Quantum vacuum instability near rotating stars

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We discuss the Starobinskii-Unruh process for the Kerr black hole. We show how this effect is related to the theory of squeezed states. We then consider a simple model for a highly relativistic rotating star and show that the Starobinskii-Unruh effect is absent.

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I. INTRODUCTION

After decades of investigation, confusion still remains concerning the nature of the quantum vacuum and the instabilities that may afflict it. Early investigators regarded the breakdown of the quantum vacuum state in the presence of strong external fields as paradoxical (e.g., the Klein paradox, the Schiff-Snyder-Weinberg paradox). In more recent years, however, the creation of particles through such instabilities has been treated as a real and possibly observable phenomenon [1].

One of the most intensively studied examples of vacuum instability is the Hawking black hole evaporation process [2] where a gravitational field causes thermal particle production. This process, and its distinctly thermal character, are associated with the existence of an event horizon around the black hole. Related to the Hawking effect, but predating its discovery, is the prediction that particles will be produced by the rotational motion of the black hole—the so-called Starobinskii-Unruh process [3]. This particular vacuum instability arises because of the existence of an ergosphere in which particles may reside with negative energy as measured from the asymptotic region away from the body. Such an ergosphere leads to the classical phenomenon of wave amplification known as superradiance; the Starobinskii-Unruh effect is the quantum counterpart of this.

Hawking's treatment of black hole quantum processes provides an elegant unified description of both of the above effects, and it is therefore tempting to attribute both types of radiation to essentially the same origin. Nevertheless, there remains considerable uncertainty as to whether the Starobinskii-Unruh effect is primarily a consequence of the event horizon, or the ergosphere. The issue becomes relevant when consideration is given to the possibility of very compact rapidly rotating stars that might have an ergosphere but no event horizon. One is led to the question: Would the quantum vacuum in the vicinity of such an object be stable, or might one expect the Starobinskii-Unruh effect to occur in that case too?

In this paper, we study a particular model for a rotating star, and conclude that there is no particle creation. In the language of curved space quantum field theory we are investigating a case in which there is a natural Killing vector which is timelike (though not hypersurface-orthogonal) in part but not all of the space-time. Although our model is somewhat artificial, it has the virtue of permitting a detailed treatment, and therefore leading to a reasonably secure conclusion.

We shall start by reviewing the phenomenon of classical superradiance and the Starobinskii-Unruh effect. We shall then show how superradiance can be expressed as a squeezing of the vacuum before going on to study our model in which we consider a quantized scalar field above a reflecting surface inside the ergosphere of Kerr space-time. Note, however, that we make no assumptions concerning the metric inside the reflecting surface, in particular, there may or may not be an event horizon inside.

II. SUPERRADIANCE

In this section we will briefly review the solution of the scalar wave equation in the Kerr metric [5] and the classical phenomena of superradiance [6].

The Klein-Gordon equation for a massless scalar field $\Phi(x)$ is

$$\frac{\partial}{\partial x^\mu}\left(g^{1/2}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\Phi=0. \tag{2.1}$$

We are interested in the Kerr metric which in Boyer–Lindquist coordinates has the form

$$ds^2=-\left(1-\frac{2mr}{\rho^2}\right)dt^2-\frac{4mra}{\rho^2}\sin^2\theta\,d\phi\,dt+\frac{\rho^2}{\triangle}\,dr^2 +\rho^2\,d\theta^2+\left[r^2(r^2+a^2)+2mra^2\right]\frac{\sin^2\theta}{\rho^2}d\phi^2 \tag{2.2}$$

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where $\triangle = (r - r_+)(r - r_-) = r^2 - 2mr + a^2$ with $r_+ = M + \sqrt{M^2 - a^2}$ the horizon radius and $\rho^2 = r^2 + a^2\cos^2\theta$. As is well known, in the Kerr metric Eq. (2.1) is separable [5] and gives rise to solutions of the form

$$\Phi(x) = \frac{N_\omega}{(r^2+a^2)^{1/2}} e^{-i\omega t + im\phi} S_{lm}(\theta) R_{\omega lm}(r), \tag{2.3}$$

where N_ω is a normalization factor, $S_{lm}(\theta)$ is a spheroidal harmonic, l and m are integers, and $|m| \le l$. It is convenient to define a new radial coordinate r_* by

$$\frac{dr_*}{dr} = \frac{r^2+a^2}{\triangle}, \tag{2.4}$$

which ranges over the entire real line, pushing the horizon off to minus infinity. In terms of this coordinate the radial equation takes the form

$$\left(\frac{d^2}{d^2 r_*} - V_{\omega lm}(r)\right) R_{\omega lm}(r) = 0. \tag{2.5}$$

In the asymptotic regions $r_* \to \pm\infty$ the potential V reduces to

$$V_{\omega lm}(r) \to \begin{cases} -\omega^2, \\ -(\omega - m\Omega_h)^2, \end{cases} \tag{2.6}$$

where $\Omega_h = a/(2Mr_+)$ is the angular velocity of the horizon.

