and a wavy-dashed line, respectively. (We reserve a solid line for system propagators if needed.) The coupling terms (vertices) $v(x)$ and $v(x')$ are depicted by a solid and an open circle on the Feynman graphs accompanied by v_1 and v'_2, etc. The

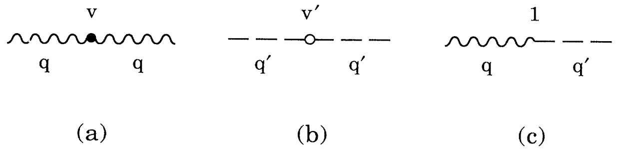

FIG. 1. Propagators for the bath variables qq, $q'q'$, and qq' are denoted in (a), (b), and (c), respectively, by a wavy, a dashed, and a wavy-dashed line. The vertices v and v' are denoted by a solid and an open circle, respectively.

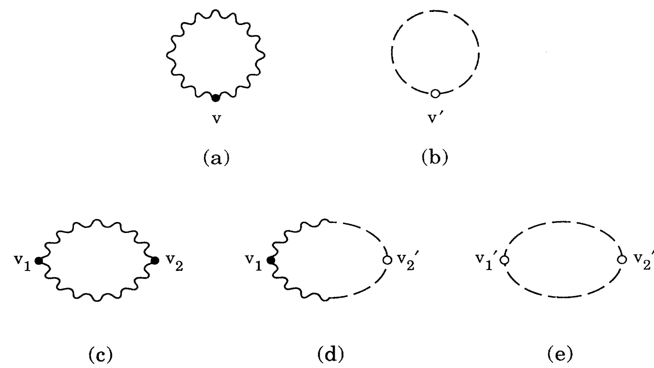

FIG. 2. Quadratic coupling diagrams: v_1, v_2' denote $v(x(s_1)), v(x'(s_2))$, etc. (a) and (b) depict Eq. (2.22) and its primed identical; (c) and (d) depict Eqs. (2.23) and (2.25); and (e) depicts the primed identical of (2.23).

subscripts 1,2 denote time sequence s_1, s_2 in the variables. The average $\langle \cdots \rangle_0$ is obtained by closing the bath-propagator lines upon themselves. These Feynman rules can be shown to be identical to those of the Schwinger-Keldysh (or closed-time-path or "in-in") formalism [29]. The influence action is obtained by adding all the one-particle-irreducible vacuum diagrams. For the $\lambda\phi^4$ theory, the diagrams have been calculated before for the construction of the coarse-grained effective action in the in-out (Schwinger-DeWitt) [11] and the in-in (Schwinger-Keldysh) formalisms [8,31].

In the following subsections, we discuss separately the cases where $f(x)$ is coupled to q_n^k with $k=2,3,4$, i.e., quadratic, cubic, and quartic in the environmental coordinates. The use of the Feynman rules and the perturbative method is amply exemplified in the quadratic case, for which we will also derive the master equation in Sec. IV. Explicit forms of the influence functional for the other two cases are also derived, not so much for the illustration of the technique, as it is a straightforward extension, but mainly for the convenience of readers who may want to use these results directly for their particular problems.

A. Quadratic coupling

For an interaction of the form (2.3) with $k=2$, i.e., with vertex function

$$v_n(x)q_n^2 = -\lambda C_n f(x) q_n^2 , \tag{2.21}$$

we find by using the above equations that the term of order λ in (2.16) introduces a renormalization of the potential, which is [Figs. 2(a) and 2(b)]

$$\langle S_{\rm int}[x,q_n]\rangle_0 = \int_0^t ds\{-\delta V_n(x)\} , \tag{2.22}$$

where

$$\delta V_n(x) = \delta\lambda_n f(x) = \hbar\lambda C_n \nu_n^{(1)}(0) f(x) = \frac{\hbar\lambda C_n}{2m_n\omega_n} z f(x) .$$

The contribution of terms in the second set of curly brackets in (2.16) is nonlocal in time and given by [Fig. 2(c)]

$$\frac{i}{2\hbar}\{\langle (S_{\rm int}[x,q_n])^2\rangle_0 - \langle S_{\rm int}[x,q_n]\rangle_0^2\} = \int_0^t ds_1 \int_0^t ds_2 \tfrac{1}{2}\lambda^2 f(x(s_1))\{-\eta_n^{(2)}(s_1-s_2)\mathrm{sgn}(s_1-s_2) + i\nu_n^{(2)}(s_1-s_2)\} f(x(s_2)) , \tag{2.23}$$

where

$$\eta_n^{(2)}(s) = 4C_n^2 \eta_n^{(1)}(s)\nu_n^{(1)}(s) = -2\frac{C_n^2}{(2m_n\omega_n)^2} z \sin(2\omega_n s) \tag{2.24a}$$

and

$$\nu_n^{(2)}(s) = 2C_n^2\{[\nu_n^{(1)}(s)]^2 - [\eta_n^{(1)}(s)]^2\} = \frac{C_n^2}{(2m_n\omega_n)^2}\{(z^2+1)\cos(2\omega_n s) + (z^2-1)\} . \tag{2.24b}$$

Terms in the last set of curly brackets in (2.16) are obtained by replacing x and x' in (2.23) [Fig. 2(d)]. The ("mixed") terms in the third set of curly brackets in (2.16) become [Fig. 2(e)]

$$-\frac{i}{\hbar}\{\langle S_{\rm int}[x,q_n] S_{\rm int}[x',q_n']\rangle_0 - \langle S_{\rm int}[x,q_n]\rangle_0 \langle S_{\rm int}[x',q_n']\rangle_0\} = \int_0^t ds_1 \int_0^t ds_2 \lambda^2 f(x(s_1))\{\eta_n^{(2)}(s_1-s_2) - i\nu_n^{(2)}(s_1-s_2)\} f(x'(s_2)) . \tag{2.25}$$

Adding up these contributions (diagrams $a-b+c+d-2e$), the influence action for the quantum Brownian motion with quadratic coupling to a general environment is, to second order in λ, given by

$$\delta A[x,x']=\sum_n \delta A_n[x,x']=\int_0^t ds\{-\delta V(x)\}-\int_0^t ds\{-\delta V(x')\}$$
$$-\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\eta^{(2)}(s_1-s_2)[f(x(s_2))+f(x'(s_2))]$$
$$+i\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\nu^{(2)}(s_1-s_2)[f(x(s_2))-f(x'(s_2))]\ , \qquad (2.26)$$

where

$$\delta V(x)=\sum_n \delta V_n(x)\ ,\quad \eta^{(2)}(s)=\sum_n \eta_n^{(2)}(s)\ ,\quad \nu^{(2)}(s)=\sum_n \nu_n^{(2)}(s) \qquad (2.27)$$

are the potential shift, dissipation kernel, and noise kernel, respectively.

B. Cubic coupling

We now consider an interaction of the form (2.3) with $k=3$, i.e., with a vertex function [Figs. 3(a) and 3(b)],

$$v_n(x)q_n^3=-\lambda C_n f(x)q_n^3\ . \qquad (2.28)$$

The one-loop terms are identically zero because the average of q_n^k with odd k vanishes (this manifests diagrammatically in the fact that there is no way to form a one-loop closed diagram with an odd number of legs in its vertex):

$$\langle S_{\rm int}[x,q_n]\rangle_0=\langle S_{\rm int}[x',q_n']\rangle_0=0\ . \qquad (2.29)$$

For the quadratic terms in (2.16) [Figs. 3(c) and 3(d)],

$$\frac{i}{2\hbar}\{\langle(S_{\rm int}^2[x,q_n])^2\rangle_0-(\langle S_{\rm int}[x,q_n]\rangle_0)^2\}$$
$$=\int_0^t ds_1\int_0^t ds_2\tfrac{1}{2}\lambda^2 f(x(s_1))\{-\eta_n^{(3)}(s_1-s_2){\rm sgn}(s_1-s_2)+i\nu_n^{(3)}(s_1-s_2)\}f(x(s_2))\ , \qquad (2.30)$$

where

$$\eta_n^{(3)}(s)=-\frac{3\hbar\lambda^2C_n^2}{2(2m_n\omega_n)^3}\{(9z^2-3)\sin\omega_n s+(3z^2+1)\sin(3\omega_n s)\} \qquad (2.31a)$$

and

$$\nu_n^{(3)}(s)=\frac{3\hbar\lambda^2C_n^2}{2(2m_n\omega_n)^3}z\{(z^2+3)\cos\omega_n s+(9z^2-3)\cos(3\omega_n s)\}\ . \qquad (2.31b)$$

