Gravitational radiation from a particle in circular orbit around a black hole. I. Analytical results for the nonrotating case

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Among the most promising and interesting sources of gravitational waves for interferometric detectors, such as the ground-based Laser Interferometer Gravitational-wave Observatory (LIGO)/VIRGO system and the proposed space-based Laser Gravitational-Wave Observatory in Space (LAGOS), is the last several minutes of inspiral of a compact binary (one made of neutron stars and/or black holes). This paper is the first in a series that will carry out detailed calculations relevant to such binaries, in the case where one body is a small-mass black hole or neutron star and the other is a much more massive black hole, and the orbit is circular (aside from its gradual inspiral). These papers will focus primarily on the emitted waveforms and especially their phasing — which is crucial for extraction of information from the detectors' measurements. This first paper is restricted to the case where the massive black hole is nonrotating. The paper begins by bringing the already well-developed formalisms for computing the waveforms (the "Regge-Wheeler" and "Teukolsky" formalisms) into a combined form that is particularly well suited both for high accuracy numerical calculations (to be carried out in paper II), and for analytic calculations (this paper). Then analytic solutions to the formalism's equations are found in the limiting case of orbits with large radii r_0, and correspondingly small values of $v = (M/r_0)^{1/2} = (M\Omega)^{1/3}$. Here M is the mass of the large black hole; v and Ω are, respectively, the orbital linear and angular velocities as measured far from the hole. In particular, (i) the leading-order (in v) contribution of each spherical harmonic to the waveforms and to the energy loss is computed analytically, and (ii) the full waveforms and full energy loss are computed analytically up through fractional corrections of order v^3 beyond Newtonian, i.e. up through post$^{3/2}$-Newtonian order. It is shown that propagation of the waves through the intermediate zone (which connects the near zone to the wave zone) distorts the waveforms and changes their power (and hence phasing), at post$^{3/2}$-Newtonian order, in ways that have not previously been computed — except abstractly and nonconcretely as formal "tail terms" in the waves. It is demonstrated that these post$^{3/2}$-Newtonian corrections will be of considerable importance for the extraction of information from the waveforms that LIGO/VIRGO expects to measure.

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I. INTRODUCTION AND OVERVIEW

A. LIGO/VIRGO and coalescing compact binaries: some background

With the construction of the American LIGO (Laser Interferometer Gravitational-wave Observatory) [1] and the French-Italian VIRGO [2] projects now approved and almost underway, it is quite plausible that direct detection of gravitational waves will be achieved in the near future. Still subject to approval is the British-German GEO project, and other contributions to the worldwide network may be provided by the Japanese and/or the Australians. Further ahead lies the possibility of space-based detectors, such as the proposed LAGOS (Laser Gravitational-wave Observatory in Space) [3] project. While ground-based interferometers have good sensitivity in the frequency band between 10 Hz and 1000 Hz, detectors such as LAGOS would have good sensitivity between 10^{-4} Hz and 0.1 Hz.

In the mean time, many theorists will be busy gathering a better understanding of the potential sources of gravitational waves (for a review, see Ref. [3]). Among them, the Caltech Relativity Group, together with L. S. Finn of Northwestern University, is currently hard at work in an effort to calculate (in greater detail than previously) the gravitational emissions of coalescing compact binaries, and their relevance to future observations by interferometric detectors. A detailed overview of their results may be found in Ref. [4]; this series of papers is part of that group effort.

Among the possible sources identified thus far, coalescing compact binaries have attracted a great deal of attention in recent years (see, e.g., Krolak and Schutz [5]). One of the main reasons is that the gravitational signal emitted from a compact binary will sweep a broad range of frequencies as the companions spiral in together; since LIGO/VIRGO has maximum sensitivity in the frequency band corresponding to the late evolution of the system — its last several minutes, when the signal is strongest and most interesting — this signal is ideally suited for detection. Another reason comes from the fact that the number of coalescences expected to be detected by LIGO/VIRGO should be large enough to be of practical interest [6, 7].

In addition, compact binaries are relatively clean systems, well suited to detailed and accurate theoretical modeling. As for the entire life of the binary system, so also for its last several minutes, the orbital time scale is much shorter than the inspiral time scale; the binary spends many revolutions at a given orbital separation and frequency. Furthermore, the orbit can be assumed to be circular, for any initial amount of eccentricity would have been dissipated by the emission of gravitational radiation [8]. It is therefore quite justified to suppose that, during the last several minutes, the orbit is at all times circular, with a radius adiabatically decaying due to radiation reaction. Another source of simplification arises from the fact that tidal interactions are not expected to be important, except at the very last stage of the binary's evolution [9]; the two companions may therefore be adequately modeled as compact bodies with vanishing multipole moments, except for their masses and spins.