We can consider two classes of solutions to (2.5). Waves from $\mathcal{I}^-$ will be partially scattered back to $\mathcal{I}^+$ by V and partially transmitted through to $\mathcal{H}^+$. Similarly waves may propagate from $\mathcal{H}^-$ and be scattered into either asymptotic region. By virtue of (2.6) these two classes of solution will have the asymptotic form

$$R^+_{\omega lm}(r) \sim \begin{cases} B^+_{\omega lm} e^{-i\tilde\omega r_*}, & r_* \to -\infty, \\ e^{-i\omega r_*} + A^+_{\omega lm} e^{i\omega r_*}, & r_* \to \infty, \end{cases} \tag{2.7}$$

and

$$R^-_{\omega lm}(r) \sim \begin{cases} e^{i\tilde\omega r_*} + A^-_{\omega lm} e^{-i\tilde\omega r_*}, & r_* \to -\infty, \\ B^-_{\omega lm} e^{i\omega r_*}, & r_* \to \infty, \end{cases} \tag{2.8}$$

where $\tilde\omega = \omega - m\Omega_h$, $\omega > 0$.

Suppressing the subscripts for convenience, the coefficients A^+, A^-, B^+, B^- satisfy the relations

$$|A^+|^2 = 1 - \frac{\tilde\omega}{\omega}|B^+|^2, \tag{2.9}$$

$$|A^-|^2 = 1 - \frac{\omega}{\tilde\omega}|B^-|^2, \tag{2.10}$$

$$\omega B^- = \tilde\omega B^+, \tag{2.11}$$

$$A^{+*} B^- = -\frac{\tilde\omega}{\omega} A^- B^{+*}. \tag{2.12}$$

Equations (2.9) and (2.10) show that for $\tilde\omega < 0$, $|A^+|^2 > 1$ and $|A^-|^2 > 1$, so these modes from $\mathcal{I}^-$ and $\mathcal{H}^-$ are reflected to $\mathcal{I}^+$ and $\mathcal{H}^+$ with an amplitude greater than they had initially. This is the classical phenomenon of superradiance. In the next section we shall discuss its quantum field theoretic analogue.

III. THE STAROBINSKII-UNRUH PROCESS

Before we discuss our model rotating star we must set the scene by discussing the Starobinskii-Unruh process for the Kerr black hole. We will follow the method of Ford [7] and hope to clarify it as well as show the connection between the Starobinskii-Unruh process and squeezed states which has not previously been elucidated.

The quantization of a scalar field in Kerr spacetime is achieved first by finding a complete, orthonormal set of solutions to (2.1). We take as our "in" quantization basis

$$R^{\rm in}_{\omega lm} = \frac{e^{-i\omega t} e^{im\phi} S_{\omega lm}(\theta) R^+_{\omega lm}(r)}{2\pi(2\omega)^{1/2}(r^2+a^2)^{1/2}}, \qquad \forall\tilde\omega,$$

$$R^{\rm out}_{\omega lm} = \frac{e^{-i\omega t} e^{im\phi} S_{\omega lm}(\theta) R^-_{\omega lm}(r)}{2\pi(2\tilde\omega)^{1/2}(r^2+a^2)^{1/2}}, \qquad \tilde\omega > 0, \tag{3.1}$$

$$R^{\rm out}_{-\omega l-m} = \frac{e^{i\omega t} e^{-im\phi} S_{\omega lm}(\theta) R^-_{-\omega l-m}(r)}{2\pi(-2\tilde\omega)^{1/2}(r^2+a^2)^{1/2}}, \qquad \tilde\omega < 0,$$

where we have used the property $S_{\omega lm}(\theta) = S_{-\omega l-m}(\theta)$. These solutions are orthonormal in the Klein-Gordon scalar product, that is