The last term is obtained by replacing x by x' in (2.30) [Figs. 3(e) and 3(f)]. The mixed term in (2.16) [Figs. 3(g) and 3(h)] is

$$-\frac{i}{\hbar}\{\langle S_{\rm int}[x,q_n]S_{\rm int}[x',q_n']\rangle_0-\langle S_{\rm int}[x,q_n]\rangle_0\langle S_{\rm int}[x',q_n']\rangle_0\}$$
$$=\int_0^t ds_1\int_0^t ds_2\lambda^2 f(x(s_1))\{\eta_n^{(3)}(s_1-s_2)-i\nu_n^{(3)}(s_1-s_2)\}f(x'(s_2))\ . \qquad (2.32)$$

Adding these contributions, we get the following influence action up to the second order in λ:

$$\delta A[x,x']=\sum_n \delta A_n[x,x']$$
$$=-\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\eta^{(3)}(s_1-s_2)[f(x(s_2))+f(x'(s_2))]$$
$$+i\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\nu^{(3)}(s_1-s_2)[f(x(s_2))-f(x'(s_2))]\ , \qquad (2.33)$$

where the dissipation kernel $\eta^{(3)}(s)$ and the noise kernel $\nu^{(3)}(s)$ are obtained by summing (2.31) over all the oscillators of the environment.

C. Quartic coupling

For an interaction of the form (2.3) with q^4 coupling, i.e., with a vertex function [Figs. 4(a) and 4(b)],

$$v_n(x)q_n^4=-\lambda C_n f(x)q_n^4\ . \qquad (2.34)$$

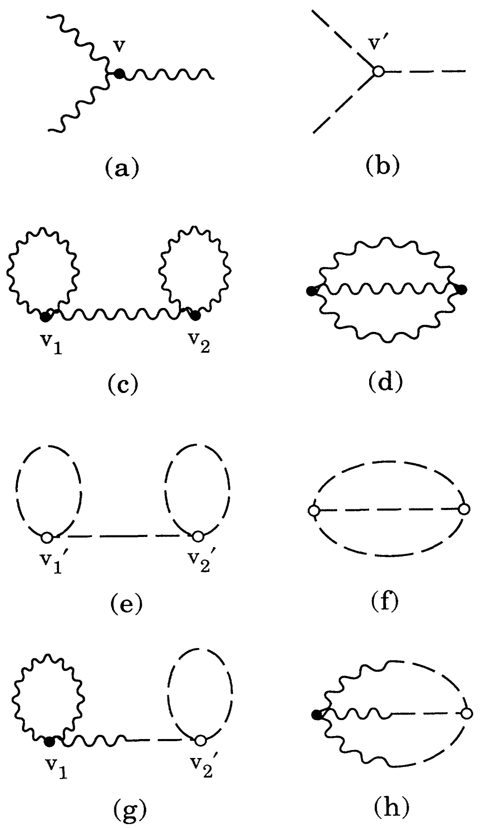

FIG. 3. Cubic coupling diagrams.

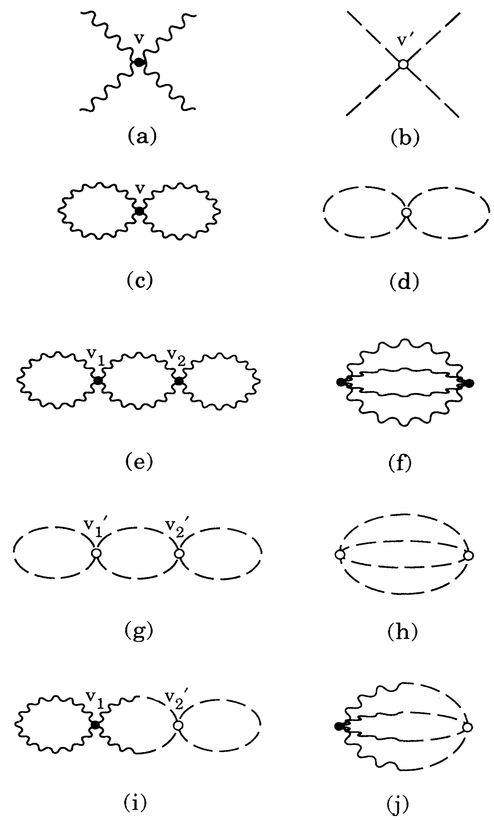

FIG. 4. Quartic coupling diagrams.

The lowest-order terms introduce a renormalization of the potential

$$\langle S_{\text{int}}[x,q_n]\rangle_0=\int_0^t ds\{-\delta V_n(x)\}\ , \tag{2.35}$$

where

$$\delta V_n(x)=\delta\lambda_n f(x)=4\hbar^2\lambda C_n(2m_n\omega_n)^2 z^2 f(x)\ . \tag{2.36}$$

The terms second order in λ are depicted by Figs. 4(c)–4(j). The complete result for the influence action is

$$\begin{aligned}\delta A[x,x']&=\sum_n \delta A_n[x,x']\\&=\int_0^t ds\{-\delta V(x(s))\}-\int_0^t ds\{-\delta V(x'(s))\}\\&\quad-\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\eta^{(4)}(s_1-s_2)[f(x(s_2))+f(x'(s_2))]\\&\quad+i\int_0^t ds_1\int_0^{s_1} ds_2\lambda^2[f(x(s_1))-f(x'(s_1))]\nu^{(4)}(s_1-s_2)[f(x(s_2))-f(x'(s_2))]\ ,\end{aligned} \tag{2.37}$$

where the dissipation and noise kernels are given, respectively, by

$$\eta_n^{(4)}(s)=-24\hbar^2\frac{C_n^2}{(2m_n\omega_n)^4}z\{(2z^2-1)\sin 2\omega_n s+\tfrac{1}{2}(z^2+1)\sin 4\omega_n s\}\ , \tag{2.38a}$$

$$\nu_n^{(4)}(s)=24\hbar^2\frac{C_n^2}{(2m_\omega\omega_n)^4}\{\tfrac{1}{8}(3-10z^2+7z^4)+\tfrac{1}{2}(3z^4+z^2-1)\cos 2\omega_n s+\tfrac{1}{8}(1+6z^2+z^4)\cos 4\omega_n s\}\ . \tag{2.38b}$$

III. FLUCTUATION-DISSIPATION RELATION

In this section we will discuss the properties of the dissipation and noise kernels for the three nonlinear-coupling cases. We will analyze their high- and low-temperature behavior and deduce a general fluctuation-dissipation relation valid for all cases.

