# Supersymmetry and positive energy in classical and quantum two-dimensional dilaton gravity

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An $N = 1$ supersymmetric version of two-dimensional dilaton gravity coupled to matter is considered. It is shown that the linear dilaton vacuum spontaneously breaks half the supersymmetries, leaving broken a linear combination of left and right supersymmetries which squares to time translations. Supersymmetry suggests a spinorial expression for the Arnowitt-Deser-Misner energy $M$, as found by Witten in four-dimensional general relativity. Using this expression it is proven that $M$ is non-negative for smooth initial data asymptotic (in both directions) to the linear dilaton vacuum, provided that the (not necessarily supersymmetric) matter stress tensor obeys the dominant energy condition. A *quantum* positive-energy theorem is also proven for the semiclassical large-$N$ equations, despite the indefiniteness of the quantum stress tensor. For black-hole spacetimes, it is shown that $M$ is bounded from below by $e^{-2\phi_H}$, where $\phi_H$ is the value of the dilaton at the apparent horizon, provided only that the stress tensor is positive outside the apparent horizon. This is the two-dimensional analogue of an unproven conjecture due to Penrose. Finally, supersymmetry is used to prove positive-energy theorems for a large class of generalizations of dilaton gravity which arise in consideration of the quantum theory.

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## I. INTRODUCTION

Two-dimensional dilaton gravity has recently been found to be a useful laboratory for studying quantum gravity in a simplified context [1,2]. Before coupling to matter, it is a theory with no local degrees of freedom: The two gauge degrees of freedom and two constraints absorb the three components of the metric together with the dilaton.[1] Thus one can study the interesting global aspects of gravity in isolation from the complicated dynamics of propagating gravitons.

In adopting two-dimensional dilaton gravity as a model for four-dimensional gravity, it is important to know what features the two theories have in common. For example, is there a positive-energy theorem for the two-dimensional case? In the present paper, we address this question following Witten's supersymmetric proof of the positive-energy theorem in four dimensions [3–5]. The basic idea is simple. In a supersymmetric theory, $H = Q^2$ and so must be positive. In Sec. II we review the supersymmetric version of dilaton gravity and show that the linear dilaton vacuum is the unique supersymmetric solution and that this vacuum preserves a nonchiral combinations of the supersymmetries. In Sec. III we prove that all smooth initial data with non-negative stress tensors on a spacelike slice asymptotic to the linear dilaton vacuum (in both directions) have positive energy. In Sec. IV positivity is proven for a scalar matter sector governed by the action $S_Z = \int [ -\frac{1}{2}(\nabla Z)^2 + QRZ ]$, despite the fact the as-

sociated stress tensor is indefinite. The Bondi mass is also shown to be positive, after noting a correction term linear in the $Z$ field. This result is then used in Sec. V to prove a quantum positive-energy theorem for the large-$N$ equations of [2]. While it is generally believed (or hoped) that positivity of the total energy remains valid at the quantum level, this is apparently the first example for which a theorem has been established. In Sec. VI we consider initial data on spacelike slices bounded by an apparent horizon on one end and asymptotic to the linear dilaton vacuum on the other. This corresponds to a black hole. A simple proof is given that in this case the mass is bounded by $e^{-2\phi_H}$, where $\phi_H$ is the value of the dilaton at the horizon. This establishes the two-dimensional analogue of Penrose's conjecture that the mass of a three-dimensional initial data set is bounded by the square root of the area of the horizons, whose validity is related to cosmic censorship [6]. Finally in Sec. VII the most general supersymmetric power-counting renormalizable theory of dilaton gravity involving three arbitrary functions of the dilaton coupled to matter is considered. Supersymmetry is used to prove a positive-energy theorem for a large subset of these theories. In conclusion, positivity of the energy is a robust feature of dilaton gravity. This increases our confidence that two-dimensional dilaton gravity is a good toy model for four-dimensional gravity.

The possibility of further applications to the quantum theory is a key motivation for our investigations. Recently, it has become clear that better control over higher-order quantum corrections is essential for understanding the problem of two-dimensional black-hole formation and/or evaporation. Typically, (extended) supersymmetry has been very useful in this regard. For example, our result that the linear dilaton vacuum is supersymmetric strongly suggests that it is an exact solution of the full quantum theory.

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[1]By the same counting Liouville gravity without a dilaton has minus one degrees of freedom, a curious fact which obscures analogies to four-dimensional gravity.