$$\begin{aligned}(R^{\rm in}_{\omega lm}, R^{\rm in}_{\omega' l' m'}) &= (R^{\rm out}_{\omega lm}, R^{\rm out}_{\omega' l' m'}) \\ &= (R^{\rm out}_{-\omega l-m}, R^{\rm out}_{-\omega' l'-m'}) \\ &= \delta(\omega - \omega')\delta_{ll'}\delta_{mm'},\end{aligned} \tag{3.2}$$

where

$$(\phi_1, \phi_2) = i\int_\Sigma \phi_1^* \overleftrightarrow{\partial_\mu} \phi_2 \sqrt{-g}\, d\Sigma^\mu \tag{3.3}$$

and all other inner products vanish.

In (3.1) for $\tilde\omega < 0$ we have a negative energy wave propagating to $\mathcal{I}^+$. This is a consequence of ∂_t not being a globally timelike Killing vector. ∂_t is spacelike in the ergosphere; however, the combination $\partial_t + \Omega\partial_\phi$, where $\Omega = -g_{t\phi}/g_{\phi\phi}$, is timelike down to the horizon upon which it becomes null. Observers following integral curves of this timelike vector field are locally nonrotating observers (LNRO's). A LNRO near the horizon would measure the frequency of the superradiant modes in (3.1) to be $-\tilde\omega = -\omega + m\Omega_h$ [where $\Omega_h = \Omega(r = r_+)$]. Since $\tilde\omega < 0$ for superradiant modes the LNRO would see positive frequency waves for all modes. For $R^{\rm in}_{\omega lm}$ all modes are positive frequency at $\mathcal{I}^+$ and $\mathcal{I}^-$. A LNRO near the horizon measures $\tilde\omega$ for the frequency and thus sees negative frequency modes in the superradiant regime. We will assume that (2.5) has no complex frequency eigenvalues. This should be a reasonable assumption since computer searches [8] have not revealed any complex frequency modes. Also it has recently been shown analytically [9] that (2.1) has no unstable solutions [i.e., Im$(\omega) > 0$].

The scalar field may now be expanded in terms of the mode solutions (3.1). We find

$$\Phi(x) = \sum_{lm}\int_0^\infty d\omega\,(a^{\text{in}}_{\omega lm}R^{\text{in}}_{\omega lm} + a^{\dagger\text{in}}_{\omega lm}R^{*\text{in}}_{\omega lm}) + \sum_{lm}\int_{\omega_{\min}}^\infty d\omega\,(a^{\text{out}}_{\omega lm}R^{\text{out}}_{\omega lm} + a^{\dagger\text{out}}_{\omega lm}R^{*\text{out}}_{\omega lm}) + \sum_{lm}\int_0^{\omega_{\min}} d\omega\,(a^{\text{out}}_{-\omega l-m}R^{\text{out}}_{-\omega l-m} + a^{\dagger\text{out}}_{-\omega l-m}R^{*\text{out}}_{-\omega l-m}). \tag{3.4}$$

We promote the expansion coefficients to operators obeying the usual commutation relations

$$[\hat{a}^{\text{in}}_{\omega lm}, \hat{a}^{\dagger\text{in}}_{\omega' l' m'}] = [\hat{a}^{\text{out}}_{\omega lm}, \hat{a}^{\dagger\text{out}}_{\omega' l' m'}] = [\hat{a}^{\text{out}}_{-\omega l-m}, \hat{a}^{\dagger\text{out}}_{-\omega' l'-m'}] = \delta(\omega-\omega')\delta_{ll'}\delta_{mm'}, \tag{3.5}$$

with all other commutators vanishing. By using the asymptotic expressions (2.7) and (2.8) we can see that $R^{\text{in}}_{\omega lm}$ describes unit incoming flux from $\mathcal{I}^-$ and zero outgoing flux from $\mathcal{H}^-$ while $R^{\text{out}}_{\omega lm}$, $R^{\text{out}}_{-\omega l-m}$ describes unit outgoing flux from $\mathcal{H}^-$ and zero incoming flux from $\mathcal{I}^-$. Therefore $\hat{a}^{\dagger\text{in}}_{\omega lm}$ and $\hat{a}^{\dagger\text{out}}_{\omega lm}$, $\hat{a}^{\dagger\text{out}}_{-\omega l-m}$ will create particles from $\mathcal{I}^-$ and $\mathcal{H}^-$ respectively. Thus we can define a vacuum state $|0,0\rangle_{\text{in}}$ by

$$\hat{a}^{\text{in}}_{\omega lm}|0,0\rangle_{\text{in}} = \hat{a}^{\text{out}}_{\omega lm}|0,0\rangle_{\text{in}} = \hat{a}^{\text{out}}_{-\omega l-m}|0,0\rangle_{\text{in}} = 0, \tag{3.6}$$

which corresponds to an absence of particles from $\mathcal{I}^-$ and $\mathcal{H}^-$.