It is convenient to write the kernels $\eta^{(k)}(s)$ in the real part of the influence action (2.26), (2.33), and (2.37) as

$$\eta^{(k)}(s)=\frac{\partial}{\partial s}\gamma^{(k)}(s)\ , \tag{3.1}$$

where the kernels $\gamma^{(k)}(s)$ are the dissipation kernels that will appear in the fluctuation-dissipation relations below. By examining the explicit expressions for these kernels given in (2.24a), (2.31a), and (2.38a), we realize that $\gamma^{(k)}(s)$ can be written as a sum of various contributions:

$$\gamma^{(k)}(s)=\sum_l \gamma_l^{(k)}(s)\ , \tag{3.2}$$

where the sum is over even (odd) values of l when k is even (odd). To derive the explicit forms of each dissipation kernel, it is useful to define first the spectral density functions:

$$I^{(k)}(\omega)=\sum_n \delta(\omega-\omega_n)k\pi\hbar^{k-2}\frac{\lambda^2C^2(\omega)}{(2m\omega)^k}\ . \tag{3.3}$$

In terms of these functions, the dissipation kernels can be written as

$$\gamma_l^{(k)}(s)=\int_0^{+\infty}\frac{d\omega}{\pi}\frac{1}{\omega}I^{(k)}(\omega)M_l^{(k)}(z)\cos l\omega_n s\ , \tag{3.4}$$

where $M_l^{(k)}(z)$ are temperature-dependent factors given by

$$\begin{aligned}
&M_0^{(2)}(z)=0\ ,\\
&M_2^{(2)}(z)=\tfrac{1}{2}z\ ,\\
&M_1^{(3)}(z)=\tfrac{3}{2}(3z^2-1)\ ,\\
&M_3^{(3)}(z)=\tfrac{1}{6}(3z^2+1)\ ,\\
&M_0^{(4)}(z)=0\ ,\\
&M_2^{(4)}(z)=3z(z^2-1)\ ,\\
&M_4^{(4)}(z)=\tfrac{3}{4}z(z^2+1)\ .
\end{aligned} \tag{3.5}$$

Analogously, the noise kernels $\nu^{(k)}(s)$ can also be written as a sum of various contributions:

$$\nu^{(k)}(s)=\sum_l \nu_l^{(k)}(s)\ , \tag{3.6}$$

where the sum runs again over even (odd) values of l for k even (odd). The kernels $\nu_l^{(k)}(s)$ can be written as

$$\nu_l^{(k)}=\hbar\int_0^{+\infty}\frac{d\omega}{\pi}I^{(k)}(\omega)N_l^{(k)}(z)\cos l\omega s\ , \tag{3.7}$$

where the temperature-dependent factors $N_l^{(k)}(z)$ are given by

$$\begin{aligned}
&N_0^{(2)}(z)=\tfrac{1}{2}(z^2-1)\ ,\\
&N_2^{(2)}(z)=\tfrac{1}{2}(z^2+1)\ ,\\
&N_1^{(3)}(z)=\tfrac{3}{2}z(3z^2-1)\ ,\\
&N_3^{(3)}(z)=\tfrac{1}{2}z(z^2+3)\ ,\\
&N_0^{(4)}(z)=\tfrac{1}{8}(7z^4-10z^2+3)\ ,\\
&N_2^{(4)}(z)=\tfrac{1}{2}(3z^4+z^2-2)\ ,\\
&N_4^{(4)}(z)=\tfrac{1}{8}(z^4+6z^2+1)\ .
\end{aligned} \tag{3.8}$$

To understand the physical meaning of the noise kernels of different orders, we can think of them as being associated with l independent stochastic sources that are coupled to the Brownian particle through interaction terms of the form

$$\int_0^t ds\sum_l \xi_l^{(k)}(s)f(x)\ . \tag{3.9}$$

This type of coupling generates a stochastic force in the associated Langevin equation:

$$F_{\xi_l}^{(k)}(s)=-\xi_l^{(k)}(s)f'(x)\ ,$$

which corresponds to multiplicative noise [34,36]. The stochastic sources $\xi_l^{(k)}$ have the probability distribution

$$P[\xi_l^{(k)}]=N_l^{(k)}\exp\left\{-\frac{1}{\hbar}\int_0^t ds_1\int_0^t ds_2\tfrac{1}{2}\xi_l^{(k)}(s_1)\times\nu_l^k(x_1-s_2)^{-1}\times\xi_l^{(k)}(s_2)\right\}, \tag{3.10}$$

which generates the correlation functions

$$\begin{aligned}
&\langle\xi_l^{(k)}(s)\rangle=0\ ,\\
&\langle\xi_l^{(k)}(s)\xi_l^{(k)}(s')\rangle=\nu(s-s')\ .
\end{aligned} \tag{3.11}$$

To every stochastic source we can associate a dissipative term that is present in the real part of the influence action. The dissipative and noise kernels are related by generalized fluctuation-dissipation relations of the form

$$\nu_l^{(k)}(t)=\int_{-\infty}^{+\infty}ds\,K_l^{(k)}(t-s)\gamma_l^{(k)}(s)\ , \tag{3.12}$$

where the kernel $K_l^{(k)}(s)$ is

$$K_l^{(k)}(s)=\int_0^{+\infty}\frac{d\omega}{\pi}L_l^{(k)}(z)l\omega\cos l\omega s \tag{3.13}$$

and the temperature-dependent factor $L_l^{(k)}$ is given by

$$L_l^{(k)}(z)=\frac{N_l^{(k)}(z)}{M_l^{(k)}(z)}\ . \tag{3.14}$$

A fluctuation-dissipation relation of the form (3.12) exists for the linear case where the temperature-dependent factor appearing in (3.13) is simply $L^{(1)}=z$. The fluctuation-dissipation kernels $K_l^{(k)}$ have rather complicated forms except in some special cases. In the high-temperature limit, which is characterized by the condition $k_B T \gg \hbar\Gamma$, where Γ is the cutoff frequency of the environment, we can replace

$$z = \coth\frac{1}{2}\beta\hbar\omega \to \frac{2}{\beta\hbar\omega} , \tag{3.15}$$

and we obtain

$$L_l^{(k)}(z) \to \frac{2k_B T}{\hbar}\frac{1}{\omega} . \tag{3.16}$$

In the limit $\Gamma \to +\infty$, we get the general result

$$K_l^{(k)}(s) = \frac{2k_B T}{\hbar}\delta(s) , \tag{3.17}$$

which tells us that at high temperature there is only one form of fluctuation-dissipation relation, which is the famous Kubo relation [2,27]

$$\nu_l^{(k)}(s) = \frac{2k_B T}{\hbar}\gamma_l^{(k)}(s) . \tag{3.18}$$

In the zero-temperature limit, characterized by $z \to 1$, we have

$$L_l^{(k)}(z) \to l . \tag{3.19}$$

The fluctuation-dissipation kernel becomes k independent and hence identical to the one for the linear-coupled case:

$$K(s) = \int_0^{+\infty}\frac{d\omega}{\pi}\omega\cos\omega s . \tag{3.20}$$

It is interesting to note that the fluctuation-dissipation relations for the linear- and nonlinear-dissipation models are exactly identical both in the high- and zero-temperature limits. In other words, the fluctuation-dissipation relation is not very sensitive to the different system-bath couplings at both high- and zero-temperature limits. It reflects a categorical relation (back reaction) between the stochastic stimulation (fluctuation noise) of the environment and the averaged response of a system (dissipation) which has a much deeper and universal meaning than that manifested in specific cases or under special conditions.

A given environment is characterized by the spectral densities $I^{(k)}(\omega)$, and it is clear that if these functions are appropriately chosen, the form of the noise and dissipation kernels can be simplified considerably. For example, if the spectral density is

$$I^{(k)}(\omega) \sim \omega^k , \tag{3.21}$$

the noise and dissipation kernels become local kernels in the high-temperature limit. In that case we have

$$\gamma_l^{(k)}(s) \sim (k_B T)^{k-1}\delta(s) \tag{3.22}$$

and

$$\nu_l^{(k)}(s) \sim (k_B T)^k\delta(s) . \tag{3.23}$$

Note that (3.22) depends upon the temperature and will produce a temperature-dependent friction term in the effective equations of motion. On the other hand, if the spectral density is the same linear function for all (k),

$$I^{(k)}(\omega) \sim \omega , \tag{3.24}$$

the dissipation kernel becomes local in the low-temperature limit,

$$\gamma^{(k)} \sim \delta(s) , \tag{3.25}$$

but the noise remains colored because of the nontrivial fluctuation-dissipation relation (3.12).

A peculiarity of our results is that the noise sources $\xi_0^{(2)}(s)$ and $\xi_0^{(4)}(s)$ given by (3.11) have no dissipation counterparts ($\gamma_0^{(2)} = \gamma_0^{(4)} = 0$), and there is no way to form any kind of fluctuation-dissipation relation. In these cases the noise correlation function is constant, which means that the Fourier transform of $\xi_0^{(2k)}(\omega)$ is a random variable with a white-noise correlation function: i.e.,

$$\begin{aligned} &\langle \xi_0^{(k)}(\omega) \rangle_{\xi_0^{(k)}} = 0 , \\ &\langle \xi_0^{(k)}(\omega_1)\xi_0^{(k)}(\omega_2) \rangle_{\xi_0^{(k)}} = \gamma_f\delta(\omega_1 - \omega_2) . \end{aligned} \tag{3.26}$$

It is also worth noting that $\xi_0^{(k)}(s)$ vanishes in the low-temperature limit.