An accurate prediction is not just an unnecessary luxury since it allows, through matched filtering [3, 10], a precise estimation of the source parameters, such as the companions' masses and spins, the orbital inclination, and the distance to the source. The predicted waveforms are functions of these parameters; matched filtering therefore cross correlates the detector output with the prediction, and varies the values of the parameters until a good cross correlation is obtained, thus identifying the values of the source parameters. Of particular importance is phase accuracy [4]; any phase shift between the detected and predicted waveforms, due to inaccurate theoretical calculations, would give rise to a poor cross correlation, and hence to a poor extraction of the source parameters.

Apart from determining the waveforms themselves, the key issue is therefore to calculate how the wave frequency changes with time. Since the wave frequency is intimately related to the orbital frequency, which is itself directly related to the orbital energy, the issue is to determine how much energy is radiated away by the gravitational waves.

At the crudest level, the gravitational luminosity (the amount of energy carried away per unit time) may be estimated using Einstein's quadrupole formula, which supposes small relative velocities. This yields [8]

$$(dE/dt)_N = \tfrac{32}{5}(\mu/M)^2(M\Omega)^{10/3}, \tag{1.1}$$

where μ and M are, respectively, the reduced and total masses of the system, and Ω is the orbital angular velocity. The subscript N stands for "Newtonian," and Eq. (1.1) is indeed a reliable estimate for the early stages of the inspiral, when both potential and kinetic energies are Newtonian.

As the companions get closer together, and as the relative velocities increase, Eq. (1.1) is no longer accurate. Wagoner and Will [11], using the tools of post-Newtonian theory, have calculated the first-order correction to the Newtonian luminosity. They find

$$dE/dt \simeq (dE/dt)_N \left\{1 - \left(\tfrac{1247}{336} + \tfrac{35}{12}\tfrac{\mu}{M}\right) v^2\right\}, \tag{1.2}$$

where $v = r_0\Omega = (M/r_0)^{1/2} = (M\Omega)^{1/3}$ is the orbital linear velocity (r_0 is the orbital separation; I have translated their result for circular orbits into the notation of this paper). As a short calculation shows, the correction term becomes significant (of order 1/10) when the gravitational-wave frequency $\nu \simeq 25$ Hz (considering a 1.4 $M_\odot$ neutron star orbiting a 10 $M_\odot$ black hole; $2\pi\nu = 2\Omega$), which lies well within the LIGO/VIRGO frequency band.

B. This paper

This series of papers addresses the following question: When the wave frequency is such that the correction term in Eq. (1.2) is important, how important are those higher-order terms which are not included in that equation?

In this paper, I show that for the limiting case where $\mu \ll M$, Eq. (1.2) generalizes to

$$dE/dt = (dE/dt)_N \left\{1 - \tfrac{1247}{336}v^2 + 4\pi v^3 + O(v^4)\right\} \tag{1.3}$$

if the black hole is nonrotating. The new term $4\pi v^3$ originates from the propagation of the gravitational waves through the intermediate zone (which connects the near zone to the wave zone), in the gravitational field of the black hole; it is a consequence of the presence of "tail terms," or "hereditary terms," in the waves [12]. I also calculate the gravitational waveforms, to the same degree of accuracy; their expressions are given in Sec. VI.

The formalism presented in this paper is also well suited for numerical calculations, which are carried out in the second paper in this series, paper II [13]. In particular, paper II provides high-precision values for dE/dt for a wide range of orbital frequencies; from those numerical results, the expansion of $(dE/dt)/(dE/dt)_N$ in powers of v is extended to higher orders.

The message of Eq. (1.3) is clear. When $\nu = 25$ Hz and the magnitude of the v^2 term is 0.0914, the magnitude of the v^3 term is already as large as 0.0486, that is, approximately 47% the size of the first correction term. As the frequency increases, so also does the percentage difference; the two terms become equal to 0.3237 when $\nu \simeq 167$ Hz, which lies well within the LIGO/VIRGO frequency band.

Conventional wisdom has it that the post-Newtonian calculation of the gravitational waveforms and luminosity is sufficiently accurate for LIGO/VIRGO's purposes. Equation (1.3) shows that this is not true. Higher-order corrections, presumably a fairly large number of supplementary terms, are clearly necessary. How the higher-order terms affect matched-filtering and information-extraction strategies will be discussed elsewhere [14].

In this paper, Eq. (1.3) and the waveforms of Sec. VI are derived using the framework of gravitational perturbations of the Schwarzschild geometry (see, e.g., Chandrasekhar [15]). This is the reason for which the reduced mass μ is assumed to be much smaller than the total mass M: linearized gravitational perturbation theory cannot handle finite mass ratios. The situation considered here is therefore that of a point particle of mass $\mu \ll M$ orbiting a nonrotating black hole of mass M. The methods used here are similar to that of Galt'sov *et al.* [16] who considered the more general case of elliptical orbits; though restricted to circular orbits, the calculations of this paper are pushed to a higher degree of accuracy than previously.

I begin in Sec. II by recalling the fundamental equations from the literature. I adopt the formalism of Newman and Penrose [17], in which the gravitational perturbations are described by the (complex) Weyl scalar Ψ_4, which is governed by an inhomogeneous Teukolsky equation [18]. The source term for this equation is obtained from the stress-energy tensor of a point particle following a circular orbit in the unperturbed Schwarschild geometry. The Teukolsky equation is solved by means of a Green's function, and the gravitational waveforms and luminosity are extracted from the solution. The basic formalism is presented in the subsections A, B, and C.