1569

## II. SUPERSYMMETRIC DILATON GRAVITY

The $N=1$ supersymmetric extension of dilaton gravity can be worked out using the superfields found by Howe [7], whose notation we follow, except for the sign of $R$. The supersymmetric version of a closely related theory has been described by Chamseddine [8]. Supersymmetric dilaton gravity is described by the superspace action

$$S_G = \frac{i}{2\pi} \int d^2x \, d^2\theta \, E e^{-2\Phi}[S + 2iD_\alpha \Phi D^\alpha \Phi - 4\lambda] , \quad (1)$$

where the superfields are given by [7]

$$\Phi = \phi + i\bar{\theta}\Lambda + \frac{i}{2}\bar{\theta}\theta F ,$$

$$E = e\left[1 + \frac{i}{2}\bar{\theta}\gamma^\mu\chi_\mu + \bar{\theta}\theta\left(\frac{i}{4}A + \frac{1}{8}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma_5\chi_\nu\right)\right] ,$$

$$S = A + \bar{\theta}\Psi + \frac{i}{2}\bar{\theta}\theta C ,$$

$$C = -R - \frac{1}{2}\bar{\chi}_\mu\gamma^\mu\Psi + \frac{i}{4}\epsilon^{\mu\nu}\bar{\chi}_\mu\gamma_5\chi_\nu A - \frac{1}{2}A^2 ,$$

$$\Psi = -2i\epsilon^{\mu\nu}\gamma_5 D_\mu\chi_\nu - \frac{i}{2}\gamma^\mu\chi_\mu A ,$$

$$E_\mu{}^a = e_\mu{}^a + i\bar{\theta}\gamma^a\chi_\mu + \frac{i}{4}\bar{\theta}\theta e_\mu{}^a A , \quad (2)$$

$$E_\mu{}^\alpha = \frac{1}{2}\chi_\mu{}^\alpha + \frac{1}{2}(\bar{\theta}\gamma_5)^\alpha\omega_\mu - \frac{1}{4}(\bar{\theta}\gamma_\mu)^\alpha A$$

$$\qquad - \bar{\theta}\theta\left[\frac{3i}{16}\chi_\mu{}^\alpha A + \frac{1}{4}(\gamma_\mu\Psi)^\alpha\right] ,$$

$$E_\beta{}^a = i(\bar{\theta}\gamma^a)_\beta ,$$

$$E_\beta{}^\alpha = \delta_\beta{}^\alpha\left[1 - \frac{i}{8}\bar{\theta}\theta A\right] ,$$

and

$$D_\mu\chi_\nu = \partial_\mu\chi_\nu - \frac{1}{2}\omega_\mu\gamma_5\chi_\nu ,$$

$$\omega_\mu = -e_{a\mu}\epsilon^{\nu\rho}\partial_\nu e_\rho{}^a + \frac{1}{2}\bar{\chi}_\mu\gamma_5\gamma^\nu\chi_\nu . \quad (3)$$

$a,b$ are tangent space indices, $\mu,\nu$ are spacetime indices, $\alpha,\beta$ are spinor indices, $\epsilon_{01}=1$, and $D$ is the superderivative. All spinors are Majorana. $\gamma_5$ is, in a notational abuse, equal to $\gamma^0\gamma^1$. Further details can be found in [7].

The bosonic part of the action in component form is

$$S_G = \frac{1}{2\pi} \int d^2x \, ee^{-2\phi}[R + 4(\nabla\phi)^2 + 2AF - 4F^2$$

$$\qquad + 2\lambda A - 8\lambda F] . \quad (4)$$

The equations of motion for the auxiliary fields $A$ and $F$ are

$$A = 0 ,$$
$$F = -\lambda . \quad (5)$$

Substituting into (4), one finds

$$S_G = \frac{1}{2\pi} \int d^2x \, ee^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2] . \quad (6)$$

The supersymmetry variations of the Fermi fields are

$$\delta\Lambda = (\nabla\phi + F)\eta ,$$

$$\delta\chi_\mu = (2D_\mu + \frac{1}{2}\gamma_\mu A)\eta . \quad (7)$$

A supersymmetric field configuration is one for which all supersymmetry variations vanish. Setting the background Fermi fields to zero and the auxiliary fields to their constant values (5) implies

$$(\nabla\phi - \lambda)\eta = 0 , \quad (8)$$

$$2D_\mu\eta = 0 . \quad (9)$$

The integrability condition for (9) is that the curvature vanishes and the metric is therefore flat. Equation (8) then implies that $(\nabla\phi)^2 = \lambda^2$. By an appropriate choice of coordinates, the general solution of (8) can then be written in the form

$$\phi = -\lambda\sigma, \quad g_{\mu\nu} = \eta_{\mu\nu} , \quad (10)$$

where $\sigma$ is a spatial coordinate in the 1 direction. The spinor $\eta$ for which the variations vanish obeys (in these coordinates)

$$(\gamma^1 + 1)\eta = 0, \quad \eta = \text{const} . \quad (11)$$

This is true for one combination of the original two supersymmetries.