IV. QUANTUM MASTER EQUATION

In this section we derive the quantum master equation for three special cases that belong to those classes of models we have described above. We analyze first the case in which the dissipation and noise kernels are local. In this limit we can derive a master equation without assuming any particular form for the function $f(x)$ in the interaction term between the system and bath [see Eq. (2.3)]. The second case we analyze corresponds to a coupling that is linear in the system [i.e., $f(x) = x$] but nonlinear in the environment coordinates. For an arbitrary spectral density, this will produce nonlocal dissipation and colored noise. We finally analyze the case in which the coupling is quadratic in the system, i.e., $f(x) = x^2$, and nonlinear in the bath. The techniques we use to derive the master equations are based on the path-integral representation of the evolution operator given in (2.5) and are similar to what was derived in paper I. Their use is, however, not limited to the cases discussed here since they can be generalized to treat more general couplings or higher orders in perturbation theory.

A. Local-dissipation and white-noise regime

As we have seen from the above, the dissipation and noise kernels become local in the high-temperature regime if the spectral density of the environment satisfies the condition (3.21). To be more specific, we will assume that the spectral density can be written as

$$I(\omega) = \gamma_0\omega\left[\frac{\omega}{\Lambda}\right]^{k-1} , \tag{4.1}$$

where γ_0 is going to play the role of the damping rate and Λ is a frequency scale associated with the environment (note that the dimensionless constant λ^2 has been absorbed into γ_0). Thus the above is just the typical spectral density of a supra-Ohmic environment. In paper I we studied linear models with generic spectral densities (including supra-Ohmic). It is worth noting that, for the linear case, the noise and dissipation kernels become purely local [satisfying (3.22) and (3.23)] only for the Ohmic environment in the high-temperature regime. By contrast, to get a local influence functional with nonlinear coupling, we must consider supra-Ohmic environments. To obtain the effective action that appears in the propagator (2.5), we can use Eqs. (2.26), (2.33), and (2.37) for $k=2,3,4$, respectively, and write

$$\begin{aligned} A[x,x'] &= S[x]-S[x']+\delta A[x,x'] \\ &= \int_0^t ds \left\{ \frac{1}{2}\dot{x}^2 - V_{\text{ren}}(x) - \frac{1}{2}\dot{x}'^2 + V_{\text{ren}}(x') \right. \\ &\quad \left. -2\gamma_0 \left[\frac{k_B T}{\hbar\Lambda}\right]^{k-1} [f(x)-f(x')] \left[\left[\frac{\partial f(x)}{\partial x}\dot{x} + \frac{\partial f(x)}{\partial x'}\dot{x}' \right] - i\frac{k_B T}{\hbar}[f(x)-f(x')] \right] \right\} , \end{aligned} \tag{4.2}$$

where we introduced the renormalized potential $V_{\text{ren}}=V+\delta V$ with the counterterm δV given in (2.22) and (2.35). We also neglected (as an approximation) the effect of the kernels $\nu_0^{(k)}$ for even values of k (note that these kernels vanish at low temperatures but not at high temperatures).

In this case the master equation can be derived from the path integral by using standard techniques (see, e.g., [1,27,28]). We give an alternative and simpler derivation of this equation based on the following observation. The effective action (4.2) can be regarded as the action of a quantum-mechanical problem with two degrees of freedom x and x', one of which (x') has a negative kinetic energy. The potential terms contain velocity-dependent interactions such as that between a charged particle and an electromagnetic field. Therefore one can associate a Schrödinger equation to the evolution operator in (2.4). In this equation the reduced density matrix plays the role of the "wave function" and the "effective Hamiltonian" is the one associated with the effective action (4.2):

$$i\hbar\frac{\partial}{\partial t}\rho_r(x,x',t) = \hat{H}_\rho(x,x')\rho_r(x,x',t) , \tag{4.3}$$

where

$$\begin{aligned} \hat{H}_\rho(x,x') &= H_{\text{ren}}(x) - H_{\text{ren}}(x') \\ &\quad - i\hbar\gamma_0 \left[\frac{k_B T}{\hbar\Lambda}\right]^{k-1} [f(x)-f(x')] \\ &\quad \times \left[\frac{\partial f(x)}{\partial x}\frac{\partial}{\partial x} - \frac{\partial f(x')}{\partial x'}\frac{\partial}{\partial x'} \right] \\ &\quad - i\gamma_0 \left[\frac{k_B T}{\hbar\Lambda}\right]^{k-1} \left[\frac{2k_B T}{\hbar}\right] [f(x)-f(x')]^2 , \end{aligned} \tag{4.4}$$

where the renormalized Hamiltonian is $H_{\text{ren}}(x) = (-\hbar^2/2)\partial_x^2 + V_{\text{ren}}(x)$. The last two terms are responsible for friction and diffusion, respectively, and are related by the fluctuation-dissipation relations. Their effect can also be appreciated by analyzing the equation for the Wigner function $W(X,p,t)$, which is defined as

$$W(X,p,t) = \int_{-\infty}^{+\infty} d\Delta\, \rho_r \left[X-\frac{\Delta}{2}, X+\frac{\Delta}{2}, t \right] \exp\left[\frac{i}{\hbar}p\Delta \right] . \tag{4.5}$$

In fact, from (4.3) we can derive the Wigner equation, which is, in general, of the Krammers-Moyal form [32,36] since, because of the nonlinearities, it contains higher-derivative terms. The generic form of this equation will be written in the final section. Here, we will just consider a specific case, i.e., for an anharmonic oscillator with the potential

$$V(x) = \tfrac{1}{2}\Omega_0^2 x^2 + Cx^4 , \tag{4.6}$$

and a biquadratic coupling in the system and bath coordinates, i.e.,

$$S_{\text{int}}[x,q_n] = \int_0^t ds\{-\lambda C_n x^2 q_n^2\} . \tag{4.7}$$

Then the counterterm δV is

$$\delta V(x) = \tfrac{1}{2}\delta\Omega^2 x^2 + \delta C x^4 , \tag{4.8}$$

which contains a frequency shift and a coupling-constant renormalization given by

$$\delta\Omega^2 = \hbar\int_0^{+\infty} \frac{d\omega}{\pi}\rho_D(\omega)\frac{\pi\lambda C(\omega)}{m\omega} , \tag{4.9}$$

$$\delta C = -\gamma_0\delta(0) . \tag{4.10}$$

In this case the Wigner equation is

$$\begin{aligned} \frac{\partial}{\partial t}W &= \{H_{\text{ren}}, W\}_{\text{PB}} - \frac{\hbar^2}{4!}\frac{\partial^3 V_{\text{ren}}}{\partial X^3}\frac{\partial^3 W}{\partial p^3} \\ &\quad + 2\gamma_0 X\frac{\partial}{\partial p}\left[Xp + \frac{\hbar^2}{4}\frac{\partial^2}{\partial X\,\partial p} \right] W \\ &\quad + 2k_B T\gamma_0 X^2\frac{\partial^2}{\partial p^2}W . \end{aligned} \tag{4.11}$$

The above equation has two terms with third derivatives, both originating from nonlinearities. The first one can be associated with the nonlinear potential $V_{\rm ren}$. In fact, the first two terms on the right-hand side of (4.11) give the usual Wigner equation of an isolated anharmonic oscillator. The new term with third derivatives originates from the nonlinearity in the friction term of the master equation. Indeed, it can be shown that the nonlinear friction term in the master equation (4.3) always generates a term with an odd number of derivatives in the Wigner equation (and a cross derivative). Equation (4.11) also contains a "normal" friction term (with a single p derivative) and a "normal" diffusion term (the last one); both are different from the ones obtained in paper I for the linear case. In fact, the diffusion and friction coefficients appearing in (4.11) are position dependent (both of them are proportional to X^2). This is clearly an effect associated with the multiplicative noise produced by the nonlinear nature of the coupling. Furthermore, one can see that, as a consequence of the fluctuation-dissipation relation, the dissipation and diffusion coefficients are simply related by a factor $2k_BT$.