We can show that the mode functions defined by

$$\begin{aligned} S^{\text{out}}_{\omega lm} &= A^{+\,*}R^{\text{in}}_{\omega lm} + B^{-\,*}\left(\tfrac{\omega}{\tilde\omega}\right)^{1/2}R^{\text{out}}_{\omega lm}, \\ S^{\text{in}}_{\omega lm} &= A^{-\,*}R^{\text{out}}_{\omega lm} + B^{+\,*}\left(\tfrac{\tilde\omega}{\omega}\right)^{1/2}R^{\text{in}}_{\omega lm}, \end{aligned} \qquad \tilde\omega > 0, \tag{3.7}$$

$$\begin{aligned} S^{\text{in}}_{-\omega l-m} &= A^{-}R^{\text{out}}_{-\omega l-m} - B^{+}\left(\tfrac{-\tilde\omega}{\omega}\right)^{1/2}R^{*\text{in}}_{\omega lm}, \\ S^{\text{out}}_{\omega lm} &= A^{+\,*}R^{\text{in}}_{\omega lm} - B^{-\,*}\left(\tfrac{\omega}{-\tilde\omega}\right)^{1/2}R^{*\text{out}}_{-\omega l-m}, \end{aligned} \qquad \tilde\omega < 0 \tag{3.8}$$

have the asymptotic form (for $\tilde\omega < 0$)

$$S^{\text{out}}_{\omega lm}(r) \sim \begin{cases} B^{-\,*}\dfrac{\omega}{\tilde\omega}e^{i\tilde\omega r_*}, & r_* \to -\infty, \\ e^{i\omega r_*} + A^{+\,*}e^{-i\omega r_*}, & r_* \to \infty, \end{cases} \tag{3.9}$$

$$S^{\text{in}}_{-\omega l-m}(r) \sim \begin{cases} e^{i\tilde\omega r_*} + A^{-}e^{-i\tilde\omega r_*}, & r_* \to -\infty, \\ B_{+}\dfrac{\tilde\omega}{\omega}e^{i\omega r_*}, & r_* \to \infty. \end{cases} \tag{3.10}$$

We see that $S^{\text{out}}_{\omega lm}$ describes unit outgoing flux to $\mathcal{I}^+$ and zero ingoing flux to $\mathcal{H}^+$ while $S^{\text{in}}_{-\omega l-m}$ describes unit ingoing flux to $\mathcal{H}^+$ and zero outgoing flux to $\mathcal{I}^+$. Nonsuperradiant modes have similar asymptotic properties. These modes have identical inner product relations to (3.2) and hence we can write the field expansion as

$$\Phi(x) = \sum_{lm}\int_0^\infty d\omega\,(b^{\text{out}}_{\omega lm}S^{\text{out}}_{\omega lm} + b^{\dagger\text{out}}_{\omega lm}S^{*\text{out}}_{\omega lm}) + \sum_{lm}\int_{\omega_{\min}}^\infty d\omega\,(b^{\text{in}}_{\omega lm}S^{\text{in}}_{\omega lm} + b^{\dagger\text{in}}_{\omega lm}S^{*\text{in}}_{\omega lm}) + \sum_{lm}\int_0^{\omega_{\min}} d\omega\,(b^{\text{in}}_{-\omega l-m}S^{\text{in}}_{-\omega l-m} + b^{\dagger\text{in}}_{-\omega l-m}S^{*\text{in}}_{-\omega l-m}). \tag{3.11}$$

We promote the expansion coefficients to operators with commutation relations equivalent to those in (3.5). Given the asymptotic properties of the modes defined in (3.7) and (3.8) $\hat{b}^{\dagger\text{out}}_{\omega lm}$ and $\hat{b}^{\dagger\text{in}}_{\omega lm}$, $\hat{b}^{\dagger\text{in}}_{-\omega l-m}$ will create particles propagating to $\mathcal{I}^+$ and $\mathcal{H}^+$ respectively. Thus we can define a vacuum state $|0,0\rangle_{\text{out}}$ by

$$\hat{b}^{\text{out}}_{\omega lm}|0,0\rangle_{\text{out}} = \hat{b}^{\text{in}}_{\omega lm}|0,0\rangle_{\text{out}} = \hat{b}^{\text{in}}_{-\omega l-m}|0,0\rangle_{\text{out}} = 0, \tag{3.12}$$

which corresponds to an absence of particles propagating to $\mathcal{I}^+$ and $\mathcal{H}^+$.