The solution (10) is known as the linear dilaton vacuum and will be referred to herein simply as the vacuum. It spontaneously breaks Poincaré invariance (down to time translations) as well as half of the supersymmetries. Note that neither left- nor right-moving supersymmetry alone is unbroken: only a linear combination of the two, which squares to time translations.

## III. POSITIVE ENERGY
## FOR ASYMPTOTICALLY FLAT SPACETIMES

The complete solution space of pure dilaton gravity (6) consists of the vacuum together with a one parameter family of black-hole solutions [9]. Nonsingular initial data exist for the black holes only if the mass is positive, and so a positive-energy theorem is rather trivially demonstrated.

More generally, one would like to know if the energy remains positive when dilaton gravity is coupled to a general matter theory with a positive stress tensor. (Positivity was shown by construction for some special circumstances in [2].) In this section we shall show that this is indeed the case if the spacetime is asymptotic to the vacuum (10) in both spatial directions. In the next section, we shall relax this condition and prove a positive-energy theorem for black-hole spacetimes with matter.

Since the vacuum leaves one supersymmetry unbroken, a conserved global supercharge can be constructed for configurations which are asymptotic to that vacuum as $\sigma \to \pm\infty$. The standard Noether procedure leads to.

$$Q = i \int d\sigma^\mu \nabla_\mu [e^{-2\phi}\bar{\eta}\gamma_5\Lambda]$$

$$\quad = ie^{-2\phi}\bar{\eta}\gamma_5\Lambda \Big|_{-\infty}^{+\infty} . \quad (12)$$

The integral is over an arbitrary spacelike slice, and $d\sigma$ is the line element. $\eta$ is any spinor asymptotically (in both directions) obeying (11) or, in a general coordinate system, (8) and (9).

Since the square of this supercharge is a time translation, a Witten-like expression for the Arnowitt-Deser-Misner (ADM) mass can be obtained as a supersymmetry variation of the supercharge. This leads to

$$M = \int d\sigma^\mu \nabla_\mu [2e^{-2\phi}\bar{\epsilon}\gamma_5\delta_\epsilon\Lambda]$$

$$= 2e^{-2\phi}\bar{\epsilon}\gamma_5(\nabla\phi-\lambda)\epsilon|_{+\infty} , \tag{13}$$

where $\epsilon$ is any *commuting* spinor obeying the analogue of the asymptotic conditions (11) and normalized so that

$$-\bar{\epsilon}\gamma_5\epsilon|_\infty = 1 , \tag{14}$$

and $\delta_\epsilon$ denotes the supersymmetry variations (7) with $\eta$ replaced by $\epsilon$. Note that with these asymptotic conditions the boundary term at $\sigma = -\infty$ in the expression for $M$ vanishes and has been omitted in (13).

This may be compared with previous expressions for $M$ [9] by linearizing around the vacuum. Defining

$$\phi = -\lambda\sigma + \delta\phi ,$$
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \tag{15}$$

one finds

$$M = e^{2\lambda\sigma}(2\partial_1\delta\phi + \lambda h_{11})|_{+\infty} , \tag{16}$$

in agreement with previous results.

Using the equation of motion,

$$\frac{2\pi}{\sqrt{-g}}\frac{\delta S_G}{\delta g^{\mu\nu}} = -\frac{2\pi}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \equiv T_{\mu\nu} , \tag{17}$$

where $T_{\mu\nu}$ is a general matter stress tensor, one finds

$$M = \int d\sigma^\mu \nabla_\mu [2e^{-2\phi}\bar{\epsilon}\gamma_5(\nabla\phi-\lambda)\epsilon]$$

$$= \int d\sigma^\mu [\epsilon_\mu{}^\nu\bar{\epsilon}\gamma^\rho\epsilon T_{\nu\rho} + 2e^{-2\phi}(\delta_\epsilon\bar{\Lambda})\gamma_5\delta_\epsilon\tilde{\chi}_\mu] , \tag{18}$$

where

$$\delta_\epsilon\tilde{\chi}_\mu = \delta_\epsilon\chi_\mu - \gamma_\mu\delta_\epsilon\Lambda .$$

This expression is valid for any choice of $\epsilon$ satisfying the boundary conditions. Positivity can be made manifest by choosing an $\epsilon$ which obeys

$$\delta_\epsilon\tilde{\chi}_1 = (2D_1 - \gamma_1\nabla\phi + \gamma_1\lambda)\epsilon = 0 , \tag{19}$$

where 1 denotes the direction tangent to the spacelike slice. This is a first-order differential equation which may be solved for $\epsilon$. The energy is then simply

$$M = \int d\sigma^\rho \epsilon_\rho{}^\nu\bar{\epsilon}\gamma^\mu\epsilon T_{\nu\mu} . \tag{20}$$

A Fierz identity implies that

$$\bar{\epsilon}\gamma^\rho\epsilon\bar{\epsilon}\gamma_\rho\epsilon = -(\bar{\epsilon}\gamma_5\epsilon)^2 , \tag{21}$$

so that $\bar{\epsilon}\gamma^\rho\epsilon$ is a timelike vector. At infinity, it is future directed and must be so everywhere by continuity. The expression (20) for $M$ is then manifestly non-negative as long as $T_{\mu\nu}$ satisfies the dominant energy condition.