The third derivative terms are of order $\hbar^2$ and can be therefore considered (in some sense) as quantum corrections. When these terms are neglected, the Wigner equation (now semiclassical) becomes one of the Fokker-Planck type and has the equilibrium distribution function

$$W_0(X,p,t)\sim\exp[-\beta\{\tfrac{1}{2}p^2+V_{\rm ren}(X)\}] \qquad (4.12)$$

as an asymptotic solution. Thus Eq. (4.11) describes the process of relaxation to equilibrium under the influence of nonlinear dissipation and diffusion.

B. Nonlinear dissipation and colored noise for $f(x)=x$

The model analyzed in Sec. IV A illustrates the nature of the Markovian regime of the nonlinear QBM. We will now examine the simplest example with non-Markovian features. Consider the cases where the nonlinearities are restricted to the environment variables q_n; i.e., assume that the coupling is linear in the system variables, i.e., $f(x)=x$, and that the potential is harmonic:

$$V(x)=\tfrac{1}{2}\Omega_0^2x^2\ . \qquad (4.13)$$

It is easy to realize that this case, to the order of approximation that we are using here, is very similar to the one discussed in paper I in which $S_{\rm int}=\lambda\int ds\, C_nxq_n$ [i.e., $k=1$ in (2.3)]. In fact, using the perturbative approach discussed in Sec. II, we obtain an influence functional that has the same form as the one used in paper I. The only difference is that here the dissipation and noise kernels are more complicated. However, since in paper I we derived a master equation that is valid for noise and dissipation kernels of arbitrary form, that result also applies here. Therefore, for any order of k but small λ, the master equation is given by

$$i\hbar\frac{\partial}{\partial t}\rho_r(x,x',t)=\left\{-\frac{\hbar^2}{2}\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right]+\frac{1}{2}\Omega_{\rm ren}^2(x^2-x'^2)\right\}\rho_r(x,x',t)-i\hbar\Gamma(t)(x-x')\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]\rho_r(x,x',t)$$
$$-i\Gamma(t)h(t)(x-x')^2\rho_r(x,x',t)+\hbar\Gamma(t)f(t)(x-x')\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right]\rho_r(x,x',t)\ , \qquad (4.14)$$

where

$$\Gamma(t)=\frac{d_1(t)}{2\dot u_1(t)}\ , \qquad (4.15a)$$

$$\delta\Omega^2(t)=d_2(t)-2\Gamma(t)\dot u_2(t)\ , \qquad (4.15b)$$

$$f(t)=\frac{e_2(t)}{2\Gamma(t)\dot u_2(0)}\ , \qquad (4.15c)$$

$$h(t)=\dot u_2(t)f(t)+\frac{e_1(t)}{\Gamma(t)}\ , \qquad (4.15d)$$

and

$$d_i(t)=2\int_0^t ds\,\eta^{(m)}(t-s)u_i(s)\ , \qquad (4.16a)$$

$$e_i(t)=\int_0^t ds\,\nu^{(m)}(t-s)u_i(s)\ . \qquad (4.16b)$$

In the above formulas (4.15), we only retained terms up to order λ^2. In the same spirit, the elementary functions $u_1(s)$ and $u_2(s)$ satisfy a simple equation

$$\frac{d^2}{ds^2}u_i(s)+\Omega_0^2u_i(s)=0\ , \qquad (4.17)$$

with the boundary conditions

$$u_1(s=0)=1=u_2(s=t)\ ,\quad u_1(s=t)=0=u_2(s=0)\ . \qquad (4.18)$$

An obvious solution is

$$u_1(s)=\frac{\sin\Omega_0(t-s)}{\sin\Omega_0t}\ ,\quad u_2(s)=\frac{\sin\Omega_0s}{\sin\Omega_0t}\ . \qquad (4.19)$$

Therefore the coefficients appearing in (4.14) can be explicitly written as

$$\delta\Omega^2(t)=2\int_0^t ds\frac{\partial}{\partial s}\gamma^{(m)}(s)\cos\Omega s\ , \qquad (4.20a)$$

$$\Gamma(t)=-\frac{1}{\Omega}\int_0^t ds\frac{\partial}{\partial s}\gamma^{(m)}(s)\sin\Omega s\ , \qquad (4.20b)$$

$$\Gamma(t)f(t)=\frac{1}{\Omega}\int_0^t ds\,\nu^{(m)}(s)\sin\Omega s\ , \qquad (4.20c)$$

$$\Gamma(t)h(t)=\int_0^t ds\,\nu^{(m)}(s)\cos\Omega s\ . \qquad (4.20d)$$

We conclude that in this case the master equation is identical to that of linear QBM and that the nonlinearities in the environment do not induce nonlinear behavior in the system (to the order of approximation we are using here). It is worth noting that the effect of the nonlocal dissipation and colored noise is to introduce time-dependent coefficients in the master equation and also to introduce a new "anomalous" diffusion term [the last one in (4.14)].

C. Nonlocal dissipation and colored noise for quadratic coupling cases $f=x^2$

We will now turn our attention to the analysis of a nonlinear case $[f(x)=x^2]$ that also exhibits non-Markovian behavior. We assume that the Brownian particle is an anharmonic oscillator described by (4.6) and that the system is coupled to the bath biquadratically as in (4.7). We will not make any assumption about the nature of the spectral density of the environment. Therefore the noise and dissipation kernels will be left arbitrary. The derivation of the master equation is very similar to what we presented in paper I for the linear case. It can be viewed as a generalization of (4.3) to the non-Markovian regime or, equivalently, a generalization of (4.14) to the truly nonlinear case. Thus one expects the master equation to differ from (4.3) by the existence of time-dependent coefficients and new "anomalous" diffusion terms [one of which should generalize the last term in (4.14)].

Consider the operator that propagates the reduced density matrix from the initial instant to time $t+dt$. The path integral can be decomposed in two pieces:

$$\int_{(0;x_i,x_i')}^{(t+dt;x_f,x_f')} Dx\, Dx' \exp(iA[x,x'])$$
$$=N(t)\int_{-\infty}^{+\infty} dx_m \int_{-\infty}^{+\infty} dx_m' \int_{(t;x_m,x_m')}^{(t+dt;x_f,x_f')} D\tilde{x}\, D\tilde{x}' \exp(iA[\tilde{x},\tilde{x}'])$$
$$\times \int_{(0;x_i,x_i')}^{(t;x_m,x_m')} D\bar{x}\, D\bar{x} \exp(iA[\bar{x},\bar{x}'])\exp(iA_i[\bar{x},\bar{x}',\tilde{x},\tilde{x}']) , \tag{4.21}$$

where the histories $\bar{x}(\tau)$ and $\tilde{x}(\tau)$ are functions defined, respectively, on the $(0,t)$ and $(t,t+dt)$ intervals satisfying the boundary conditions $\bar{x}(0)=x_0$, $\bar{x}(t)=x_m=\tilde{x}(t)$, and $\tilde{x}(t+dt)=x_f$. In the limit $dt\to 0$, the path integral over $(\tilde{x},\tilde{x}')$ is proportional to the value of the integrand evaluated on the "straight-line histories" defined by

$$\tilde{x}(s)=x_m+(x_f-x_m)\frac{s=t}{dt}=x_m+\beta_x\frac{s-t}{dt} . \tag{4.22}$$