Equations (3.7) and (3.8) represent the Bogoliubov transformation between our two sets of complete modes. For superradiant modes they give rise to the operator relations

$$\begin{aligned} \hat{a}^{\text{in}}_{\omega lm} &= A^{+\,*}\hat{b}^{\text{out}}_{\omega lm} - B^{+\,*}\left(\tfrac{-\tilde\omega}{\omega}\right)^{1/2}\hat{b}^{\dagger\text{in}}_{-\omega l-m}, \\ \hat{a}^{\text{out}}_{-\omega l-m} &= A^{-}\hat{b}^{\text{in}}_{-\omega l-m} - B^{-}\left(\tfrac{\omega}{-\tilde\omega}\right)^{1/2}\hat{b}^{\dagger\text{out}}_{\omega lm}. \end{aligned} \tag{3.13}$$

For nonsuperradiant modes the equivalent relations do not mix conjugated and nonconjugated operators. This means that $|0,0\rangle_{\text{out}}$ and $|0,0\rangle_{\text{in}}$ are equivalent vacua for these modes. We can now calculate the average number of outgoing particles spontaneously emitted into the superradiant modes. For any superradiant mode this is given by

$$\langle N\rangle = {}_{\text{in}}\langle 0,0|\hat{b}^{\dagger\text{out}}\hat{b}^{\text{out}}|0,0\rangle_{\text{in}} = |A_+|^2 - 1. \tag{3.14}$$

It is possible to express the state $|0,0\rangle_{\text{in}}$ in terms of the theory of squeezed states [10]. Temporarily dropping subscripts for convenience, we can write equations (3.13) as

$$\hat{a}^{\text{in}} = u\hat{b}^{\text{out}} + v\hat{b}^{\dagger\text{in}}, \quad \hat{a}^{\dagger\text{out}} = w\hat{b}^{\text{out}} + z\hat{b}^{\dagger\text{in}}, \tag{3.15}$$

where

$$u = A^{+\,*}, \qquad v = -B^{+\,*}\left(\frac{-\tilde{\omega}}{\omega}\right)^{1/2}, \qquad w = -B^{-}\left(\frac{\omega}{-\tilde{\omega}}\right)^{1/2}, \qquad z = A^{-\,*} \tag{3.16}$$

and, with the help of (2.9)–(2.12), the following relations can be verified:

$$u = z^*, \quad v = w^*, \quad u^*u - v^*v = 1, \quad z^*z - w^*w = 1. \tag{3.17}$$

These relations allow us to introduce the new parameters r, φ and ϑ defined by the equations

$$u = e^{-i\vartheta}\cosh r, \quad v = -e^{-i(\vartheta-2\varphi)}\sinh r, \tag{3.18}$$

$$w = -e^{i(\vartheta-2\varphi)}\sinh r, \quad z = e^{i\vartheta}\cosh r, \tag{3.19}$$

where r, φ, and ϑ are real numbers and $r \geq 0$. It is possible to rewrite (3.15) as

$$\hat{a}^{\text{in}} = \hat{R}^\dagger\hat{S}^\dagger\hat{b}^{\text{out}}\hat{S}\hat{R}, \quad \hat{a}^{\dagger\text{out}} = \hat{R}^\dagger\hat{S}^\dagger\hat{b}^{\dagger\text{in}}\hat{S}\hat{R} \tag{3.20}$$

where $\hat{S}$ and $\hat{R}$ are the unitary operators

$$\hat{S}(r,\varphi) = \exp[r(e^{-2i\varphi}\hat{b}^{\text{out}}\hat{b}^{\text{in}} - e^{2i\varphi}\hat{b}^{\dagger\text{out}}\hat{b}^{\dagger\text{in}})], \tag{3.21}$$

$$\hat{R}(\vartheta) = \exp[-i\vartheta(\hat{b}^{\dagger\text{out}}\hat{b}^{\text{out}} + \hat{b}^{\dagger\text{in}}\hat{b}^{\text{in}})]. \tag{3.22}$$