Thus we have proven that the ADM energy for spacetimes asymptotic to the vacuum is always non-negative if the matter has non-negative stress tensor and vanishes if and only if the matter stress tensor does. While supersymmetry has been used to motivate expression (13), the result applies to a much broader class of theories.

## IV. POSITIVE ENERGY FOR NONMINIMALLY COUPLED CONFORMAL SCALARS

In this section we shall consider dilaton gravity coupled to matter governed by the action

$$S_Z = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[-\tfrac{1}{2}(\nabla Z)^2 + QRZ\right] , \tag{22}$$

where $Q$ is an arbitrary constant. The associated stress tensor is

$$T_{\mu\nu}^Z = \tfrac{1}{2}\nabla_\mu Z\nabla_\nu Z - \tfrac{1}{4}g_{\mu\nu}(\nabla Z)^2 + Q\epsilon_\mu{}^\alpha\epsilon_\nu{}^\beta\nabla_\alpha\nabla_\beta Z . \tag{23}$$

$T^Z$ has no particular positivity properties. The last term, which dominates for small $Z$, changes sign under $Z \to -Z$. Thus one might not expect a positive-energy theorem for the gravity-$Z$ system. On the other hand, it is easy to see that (22) can be supersymmetrized, and so the $H = Q^2$ relation suggests such a theorem nevertheless exists. In this section we shall prove that this is indeed the case.

The presence of a second derivative term in $T^Z$ implies a correction to the boundary formula for the mass of the gravity-$Z$ system:

$$M = [2e^{-2\phi}\bar{\epsilon}\gamma_5(\nabla\phi-\lambda)\epsilon - Q\bar{\epsilon}\gamma_5\nabla Z\epsilon]_\infty . \tag{24}$$

For spinors obeying the boundary conditions (11), the extra term is proportional to the spatial derivative of $Z$. If $M$ is evaluated at spatial infinity and $Z$ is asymptotically constant, this extra term vanishes.

An expression for the Bondi mass is obtained simply by evaluating (24) at right future null infinity ($\mathscr{I}_R^+$). In that case, however, the extra boundary term is not negligible. Asymptotically, $Z$ obeys the free wave equation and is given by $Z = Z_+(x^+) + Z_-(x^-)$. Thus, in general, if there is outgoing (incoming) $Z$ radiation, $\partial_- Z$ ($\partial_+ Z$) will not vanish on $\mathscr{I}_R^+$ ($\mathscr{I}_R^-$), and the extra boundary term will be nonzero.

Integrating by parts, the mass formula can be written as in (18):

$$M = \int d\sigma^\mu [\epsilon_\mu{}^\nu\bar{\epsilon}\gamma^\rho\epsilon T_{\nu\rho}^Z + 2e^{-2\phi}\delta_\epsilon\bar{\Lambda}\gamma_5\delta_\epsilon\tilde{\chi}_\mu$$

$$-Q\nabla_\mu(\bar{\epsilon}\gamma_5\nabla Z\epsilon)] . \tag{25}$$

The terms involving two derivatives of $Z$ cancel (using $\bar{\epsilon}\gamma_5\gamma_\mu\epsilon = -\epsilon_\mu{}^\nu\bar{\epsilon}\gamma_\nu\epsilon$), yielding

$$M = \int d\sigma^\mu [\epsilon_\mu{}^\nu\bar{\epsilon}\gamma^\rho\epsilon\hat{T}_{\nu\rho}^Z + 2e^{-2\phi}\delta_\epsilon\bar{\Lambda}\gamma_5\delta_\epsilon\tilde{\chi}_\mu$$

$$-Q\nabla_\rho Z\nabla_\mu(\bar{\epsilon}\gamma_5\gamma^\rho\epsilon)] , \tag{26}$$

where the reduced stress tensor

$$\hat{T}^Z_{\mu\nu} = \tfrac{1}{2}\nabla_\mu Z \nabla_\nu Z - \tfrac{1}{4}g_{\mu\nu}(\nabla Z)^2 \tag{27}$$

obeys the dominant energy condition.