With this, one can show that

$$A[x,x']\simeq\frac{1}{2dt}\beta_x^2-\frac{1}{2dt}\beta_x'^2-dt\, V(x_f)+dt\, V(x_f')+O(dt) \tag{4.23}$$

and write the mixed part of the action as

$$A_i[\bar{x},\bar{x}';\tilde{x},\tilde{x}']\simeq -dt\int_0^t ds\, J_\Sigma(s)[\bar{x}^2(s)+\bar{x}'^2(s)]+i\,dt\int_0^t ds\, J_\Delta(s)[\bar{x}^2(s)-\bar{x}'^2(s)]+O((dt)^2) , \tag{4.24}$$

where the two sources are of order λ^2 since they are proportional to the noise and dissipation kernels:

$$J_\Sigma(s)\simeq(x_f^2-x_f'^2)\eta^{(m)}(t-s)+O(dt) , \tag{4.25a}$$
$$J_\Delta(s)\simeq(x_f^2-x_f'^2)\nu^{(m)}(t-s)+O(dt) . \tag{4.25b}$$

A simple manipulation allows us to reorganize Eq. (4.21) as follows:

$$J_r(x_m,x_m',t+dt|x_i,x_i',0)\simeq(\hbar\, dt)K\int_{-\infty}^{+\infty} d\beta_x\int_{-\infty}^{+\infty} d\beta_x' \left\{1-dt\frac{i}{\hbar}[V(x_f)-V(x_f')]+dt\frac{i}{\hbar}\int_0^t ds\, J_\Sigma(s)[\langle\bar{x}^2(s)\rangle_0+\langle\bar{x}'^2(s)\rangle_0]\right.$$
$$\left.-\frac{dt}{\hbar}\int_0^t ds\, J_\Delta(s)[\langle\bar{x}^2(s)\rangle_0-\langle\bar{x}'^2(s)\rangle_0\right\}\exp\left[\frac{i}{2}[\beta_x^2-\beta_x'^2]\right]$$
$$\times J_r(x_m,x_m',t|x_i,x_i',0) , \tag{4.26}$$

where we have introduced the average

$$\langle\bar{x}^2(s)\rangle_0=N\int_{x_i}^{x_m} D\bar{x}\int_{x_i'}^{x_m'} D\bar{x}'\,\bar{x}^2(s)\exp\left[\frac{i}{\hbar}A[\bar{x},\bar{x}']\right] . \tag{4.27}$$

Using the above results, we can obtain the following differential equation for the propagator:

$$i\hbar\frac{\partial}{\partial t}J_r(x_f,x_f',t|x_i,x_i',0)=\Big\{h_{\rm ren}(x)-h_{\rm ren}(x')-\lambda^2\int_0^t ds\, J_\Sigma(s)[\langle\bar x^2(s)\rangle_0+\langle\bar x'^2(s)\rangle_0]$$
$$-i\lambda^2\int_0^t ds\, J_\Delta(s)[\langle\bar x^2(s)\rangle_0-\langle\bar x'^2(s)\rangle_0]\Big\}J_r(x_f,x_f',t|x_i,x_i',0)\ , \qquad (4.28)$$

where the renormalized Hamiltonian $h_{\rm ren}$ is constructed using the renormalized frequency Ω_r^2 defined in (4.9) and the original coupling constant C.

As the third term (dissipation) and the fourth term (noise) on the right-hand side of (4.28) are of order λ^2, we can calculate $\langle\bar x^2(s)\rangle_0$ using (4.27) with the "free" action

$$S_0[x]=\int_0^t ds\{\tfrac{1}{2}\dot x^2-\tfrac{1}{2}\Omega_r^2x^2\} \qquad (4.29)$$

and neglect both $\delta A(x,x')$ and the anharmonicity of the potential that are of order λ.

The calculation of $\langle\bar x^2\rangle_0$ is done straightforwardly by expanding the path $\bar x$ around the classical trajectories given by

$$x_{\rm cl}(s)=x_i\frac{\sin\Omega_0(t-s)}{\sin\Omega_r t}+x_f\frac{\sin\Omega_r s}{\sin\Omega_r t}\ . \qquad (4.30)$$

In fact, writing

$$\bar x(s)=x_{\rm cl}(s)+y(s)\ , \qquad (4.31)$$

we get

$$\langle\bar x^2(s)\rangle_0=N'\int_0^0 Dy[x_{\rm cl}(s)+y(s)]^2\exp\left[\frac{i}{\hbar}S_0[y]\right]=x_{\rm cl}^2(s)+iQ(s)\ , \qquad (4.32)$$

where

$$Q(s)=\hbar\frac{\sin\Omega_r s\,\sin\Omega_r(t-s)}{\Omega_r\sin\Omega_r t}\ . \qquad (4.33)$$

Using these results, Eq. (4.29) becomes

$$i\hbar\frac{\partial}{\partial t}J_r(x_f,x_f',t|x_i,x_i',0)=\Big\{h_{\rm ren}(x)-h_{\rm ren}(x')-i(x_f^2-x_f'^2)\int_0^t ds\, \nu^{(k)}(t-s)[x_{\rm cl}^2(s)-x_{\rm cl}'^2(s)+2iQ(s)]$$
$$-(x_f^2-x_f'^2)\int_0^t ds\, \eta^{(k)}(t-s)[x_{\rm cl}^2(s)+x_{\rm cl}'^2(s)]\Big\}J(x_f,x_f',t|x_i,x_i',0)\ . \qquad (4.34)$$

Our last step is to eliminate the dependence on the initial points x_i and x_i' that enter (4.34) through $x_{\rm cl}(s)$ and $x_{\rm cl}'(s)$. This can be easily done by showing that the free propagator, defined as

$$J_0(x_f,x_f',t|x_i,x_i',0)=\int_{x_i}^{x_f}Dx\int_{x_i'}^{x_f'}Dx'\exp\left[\frac{i}{\hbar}[S_0[x]-S_0[x']]\right]\ , \qquad (4.35)$$

satisfies the identities

$$x_{\rm cl}(s)J_0(x_f,x_f',t|x_i,x_i',0)=\left\{\cos\Omega_r(t-s)x_f+\frac{\sin\Omega_r(t-s)}{\Omega_r}i\hbar\frac{\partial}{\partial x_f}\right\}J_0(x_f,x_f',t|x_i,x_i',0) \qquad (4.36)$$

and

$$x_{\rm cl}'(s)J_0(x_f,x_f',t|x_i,x_i',0)=\left\{\cos\Omega_r(t-s)x_f'-\frac{\sin\Omega_r(t-s)}{\Omega_r}i\hbar\frac{\partial}{\partial x_f'}\right\}J_0(x_f,x_f',t|x_i,x_i',0)\ . \qquad (4.37)$$

Using these relations in (4.34), one finally obtains the master equation

$$i\hbar\frac{\partial}{\partial t}\rho_r(x,x',t)=\hat H_\rho(x,x',t)\rho_r(x,x',t)\ , \qquad (4.38)$$

where the time-dependent "Hamiltonian" is

$$\hat{H}_p(x,x',t)=H_{\rm ren}(x)-H_{\rm ren}(x')-ib_1(t)(x^2-x'^2)^2-a_2(t)(x^2-x'^2)i\hbar\left[x\frac{\partial}{\partial x}-x'\frac{\partial}{\partial x'}\right]$$
$$-ib_2(t)(x^2-x'^2)i\hbar\left[x\frac{\partial}{\partial x}+x'\frac{\partial}{\partial x'}\right]-a_3(t)(x^2-x'^2)(i\hbar)^2\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x'^2}\right]$$
$$-ib_3(s)(x^2-x'^2)(i\hbar)^2\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right] \quad (4.39)$$

and the effective Hamiltonian $\hat{H}_{\rm ren}(x)$ is defined as

$$\hat{H}_{\rm ren}(x)=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\tilde{\Omega}_r^2(t)x^2+\tilde{C}_r(t)x^4 , \quad (4.40)$$

with renormalized parameters given by

$$\tilde{\Omega}_r^2(t)=\Omega_t^2+\delta\Omega_2^2(t)=\Omega_0^2+\delta\Omega_1^2+\delta\Omega_2^2(t) , \quad (4.41a)$$

$$\tilde{C}_r(t)=C_0-a_1(t) , \quad (4.41b)$$

$$\delta\tilde{\Omega}_2(t)=\frac{4\hbar}{\Omega_r}\int_0^t ds\, D(s)\left[\cos\Omega_r s\,\sin\Omega_r s+\frac{\sin\Omega_r s\,\sin\Omega_r(t-s)}{\sin\Omega_r t}\right] . \quad (4.41c)$$