The operator $\hat{S}(r,\varphi)$ is a two-mode squeeze operator and the operator $\hat{R}(\vartheta)$ is a rotation operator. If we consider a function of operator arguments $F(\hat{a}^{\text{in}}, \hat{a}^{\text{out}}, \hat{a}^{\dagger\text{in}}, \hat{a}^{\dagger\text{out}})$ and a quantum state $|x_{\text{in}}\rangle$, we can show using (3.20) and the unitarity properties of (3.21) and (3.22) that

$$\begin{aligned}\langle x_{\text{in}}|F(\hat{a}^{\text{in}}, \hat{a}^{\text{out}}, \hat{a}^{\dagger\text{in}}, \hat{a}^{\dagger\text{out}})|x_{\text{in}}\rangle &= \langle x_{\text{in}}|\hat{R}^\dagger\hat{S}^\dagger F(\hat{b}^{\text{in}}, \hat{b}^{\text{out}}, \hat{b}^{\dagger\text{in}}, \hat{b}^{\dagger\text{out}})\hat{S}\hat{R}|x_{\text{in}}\rangle \\ &= \langle x_{\text{out}}|F(\hat{b}^{\text{in}}, \hat{b}^{\text{out}}, \hat{b}^{\dagger\text{in}}, \hat{b}^{\dagger\text{out}})|x_{\text{out}}\rangle,\end{aligned} \tag{3.23}$$

where

$$|x_{\text{out}}\rangle = \hat{S}(r,\varphi)\hat{R}(\vartheta)|x_{\text{in}}\rangle. \tag{3.24}$$

Since we are working in the Heisenberg picture we are interested in the state $|x_{\text{in}}\rangle$. Thus we can invert (3.24) using the properties of (3.21) and (3.22). We find

$$|x_{\text{in}}\rangle = \hat{S}(r,\varphi+\pi/2+\vartheta)\hat{R}(-\vartheta)|x_{\text{out}}\rangle. \tag{3.25}$$

For the special case where we use the in and out vacua we find

$$|0,0\rangle_{\text{in}} = \hat{S}(r,\varphi+\pi/2+\vartheta)|0,0\rangle_{\text{out}}, \tag{3.26}$$

since the rotation operator has no effect on the vacuum state. As all superradiant modes are squeezed, the in vacua can be written as

$$|0,0\rangle_{\text{in}} = \prod_{\substack{\omega l m\\ \tilde{\omega}<0}} \hat{S}_{\omega l m}(r,\varphi+\pi/2+\vartheta)|0,0\rangle_{\text{out}}. \tag{3.27}$$

Two-mode squeezed states also occur naturally in particle creation processes in expanding universes [11]. As well as their interesting noise properties the two modes of a two-mode squeezed state are as strongly correlated as quantum mechanics will allow [12].

In practice one would only be able to measure observables that depend on the outgoing particles only. Thus we are interested in finding the reduced density matrix of (3.26), which is obtained by expressing (3.26) as a density matrix in the number basis and tracing over the ingoing modes. We find

$$\rho_{\text{red}} = (1-\tanh^2 r)\sum_{n=0}^{\infty}(\tanh^2 r)^n\,|n\rangle\langle n|. \tag{3.28}$$

Using (3.16)–(3.19) we can write this as

$$\rho_{\text{red}} = \sum_{n=0}^{\infty}\frac{1}{|A^+|^2}\left(1-\frac{1}{|A^+|^2}\right)^n|n\rangle\langle n|. \tag{3.29}$$

Thus

$$P^n_{\omega l m} = \frac{1}{|A^+|^2}\left(1-\frac{1}{|A^+|^2}\right)^n$$

is the probability of finding n particles in the superradiant mode ω, l, m. This is the Starobinskii-Unruh process.

IV. SUPPRESSION OF QUANTUM SUPERRADIANCE

In this section we shall investigate the vacuum stability of a highly relativistic rotating star by considering the effect a reflecting boundary condition outside the horizon has on the Starobinskii-Unruh process. If the boundary is outside the ergosphere then the space-time is stationary and there will be a stable vacuum. We are interested in the case when the reflecting surface is sufficiently close to the horizon so that the space-time still has an ergoregion. In this case the space-time is not stationary since it does not possess a Killing vector which is everywhere timelike, and the stability of the vacuum is an open question.