The gauge choice (19) for $\epsilon$ is not useful for proving positivity of the gravity-$Z$ system. We choose instead the modified condition

$$\delta_\epsilon \tilde{\chi}_1 = \tfrac{1}{2}Qe^{2\phi}\gamma_1 \slashed{\nabla} Z \epsilon \ . \tag{28}$$

One then finds, after some algebra,

$$M = \int d\sigma^\rho \epsilon_\rho{}^\mu (1-2Q^2 e^{2\phi})\bar{\epsilon}\gamma^\nu \epsilon \hat{T}_{\mu\nu} \ . \tag{29}$$

Since $\hat{T}_{\mu\rho}$ obeys the dominant energy condition, this expression is non-negative as long as $Z$ has support only in the region where

$$2Q^2 e^{2\phi} < 1 \ . \tag{30}$$

This last restriction comes as no surprise to those familiar with the relation between dilaton gravity-$Z$ system and the large-$N$ quantum equations, to which we now turn.

## V. QUANTUM POSITIVE-ENERGY THEOREM

The classical gravity-$Z$ dynamics of the previous section is closely related to the large-$N$ quantum dynamics of dilaton gravity minimally coupled to $N$ scalars. To see this, note that the $Z$ equation of motion,

$$-\Box Z = QR \ , \tag{31}$$

can be substituted into the trace of Einstein equations to yield

$$2e^{-2\phi}[-\Box\phi + 2(\nabla\phi)^2 - 2\lambda^2] = Q^2 R \ . \tag{32}$$

This is identical to the large-$N$ quantum trace equation of [2] for

$$Q^2 = \frac{N}{24} \ . \tag{33}$$

The dilaton equation is unaffected by matter, and so it is also identical for the two cases.

In addition to these two equations, there are the constraint equations which are most easily expressed in conformal gauge:

$$g_{++} = g_{--} = 0 \ ,$$
$$g_{+-} = -\tfrac{1}{2}e^{2\rho} \ , \tag{34}$$

where $x^\pm = x^0 \pm x^1$. The $++$ constraint equation for the $(Z,\rho,\phi)$ system is

$$0 = e^{-2\phi}(4\partial_+\rho\,\partial_+\phi - 2\partial_+^2\phi) + T^Z_{++} \ , \tag{35}$$

where

$$T^Z_{++} = \tfrac{1}{2}(\partial_+ Z)^2 + Q(\partial_+^2 Z - 2\partial_+\rho\,\partial_+ Z) \ . \tag{36}$$

The large-$N$ constraint equation is

$$0 = e^{-2\phi}(4\partial_+\rho\,\partial_+\phi - 2\partial_+^2\phi) + T^Q_{++} + T^M_{++} \ , \tag{37}$$

where $T^M$ is the classical matter stress tensor and

$$T^Q_{++} = -\frac{N}{12}[(\partial_+\rho)^2 - \partial_+^2\rho] + t_+(x^+) \ . \tag{38}$$

$t_+$ is an arbitrary function of $x^+$, which is fixed by boundary conditions. A similar equation holds for $T_{--}$. If $(\rho,\phi)$ satisfy the dilaton and large-$N$ trace equations, it is always possible to find a $t_\pm$ such that the $T_{\pm\pm}$ constraint equations hold. Since the dilaton and trace equations are identical (after using the $Z$ equations of motion) for the quantum large-$N$ and classical gravity-$Z$ systems, it follows that every $(\rho,\phi,Z)$ which satisfy the classical gravity-$Z$ equations provide a $(\rho,\phi)$ which solve the quantum large-$N$ equations.

The converse is not always true: Given a solution $(\rho,\phi)$ of the quantum equations, it is not always possible to reconstruct a solution $(Z,\rho,\phi)$ of the classical equations. Attempts to do so may run into singularities in the $Z$ field. As an example, suppose one has a solution of the quantum equation such that on the null slice $x^- = x_0^-$, $\rho = 0$, and

$$T^Q_{++}(x_0^-, x^+) + T^M_{++}(x_0^-, x_+) = a\delta(x^+) \ . \tag{39}$$

We wish to find an asymptotically constant function $\partial_+ Z(x_0^-, x^+)$ such that

$$\tfrac{1}{2}[\partial_+ Z(x_0^-, x^+)]^2 + Q\,\partial_+^2 Z(x_0^-, x^+) = a\delta(x^+) \ . \tag{40}$$

The general solution is

$$\partial_+ Z = +2Q\left[\theta(-x^+)\frac{1}{x^+ + \alpha} + \theta(x^+)\frac{1}{x^+ + \beta}\right] \ , \tag{41}$$

where $1/\beta - 1/\alpha = a/2Q^2$. This is nonsingular only if $\alpha < 0$ and $\beta > 0$, which is possible only if $a > 0$, corresponding to a positive stress tensor in (39).

This is, of course, expected: One cannot hope to prove positivity for every solution of the large-$N$ equations with unrestricted $t_\pm$, since the $t_\pm$ can be chosen to correspond to negative-energy matter.