The time-dependent coefficients b_i and c_i in (4.39) are defined as

$$a_1(t)=\int_0^t ds\,\eta(s)\cos^2\Omega_r s , \quad (4.42a)$$

$$a_2(t)=\frac{1}{2\Omega_r}\int_0^t ds\,\eta(s)\sin 2\Omega_r s , \quad (4.42b)$$

$$a_3(t)=\frac{1}{\Omega_r^2}\int_0^t ds\,\eta(s)\sin^2\Omega_r s , \quad (4.42c)$$

$$b_1(t)=\Omega_r^2 b_3(t)=\int_0^t ds\,\nu(s)\cos^2\Omega_r s , \quad (4.42d)$$

$$b_2(t)=\frac{1}{2\Omega_r}\int_0^t ds\,\nu(s)\sin 2\Omega_r s . \quad (4.42e)$$

In deriving Eq. (4.38), we did not make any assumption about the nature of the spectral density of the environment. We can easily show that if the spectral density and the temperature are such that the dissipation and noise kernels become local, then the master equation (4.38) reduces to Eq. (4.4) that was derived in the local approximation. In fact, if

$$\eta(s)\sim\frac{\partial}{\partial s}\delta(s) , \quad (4.43a)$$

$$\nu(s)\sim\delta(s) , \quad (4.43b)$$

one can show that

$$a_2(t)\sim\gamma_0\left[\frac{k_B T}{\hbar\Lambda}\right] , \quad b_1(t)\sim\frac{2k_B T}{\hbar}\gamma_0\left[\frac{k_b T}{\hbar\Lambda}\right] ,$$
$$a_1(t)=a_3(t)=b_2(t)=b_3(t)=\delta\omega_2^2(t)=0 . \quad (4.44)$$

Note that the time-dependent coefficients b_i and c_i are very similar to those defined in paper I in the general equation for the linear case. In fact, Eq. (4.39) also applies to weak nonlinear-dissipation cases.

Finally, using Eq. (4.39), one can easily derive the Wigner equation

$$\frac{\partial W}{\partial t}=\{H_{\rm ren},W\}_{\rm PB}-\frac{\hbar^2}{4!}\frac{\partial^3 V_{\rm ren}}{\partial X^3}\frac{\partial^3 W}{\partial p^3}+2a_2(t)X\frac{\partial}{\partial p}\left[Xp+\frac{\hbar^2}{4}\frac{\partial^3}{\partial X\,\partial^2 p}\right]W+4\hbar b_1(t)X^2\frac{\partial^2 W}{\partial p^2}$$
$$-4\hbar b_3(t)X\frac{\partial^2 W}{\partial X\,\partial p}+2\hbar b_2(t)X\frac{\partial}{\partial p}\left[X\frac{\partial}{\partial X}-\frac{\partial}{\partial p}p\right]W-a_3(t)X\frac{\partial}{\partial p}\left[4p^2-\hbar^2\frac{\partial^2}{\partial X^2}\right]W . \quad (4.45)$$

The terms that contain third-order derivatives are quantum corrections since they are of order $\hbar^2$. In the classical limit, the Wigner equation (4.45) again becomes the Fokker-Planck equation.

V. DISCUSSION

The master equation is a very useful tool for studying many important physical problems. We shall mention a few properties of the master equation obtained here for nonlinear QBM and compare them with the results of the linear case.

The Markovian regime of linear QBM corresponds to an environment with an Ohmic spectral density [i.e., $I(\omega)\propto\omega$] in the high-temperature limit. It is only under such conditions that the noise and dissipation kernels [$\nu(s)$ and $\gamma(s)$] become proportional to the δ function and the influence functional becomes local in time. In that regime the master equation was shown to be (see [27])

$$i\hbar\dot{\rho}(x,x',t)=\Bigg[H_{\rm ren}(x)-H_{\rm ren}(x') -i\gamma_0\left[\frac{2k_BT}{\hbar}\right](x-x')^2 -i\hbar\gamma_0(x-x')\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]\Bigg]\rho(x,x',t)\ . \tag{5.1}$$

This equation contains a diffusion and a friction term (the second and third lines respectively) that are simply related by the fluctuation-dissipation relation.

In the first example of Sec. IV, we analyzed the Markovian regime of nonlinear QBM and obtained the nonlinear generalization of Eq. (5.1). As a first remark, it is worth stressing that, in the nonlinear case, the Markovian regime corresponds always to a supra-Ohmic environment [see (4.1) and note that the spectral coefficient is fixed by the degree of nonlinearity] in the high-temperature limit. In this limit the form of the resulting master equation is rather similar to (5.1) [see (4.14)] since it reads

$$i\hbar\dot{\rho}(x,x',t)=\Bigg[H_{\rm ren}(x)-H_{\rm ren}(x')-i\gamma_0\left[\frac{k_BT}{\hbar\Lambda}\right]^{k-1}\left[\frac{2k_BT}{\hbar}\right][f(x)-f(x')]^2 -i\hbar\gamma_0\left[\frac{k_BT}{\hbar\Lambda}\right]^{k-1}[f(x)-f(x')]\left[\frac{\partial f(x)}{\partial x}\frac{\partial}{\partial x}-\frac{\partial f(x')}{\partial x'}\frac{\partial}{\partial x'}\right]\Bigg]\rho(x,x',t)\ . \tag{5.2}$$

The information about the coupling between the system and environment enters (5.2) in two different ways. On the one hand, the integer k is the degree of nonlinearity in the environment coordinates. We see that a higher nonlinearity introduces more temperature dependence both in the friction and diffusion terms. On the other hand, the unspecified function $f(x)$ in (5.2) indicates how the system's coordinates appear in the coupling term. Obviously, (5.2) reduces to (5.1) in the linear case $k=1$ and $f(x)=x$. The master equation for the Markovian regime of nonlinear QBM has friction and diffusion terms that are related by the nonlinear fluctuation dissipation relations (FDR's). As in the case of high temperatures, the nonlinear FDR's are equivalent to the linear ones; the friction and diffusion terms in (5.2) turn out to be simply related by a factor of k_BT.

The properties of the master equation in the non-Markovian regime for linear QBM have been studied by many for years. However, the most general result was found only recently. It was shown (in paper I) that the non-Markovian regime of linear QBM is described by a local master equation which is similar in form to (5.1). It differs from (5.1) only in the time-dependent friction and diffusion coefficients and an extra "anomalous diffusion" term proportional to $(x-x')(\partial_x+\partial_{x'})$. In the last two examples of Sec. IV, we discussed the non-Markovian regime of nonlinear QBM and found the nonlinear generalization of the master equation obtained in paper I. We first analyzed the simple case for which the coupling is linear in the system [i.e., $f(x)=x$] and showed that in this case the master equation does not differ from the one we obtained in paper I for the linear case [see (4.14)]. Thus, to the order of approximation we are working here (second order in the coupling constant), the nonlinearities in the environment do not induce nonlinear behavior in the system (this effect does manifest in the higher orders). This property is independent of the spectral density of the environment or other details of the model (such as the temperature of the environment, for example) and therefore goes well beyond the Markovian regime. In the last example of Sec. IV, we analyzed an example of the most general case: The spectral density is general (thus in the non-Markovian regime), and the coupling is nonlinear in the system's coordinates. In that case we obtained a master equation that generalizes the one we found in paper I. It is also a generalization of (5.2) that describes the Markovian regime of nonlinear QBM. In fact, the non-Markovian equation differs from (5.2) in that it has time-dependent coefficients and some extra "anomalous diffusion terms" [see (4.38) and (4.39)]. In this case there are three anomalous diffusion terms. One is proportional to

$$[f(x)-f(x')][f'(x)\partial_x+f'(x')\partial_{x'}]\ ,$$

which is a natural nonlinear generalization of the anomalous diffusion term obtained in the linear case. The two other terms proportional to

$$[f(x)-f(x')][f''(x)\partial_x^2\pm f''(x')\partial_{x'}^2]$$

are entirely generated by the nonlinearity since they identically vanish for the linear case. The effects of these new terms will be analyzed elsewhere.