We should add that there is no equivalent to Birkhoff's theorem for a rotating star and so the space-time outside may depend on the details of the star. As we are interested in constructing a simple model, we shall take the space-time outside the star to be given by the Kerr metric. We need make no assumptions concerning the metric inside the star, in particular, there may or may not be an event horizon.

As in the previous section, we need to find two sets of modes that give rise to appropriate in and out vacua. However, now these modes must also satisfy the boundary condition that they vanish at the surface of the star, $r_* = x$. For our in vacuum basis set we choose

$$F_{\omega lm} = \frac{1}{(r^2+a^2)^{1/2}} e^{-i\omega t + im\phi} S_{lm}(\theta) G_{\omega lm}(r), \tag{4.1}$$

with

$$G_{lm\omega} = \begin{cases} \frac{1}{N^F_{\omega lm}(x)}[R^{\rm in}_{\omega lm} + \alpha_{\omega lm}(x) R^{\rm out}_{\omega lm}], & \tilde{\omega} > 0 \\ \frac{1}{N^F_{\omega lm}(x)}[R^{\rm in}_{\omega lm} + \alpha_{\omega lm}(x) R^{*\rm out}_{-\omega l-m}], & \tilde{\omega} < 0 \end{cases} \tag{4.2}$$

where $\alpha_{\omega lm}(x)$ is chosen so that the modes vanish at $r_* = x$ and $N^F_{\omega lm}(x)$ is an appropriate normalization factor. By Gauss's law we know that the inner product (3.3) of the above modes is time independent since the modes vanish on the timelike hypersurfaces $r_* = x$ and $r_* = \infty$. This means that the inner product must vanish when $\omega \neq \omega'$. Also the integrals over θ and ϕ are unaffected by the boundary condition hence we obtain

$$(F_{\omega lm}, F_{\omega' l' m'}) = \left(\int_x^\infty \frac{\omega - m\Omega}{N} |G_{\omega lm}|^2 \, dr_* \right) \times \delta(\omega - \omega') \delta_{ll'} \delta_{mm'} \tag{4.3}$$

where $\Omega = -g_{t\phi}/g_{\phi\phi}$, $N = \left(-g_{tt} + g_{t\phi}^2/g_{\phi\phi}\right)^{1/2}$, and we have used $n = (1/N)(\partial_t + \Omega\partial_\phi)$ as the unit normal to the $t =$ const hypersurfaces and numerical factors have been absorbed into the mode normalization factors. In (4.3) Ω, N, and $|G_{\omega lm}|^2$ are positive definite and hence the inner product has a greater chance of becoming negative as ω decreases and m increases which corresponds to the superradiant regime. We can define a set $\mathcal{S}(x)$ such that $(\omega, m) \in \mathcal{S}(x)$ if the inner product in (4.3) is negative. We find then that after suitable normalization the modes will satisfy:

$$(F_{\omega lm}, F_{\omega' l' m'}) = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}, \qquad (\omega, m) \notin \mathcal{S}(x), \tag{4.4}$$

$$(F_{-\omega l-m}, F_{-\omega l-m}) = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}, \qquad (\omega, m) \in \mathcal{S}(x). \tag{4.5}$$

Given that the modes (4.1) vanish on the horizon, correspond to unit incoming flux from $\mathcal{I}^-$, and satisfy the above inner product relations, they are appropriate modes to define the in vacuum. Thus we can write

$$\Phi(x) = \sum_{lm} \int_{\omega, m \notin \mathcal{S}(x)} d\omega \, [a_{\omega lm} F_{\omega lm} + a^\dagger_{\omega lm} F^*_{\omega lm}] + \sum_{lm} \int_{\omega, m \in \mathcal{S}(x)} d\omega \, [a_{-\omega l-m} F_{-\omega l-m} + a^\dagger_{-\omega l-m} F^*_{-\omega l-m}]. \tag{4.6}$$