In physically interesting situations, the $t_\pm$ are constrained. For example, solutions of the large-$N$ equations have been studied which correspond to black-hole formation and evaporation. Our results may be used to show that in these examples the Bondi mass is always positive, as follows.

A black hole can be formed by specifying that the initial data on $\mathcal{J}_L^-$ correspond to the vacuum, while on $\mathcal{J}_R^-$ one has some general incoming radiation pulse,

$$T^M_{++} = T^Q_{++} + T^M_{++} > 0 \ , \tag{42}$$

which we take to have compact support. [A shock wave corresponds to $T^M_{++} = a\delta(x^+ - x_0^+)$.] If $T^M_{++} + T^Q_{++}$ is positive, one can always construct an asymptotically constant $\partial_+ Z$ such that

$$T^Z_{++} = T^M_{++} + T^Q_{++} \tag{43}$$

on $\mathcal{J}_R^-$. The Bondi mass $M_B$ on $\mathcal{J}_R^-$ (i.e., the ADM mass) is thus positive by the theorem of the preceding section. The Bondi mass at finite values of $x^-$ may then be found from (29), integrating over the null slice of constant $x^-$. It will remain positive as long as $\partial_+ Z$ is nonsingular on

the slice and has support only in the region $e^{-2\phi} > N/12$.

When can $\partial_+ Z$ become singular? From the $Z$ equation of motion,

$$\partial_+ \partial_- Z = 2Q\,\partial_+ \partial_- \rho \; , \tag{44}$$

it is evident that if the scalar curvature diverges, $\partial_+ Z$ does as well. In fact, this condition is necessary as well as sufficient. If $\partial_+ Z$ is initially finite on $\mathcal{J}_R^-$ and $\partial_+ Z$ eventually diverges, there must be a first value of $x^-$ at which it does so. Thus $\partial_+ Z$ goes from a finite to infinite value in a finite interval, and so $\partial_+ \partial_- Z$ must diverge, together with the scalar curvature.

We thus conclude that as long as the scalar curvature is finite and the $Z$ pulse has not crossed the line $e^{-2\phi} = N/12$, $M_B$ is non-negative.

It has been shown [10,11] that when the leading edge of the pulse first intersects the line $e^{-2\phi} = N/12$, a curvature singularity appears, which then continues to the right along a spacelike trajectory. Thus every null slice which is prior to the singularity necessarily has $\partial_+ Z = 0$ in the region $e^{-2\phi} < N/12$, and positivity therefore follows from finiteness of the scalar curvature alone.

If the sequence of null slices first encounters the curvature singularity at a finite value of $x^+$ (i.e., not on $\mathcal{J}_R^+$), it is by definition a naked singularity and the end of the black hole. Our result then implies that the black-hole mass cannot become negative before it disappears. The behavior of the mass after the naked singularity will depend on the boundary condition imposed there.

On the other hand, it may be the case [12] that the null slices meet the singularity at $\mathcal{J}_R^+$. This is then the end of the spacetime, and our results imply that the Bondi mass is everywhere positive.

These statements may seem at odds with Refs. [13–16] in which it was stated, in modified versions of the large-$N$ equations, that the Bondi mass in fact does get negative before the black hole disappears. One possibility is that an analogous theorem does not exist for the modified equations. In fact, we believe, though we have not worked out the details, that the methods of Sec. VII can be used to prove $M_B > 0$ for the modified equations.

More likely, we believe the discrepancy lies in the differing definitions of $M_B$. Previous work did not include the crucial $Q\,\partial_- Z$ term,[2] which may be regarded as a quantum correction to the classical mass formula.

We do not know which is "the" correct mass formula. However, we note that supersymmetry suggests that the energy should be positive even at the quantum level, as we have indeed found to be the case with our modified $M_B$.

## VI. ENERGY BOUND FOR BLACK-HOLE SPACETIMES

In four-dimensional general relativity, there is a conjecture due to Penrose that the mass $M_i$ of an initial data

set containing apparent horizons is bounded by the square root of the total area $A_i$ of the apparent horizons [6]. The motivation behind this conjecture is that eventually the system is expected to settle down to Schwarzchild with a larger horizon area $A_f$ (by the area theorem, which assumes cosmic censorship) and less energy (having lost some in outgoing radiation). The final mass $M_f$ is then proportional to the square root of the final area so that

$$M_i \geq M_f = \left[ \frac{A_f}{16\pi} \right]^{1/2} \geq \left[ \frac{A_i}{16\pi} \right]^{1/2} \; ,$$

implying Penrose's conjecture $M > \sqrt{A/16\pi}$.