Let us now discuss some of the effects produced by these nonlinear master equations. For simplicity, we will restrict ourselves to analyzing the Markovian equation (5.2) in the study of the decoherence process. For this purpose it is convenient to use the Wigner equation associated with (5.2). In fact, for a general coupling, this equation reads

$$\dot{W}=\{H_{\rm ren},W\}_{\rm PB}+\sum_{l=1}^{\infty}\frac{\hbar^{2l}(-1)^l V_{\rm ren}^{(2l+1)}}{(2l+1)!2^{2l}}\partial_p^{2l+1}W$$

$$+2\gamma_0\left[\frac{k_bT}{\hbar\Lambda}\right]^{k-1}\sum_{j=0}^{\infty}\left[-\frac{\hbar^2}{4}\right]^j\partial_p^{2j+1}\sum_{m=0}^{j}\frac{f^{(2j-2m+1)}}{2m!(2j-2m+1)!}\left[pf^{(2m+1)}+\frac{\hbar^2 f^{(2m+2)}}{4(2m+1)}\partial_x\partial_p\right]$$

$$+\gamma_0\left[\frac{k_bT}{\hbar\Lambda}\right]^{k-1}\frac{\hbar^2}{\lambda_{db}^2}\sum_{j=0}^{\infty}(-\hbar^2)^j\partial_p^{2j+2}\sum_{m=0}^{j}\frac{f^{(2j-2m+1)}f^{(2m+1)}}{(2m+1)!(2j-2m+1)!}\ , \tag{5.3}$$

where $\lambda_{db}^2=\hbar/2Mk_BT$ is the thermal de Broglie wavelength. Here, a superscript on f denotes the order of derivatives with respect to its argument. The first line of the above Wigner equation corresponds to the unitary evolution generated by the renormalized Hamiltonian. The nonlinearities in the potential $V_{\rm ren}$ generate the higher-derivative terms. Terms in the second line originate from the nonlinear friction terms in the master equation (5.2). They contain higher-odd-order and mixed derivatives that are generated by the nonlinearities carried in $f(x)$. Terms in the last line contain an even number of derivatives with respect to the momentum variable and describe the effect of diffusion.

To study decoherence we will analyze the decay of the interference term between two Gaussian packets. This problem was extensively analyzed for linear QBM in [20]. We will assume that the wave function of the system is a superposition of two Gaussian wave packets Ψ_1 and Ψ_2 whose mean values are $\langle x\rangle=x_0\pm\Delta x$ and $\langle p\rangle=0$. The initial Wigner function can be shown to be the sum of two direct terms (W_1 and W_2) and an interference term ($W_{\rm int}$). The interference term is oscillatory and nonpositive definite. In fact, at the initial time, it can be written as

$$W_{\rm int}(x,p)=2[W_1(x,p)W_2(x,p)]^{1/2}\cos\left[\frac{p\,\Delta x}{\hbar}\right]. \tag{5.4}$$

At large times the nonlinearities will destroy the Gaussian nature of the wave packets and the Gaussian ansatz will become invalid. However, at times very short compared to the dynamical time scales, the Gaussian ansatz is still adequate. As a measure of the effectiveness of decoherence, we will use the one proposed in [20], which is the peak-to-peak ratio between the interference and direct terms in the Wigner function:

$$A_{\rm int}=-\ln\left[\frac{W_{\rm int}|_{\rm peak}}{2(W_1|_{\rm peak}W_2|_{\rm peak})^{1/2}}\right]. \tag{5.5}$$

It is possible to show that, assuming the Gaussian ansatz and using the Wigner equation, only the diffusion terms [the third line of (5.3)] directly contribute to the variation of $A_{\rm int}$. In fact, the time derivative of $A_{\rm int}$ initially turns out to be

$$\dot{A}_{\rm int}|_{t=0}=-\gamma_0\left[\frac{k_bT}{\hbar\Lambda}\right]^{k-1}\left[\frac{\Delta x}{\lambda_{db}}\right]^2\times\sum_{j=0}^{\infty}(\Delta x)^{2j}\sum_{m=0}^{j}\frac{f^{(2j-2m+1)}f^{(2m+1)}}{(2m+1)!(2j-2m+1)!}\ . \tag{5.6}$$

This equation has to be compared with the result in the linear QBM case (see [20]), which is simply given by

$$\dot{A}_{\rm int}|_{t=0}=-\gamma_0\left[\frac{\Delta x}{\lambda_{db}}\right]^2. \tag{5.7}$$

The right-hand sides of Eqs. (5.6) and (5.7) determine the initial coherence rate, which in the linear regime is directly proportional to $(\Delta x)^2$ and independent of x_0. The role of nonlinearities is again twofold. On the one hand, the nonlinearities in the environment coordinates (determined by the integer k) introduce further temperature dependence of the decoherence rate. On the other hand, the nonlinearities carried by $f(x)$ make the dependence of the decoherence rate on Δx^2 deviate from the linear one obtained in (5.7). In fact, for large values of the separation between the superposing wave packets, nonlinear effects can dominate and the rate may substantially increase beyond the values obtained in the linear regime. Some effects of the nonlinear diffusion terms appearing in the Markovian equations (5.2) and (5.3) have previously been studied by Habib [37].

Another feature of our results is also evident from Eq. (5.6). The decoherence rate not only depends nontrivially upon the distance between the centers of the two Gaussian wave packets (Δx^2), but also depends upon the value of x_0 itself (the central point between the two packets, which coincides with the mean value of the position in the initial state). This dependence resides in the derivatives of the function $f(x)$ in (5.6) since they are evaluated at x_0. As a consequence of the x_0 dependence, the spatial homogeneity of the result is lost in the nonlinear case. This is not surprising since it is just a consequence of the multiplicative nature of the noise in the nonlinear regime. Thus the stochastic force the system is interacting with does depend on the position of the system and therefore decoherence (and any other physical effect) is expected to carry this dependence too. This inhomogeneity induced by the interaction with the environment may be a desirable feature of the result (in fact, physical examples of systems that interact with a multiplicative noise are well known; see [36]), but one may also wonder if it is possible to construct models in which the interaction is nonlinear but the homogeneity is preserved. There are indeed such models. To end our discussion, let us mention just such an example of a nonlinear but homogeneous model—that of a particle (the system) interacting with a field (the environment). For simplicity, we consider a scalar field $\phi(\mathbf{r},t)$ in one dimension, with a local interaction action (in the sense that the interaction Lagrangian is proportional to the field evaluated in the position of the particle) of the form

$$S_{\rm int}=\lambda\int dt\, dq\, \phi(q,t)\delta(q-x(t)) =\lambda\int dt\, \phi(x,t)\ . \quad (5.8)$$

This model can be thought of as a local generalization of the one considered by Unruh and Zurek [19]. In fact, if we use a "dipole" approximation and expand the action (5.8) for small values of x, we recover the results of Unruh and Zurek in the first nontrivial order. If we assume that the state of the environment is homogeneous (thermal equilibrium, for example), the influence functional can be computed exactly [38,39]. The imaginary part of the influence action turns out to be

$$\mathrm{Im}\delta A[x,x']=-\frac{\lambda^2}{4}\int_0^\infty\frac{d\omega}{\pi\omega}\int_0^t ds\int_0^t ds'\cos\omega(s-s')\{\cos\omega[x(s)-x(s')]+\cos\omega[x'(x)-x'(s')] -2\cos\omega[x(s)-x'(s')]\}\ . \quad (5.9)$$

From this expression one can immediately realize that although the interaction is highly nonlinear it does not introduce inhomogeneity. In fact, in contradistinction to what happens in the examples discussed in this paper, the influence functional (5.9) is invariant under translations of the form $x\rightarrow x+a$. The behavior of the decoherence rate for this model is under study, but preliminary results obtained by Gallis [38] show that the dependence upon the separation Δx is nontrivial and that the rate saturates for large values of Δx. Detailed analysis of the above nonlinear models and the properties of their master equations in the description of various physical problems of current interest will be presented elsewhere.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation Grant No. PHY91-19726.

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