We promote the expansion coefficients to operators obeying

$$[\hat{a}_{\omega lm}, \hat{a}^\dagger_{\omega' l' m'}] = [\hat{a}_{-\omega l-m}, \hat{a}^\dagger_{-\omega' l'-m'}] = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'} \tag{4.7}$$

with all other commutators vanishing. The in vacuum $|0\rangle_{\rm in}$ is defined by $\hat{a}_{\omega lm}|0\rangle_{\rm in} = \hat{a}_{-\omega l-m}|0\rangle_{\rm in} = 0$ which corresponds to an absence of particles propagating from $\mathcal{I}^-$. To define the out vacua we consider the modes

$$H_{\omega lm} = \frac{1}{N^H_{\omega lm}(x)}[S^{\rm out}_{\omega lm} + \beta_{\omega lm}(x) S^{\rm in}_{\omega lm}], \quad \tilde{\omega} > 0, \tag{4.8}$$

$$H_{\omega lm} = \frac{1}{N^H_{\omega lm}(x)}[S^{\rm out}_{\omega lm} + \beta_{\omega lm}(x) S^{*\rm in}_{-\omega l-m}], \quad \tilde{\omega} < 0, \tag{4.9}$$

where $\beta_{\omega lm}(x)$ is chosen so that the modes vanish at $r_* = x$ and $N^H_{\omega lm}(x)$ is a normalization factor. Since these modes contain unit flux propagating to $\mathcal{I}^+$ they are appropriate to define the out vacuum. If we perform a Bogoliubov transformation between the in and out modes we find:

$$H_{\omega lm} = \frac{N^F_{\omega lm}(x)}{N^H_{\omega lm}(x)} \left[A^{+\,*} + \beta_{\omega lm}(x) B^{+\,*} \left(\frac{\tilde{\omega}}{\omega}\right)^{1/2} \right] F_{\omega lm}, \quad \tilde{\omega} > 0, \tag{4.10}$$

$$H_{\omega lm} = \frac{N^F_{\omega lm}(x)}{N^H_{\omega lm}(x)} \left[A^{+\,*} - \beta_{\omega lm}(x) B^{+\,*} \left(\frac{-\tilde{\omega}}{\omega}\right)^{1/2} \right] F_{\omega lm}, \quad \tilde{\omega} < 0. \tag{4.11}$$

The inner product of modes (4.8) and (4.9) will be the same as (4.4) and (4.5) where the set $\mathcal{S}(x)$ is unchanged. This is easily verified by (4.10) and (4.11). Since the Bogoliubov transformations (4.10) and (4.11) show no frequency mixing between in and out modes, the in and out vacua are equivalent [13] and there is no particle creation.

V. CONCLUSION

It should be stressed that the stability of the quantum vacuum in our model calculation depends crucially on the reflecting boundary conditions used. In retrospect, our result might have been expected on grounds of conservation of energy and angular momentum: as the quantum vacuum in the ergoregion is effectively separated from the body of the star, there is no way that energy or angular momentum could be communicated to the field to create particles. In the case of black hole it is possible for negative energy (as seen from infinity) to flow across the horizon giving rise to the possibility of a flux of positive energy out to infinity. In the presence of the mirror no such scenario is possible.

Although a body of the sort modeled here is physically possible, it is hardly realistic, and the question arises as to whether the vacuum stability would remain in a more physically appealing model. We believe that the mirror effectively mimics the center of coordinates of the star in the case that the modes are allowed to propagate freely through the interior. This belief was justified in the case of the Hawking effect [14] where a suitably accelerating mirror accurately reproduces the effect of modes being redshifted by propagating through the interior of a collapsing star and out the other side.

In both Hawking's calculation and ours, however, there remains some vagueness concerning the generic nature of the result if account is taken of the effects of interaction between the field and the material of the star through which they propagate. Hawking appeals to the fact that the relevant modes in his calculation are highly blueshifted, and so propagate effectively freely. If, in our calculation, the modes are allowed to couple to the material of the star, then the argument from energy and momentum conservation need no longer apply, and some particle creation in the exterior region, on these grounds, seems possible. However, the details will be very model dependent and in practice, of course, the intensity of such radiation is likely to be very low.

We should add that our result appears to contradict the conclusions of Ashtekar and Magnon [4] who have given a general argument (based on their complex structure approach to particle definition) suggesting that particle production should occur in stars with ergoregions. However, while their approach is generally accepted for static space-times, it has been criticized for stationary space-times [15] on the basis that it is the Cauchy hypersurfaces rather than the Killing vector field which is crucial for the quantization.

Finally, we should also mention that in our calculation we have neglected the inclusion of complex frequency modes of the type discussed by Vilenkin [16]. These modes form a discrete set, and if any of them fall in the superradiant regime they will give rise to a novel form of vacuum instability (classically such modes are exponentially amplified, reminiscent of a laser). The quantization of such modes has been discussed by Fulling [1] in the context of a general study of vacuum instability. We hope that our calculation will help clarify this general topic.

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