This conjecture has been proven only in special cases in four dimensions [17]. In this section an analogous inequality will be proven for two-dimensional dilaton gravity. The expected inequality can be phrased by noting that the value of $\lambda e^{-2\phi}$ at the horizon plays the role analogous to that played by $\sqrt{A/16\pi}$ in four dimensions. (Indeed, when two-dimensional dilaton gravity is derived by dimensional reduction of spherically symmetric four-dimensional black holes [2,10,18], $e^{-4\phi}$ is proportional to the area of the two spheres at constant radius.) In the absence of matter, the black-hole mass is exactly $\lambda e^{-2\phi}|_H$. Adding positive-energy matter outside the black hole should only increase the total mass, and so one expects a bound

$$M \geq \lambda e^{-2\phi}|_H \; . \tag{45}$$

Black-hole spacetimes in dilaton gravity are asymptotic to the vacuum only as $\sigma \to +\infty$ and therefore are not covered by the analysis of Sec. III. In this section we shall prove that such spacetimes satisfy the energy bound (45) provided only that the matter stress tensor is positive outside the outermost apparent horizon.

An apparent horizon is a line along which the gradient of the dilaton field becomes null [19,1]. The existence of an apparent horizon can be determined from the initial data on a spacelike slice. This is in contrast with an event horizon, whose location can be determined only when the entire spacetime is known.

In this section we will use an alternative formula for the mass

$$\widetilde{M} = e^{-2\phi} \left[ \lambda - \frac{1}{\lambda}(\nabla\phi)^2 \right] \; , \tag{46}$$

which is similar to a formula employed by Susskind and Thorlacius [19]. It is easy to check that (46) agrees asymptotically with (13), though the two are not the same at finite points. In terms of $\widetilde{M}$, the metric equation of motion (17) can be written

$$2e^{-2\phi}\epsilon_\mu{}^\rho \epsilon_\nu{}^\sigma \nabla_\rho \nabla_\sigma \phi = 2\lambda g_{\mu\nu}\widetilde{M} + T_{\mu\nu} \; . \tag{47}$$

It is then easy to show using (47) that

$$d\widetilde{M} = (d\sigma^\rho \epsilon_\rho{}^\nu) \left[ \frac{1}{\lambda}\epsilon^{\mu\sigma}\nabla_\sigma \phi \right] T_{\mu\nu} \; . \tag{48}$$

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[2]Note that if $\rho$ and $Z$ both vanish on $\mathcal{J}_L^-$, which is true in some gauge for most cases of interest, quantum mass correction is by virtue of (44) proportional to $\partial_- \rho$.

Since $\epsilon^{\mu\sigma}\nabla_\sigma\phi$ is timelike and $T_{\mu\nu}$ obeys the dominant energy condition, this implies that $\tilde{M}$ increases away from the horizon. One therefore concludes that $\tilde{M}|_H$ is less than or equal to the ADM mass $\tilde{M}|_{+\infty}$. On the other hand, since $(\nabla\phi)^2$ vanishes at a horizon,

$$\tilde{M}|_H = \lambda e^{-2\phi}|_H . \tag{49}$$

So we have established the desired inequality (45). Actually, this method easily establishes positivity of energy for the case of two asymptotically flat directions as well, but the spinorial proof was used in Sec. III because it elucidates the connection with supersymmetry and naturally generalizes to the models of Secs. IV and VII.

The theorems of this section strengthen the analogy of two-dimensional dilaton gravity with four-dimensional general relativity. It would certainly be of interest to formulate and prove (or disprove) cosmic censorship in this theory.

## VII. GENERALIZED DILATON GRAVITY

The action (4) is, of course, a very special form for the two-dimensional dynamics of a scalar field coupled to gravity. In many contexts it is of interest to consider a more general form for the dynamics. At the very least, quantum corrections will generate corrections to (4) as a power series in $e^{2\phi}$. Indeed, these corrections have been found to play a crucial role in black-hole dynamics [20,13–16]. In this section we consider the most general, power-counting renormalizable, supersymmetric action:[3]

$$S_G = \frac{i}{2\pi} \int d^2x \, d^2\theta \, E[J(\Phi)S$$

$$+ iK(\Phi)D_\alpha\Phi D^\alpha\Phi + L(\Phi)] . \tag{50}$$

Presumably, many choices of the functions $J$, $K$, and $L$ lead to sick theories with unphysical behavior. It is a difficult (but interesting) problem to characterize the dynamics of the general theory. In this section we will analyze two key properties of these theories: the existence of a supersymmetric ground state and positivity of the energy.

We first record the bosonic part of the action following from (50). The auxiliary field equations of motion are

$$F = -\frac{L}{2J'} ,$$
$$A = -\frac{L'}{J'} + \frac{2KL}{J'^2} , \tag{51}$$

where $J'$ $(L')$ is the derivative of $J$ $(L)$ with respect to $\phi$. This leads to the bosonic action

$$S_G = \frac{1}{2\pi} \int d^2x \, e \left[ JR + 2K(\nabla\phi)^2 + \frac{LL'}{2J'} - \frac{KL^2}{2J'^2} \right] . \tag{52}$$

The supersymmetry transformation laws are obtained by substituting (51) into (7). A supersymmetric vacuum is one for which there is a spinor $\epsilon$ such that

$$\delta_\epsilon\Lambda = \left[ \slashed{\nabla}\phi - \frac{L}{2J'} \right]\epsilon = 0 , \tag{53}$$

$$\delta_\epsilon\chi_\mu = \left[ 2D_\mu - \gamma_\mu \left[ \frac{L'}{2J'} - \frac{KL}{J'^2} \right] \right]\epsilon = 0 . \tag{54}$$

These equations imply that

$$k^\mu = \bar{\epsilon}\gamma^\mu\epsilon \tag{55}$$

is a Killing vector,

$$\nabla_{(\nu}k_{\mu)} = 0 , \tag{56}$$

which also generates a symmetry of $\phi$,

$$k^\mu\nabla_\mu\phi = 0 , \tag{57}$$

and is timelike:

$$k^2 < 0 . \tag{58}$$

Multiplying in (53) by $\bar{\epsilon}\gamma_5/(-k^2)^{1/2}$, one finds

$$\frac{d}{d\sigma}\phi = -\frac{L}{2J'} , \tag{59}$$

where

$$\sigma(x) = \int_{-\infty}^x dx^\mu \frac{\epsilon_{\mu\nu}k^\nu}{(-k^2)^{1/2}} \tag{60}$$

is a spatial coordinate. $\phi(\sigma)$ is then the solution of

$$\sigma = -\int \frac{2J'(\phi)}{L(\phi)} d\phi . \tag{61}$$

The curvature is determined from the integrability condition for (54). Given (53), one finds

$$R = -\tfrac{1}{2}(A^2 + 2A'F) , \tag{62}$$

where $A$ and $F$ are given in (51).

Equation (61) will have a solution for every $\sigma$ as long as $L/J'$ is everywhere finite. The reason for the absence of a nonsingular vacuum if $J'$ has zeros is evident from (50): The kinetic part of the action degenerates at zeros of $J'$. If the right-hand side of (62) is everywhere finite, then it can be solved to determine the geometry. Thus a supersymmetric vacuum exists in a wide variety of cases.

Following the steps of Sec. III, an expression for the mass at $\sigma = +\infty$ can for spacetimes asymptotic to the vacuum be found as

$$M(+\infty) = -\bar{\epsilon}\gamma_5 \left[ J'\slashed{\nabla}\phi - \frac{L}{2} \right]\epsilon , \tag{63}$$

where $\epsilon$ is asymptotically a solution of (53) and (54) normalized according to (14). Integrating by parts as in Sec. III, one finds

$$M(+\infty) = \int d\sigma^\mu (\epsilon_\mu^{\ \nu}\bar{\epsilon}\gamma^\rho\epsilon T_{\nu\rho} - J'\delta_\epsilon\bar{\Lambda}\gamma_5\delta_\epsilon\tilde{\chi}_\mu)$$

$$+ M(-\infty) , \tag{64}$$

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[3]Field redefinitions can be used to locally eliminate two of the three free functions in $S_G$. However, since global considerations may be important, we will not do this.

where

$$\delta_\epsilon \widetilde{\chi}_\mu = \delta_\epsilon \chi_\mu + \gamma_\mu \frac{K}{J'} \delta_\epsilon \Lambda \ .$$

Choosing $\epsilon$ so that $\delta_\epsilon \widetilde{\chi}_1 = 0$, one has

$$M(+\infty) = \int d\sigma^\rho \epsilon_p{}^\nu \overline{\epsilon} \gamma^\mu \epsilon T_{\nu\mu} + M(-\infty) \ . \qquad (65)$$

This is manifestly non-negative if $M(-\infty)$ is.

A sufficient condition for $M(-\infty)$ to vanish is that $J'(\phi(-\infty))$ and $L(\phi(-\infty))$ vanish. This is the case for the model of Sec. III and is true for a wide range of choices of $J$ and $L$. We do not know the necessary conditions for non-negativity of $M(+\infty)$.

We expect that similar results can be derived for generalized classical equations by adding the $Z$ field as in Sec. V. This would be relevant to some of the semiclassical models studied in [13–16].

[1] For recent reviews, see J. A. Harvey and A. Strominger, in Proceedings of the 1992 TASI Summer School in Boulder, Colorado (unpublished); S. B. Giddings, "Toy Models for Black Hole Evaporation," in Proceedings of the International Workshop of Theoretical Physics, 6th Session, Erice, Italy, 1992 (unpublished).

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