In this formalism, an important role is played by the function $R^H_{\omega\ell m}(r)$, called here the Teukolsky function (r is the usual Schwarzschild radial coordinate), which is the solution to the homogeneous Teukolsky equation corresponding to purely ingoing waves at the black-hole horizon, and a superposition of ingoing and outgoing waves at infinity. A measure of the precise amount of ingoing waves at infinity, $B^{\rm in}_{\omega\ell m}$, also plays an important role. Subsection II D explains how to obtain $R^H_{\omega\ell m}(r)$ and $B^{\rm in}_{\omega\ell m}$ by solving the Regge-Wheeler equation [19], which is handled more easily, rather than solving the Teukolsky equation directly.

The waveforms, and also the luminosity, are decomposed into spherical-harmonic modes, characterized by the values of the indices ℓ and m; the full waveforms, and the full luminosity, are recovered by summing over these indices. I present, in Sec. III, for a given mode (ℓ, m), the leading-order calculation for these quantities. In particular, subsection C shows that the power radiated by the mode (ℓ, m), when divided by the Newtonian answer (1.1), is of order $v^{2(\ell-2)}$ if $\ell+m$ is even, while it is of order $v^{2(\ell-1)}$ if $\ell+m$ is odd. Section III is a good warmup exercise for what comes next, but also serves to clarify a few subtle issues, such as identifying an appropriate approximation scheme (subsection A), and appropriate approximate boundary conditions for the Regge-Wheeler and Teukolsky functions (subsection B).

Section IV provides a detailed study of the Regge-Wheeler equation, which is there expanded in powers of $M\omega$, up to first order (ω is the wave angular frequency). A general integral solution is given for all ℓ (subsection A), and its behavior at small and large r is extracted (subsection B). The general formalism is then applied, in subsection C, to the rather important case $\ell = 2$.

In Sec. V are presented the detailed calculation of the gravitational waveforms and luminosity, to order v^3. The result for the luminosity was given in Eq. (1.3); the results for the waveforms are summarized in Sec. VI, which concludes this paper.

Throughout this paper I use geometrized units in which $c = G = 1$. The sign conventions of Newman and Penrose [17] (as summarized by Chandrasekhar [15]) are adopted; in particular, I employ the $(+,-,-,-)$ metric signature.

II. THE GENERAL FORMALISM

This section is a detailed summary, necessary for the clarity of exposition, of the relevant material gathered from the literature. The combined Regge-Wheeler-Teukolsky formalism of subsection D was suggested to me by L.S. Finn, who used similar methods for his numerical calculations.

A. The inhomogeneous Teukolsky equation and its solution

In this paper I consider gravitational radiation within the framework of gravitational perturbations of the Schwarschild geometry

$$\begin{aligned} &ds^2 = f dt^2 - f^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ &f = 1 - 2M/r, \end{aligned} \tag{2.1}$$

and shall describe these perturbations using the Newman-Penrose formalism [17]. The fundamental perturbation function is the Weyl scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, which is well suited to describe outgoing gravitational radiation; $C_{\alpha\beta\gamma\delta}$ designates the perturbed Weyl tensor, and the vectors $n^\alpha = \frac{1}{2}(1, -f, 0, 0)$, $\bar{m}^\alpha = (0, 0, 1, -i/\sin\theta)/\sqrt{2}r$, are part of an orthonormal null tetrad.

The differential equation governing Ψ_4 is obtained by first decomposing it into Fourier-harmonic components $R_{\omega\ell m}(r)$:

$$r^4\Psi_4 = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} R_{\omega\ell m}(r) {}_{-2}Y_{\ell m}(\theta,\phi) e^{-i\omega t}, \tag{2.2}$$

where ${}_{-2}Y_{\ell m}(\theta,\phi)$ denotes the spherical harmonics of spin-weight $s = -2$ (cf. Goldberg *et al.* [20]). The sums over ℓ and m are restricted to $-\ell \le m \le \ell$ and $\ell \ge 2$. The inhomogeneous Teukolsky equation reads

$$\begin{aligned} \left\{ r^2 f d^2/dr^2 - 2(r-M)d/dr + U(r) \right\} R_{\omega\ell m}(r) \\ = T_{\omega\ell m}(r), \end{aligned} \tag{2.3}$$

with

$$U(r) = f^{-1}\left[(\omega r)^2 - 4i\omega(r - 3M)\right] - (\ell-1)(\ell+2). \tag{2.4}$$

The source term $T_{\omega\ell m}(r)$ can be derived from the stress-energy tensor $T_{\alpha\beta}$ which perturbs the gravitational field. The first step is to construct the tetrad projections $T_{\alpha\beta}n^\alpha n^\beta$, $T_{\alpha\beta}n^\alpha \bar{m}^\beta$, and $T_{\alpha\beta}\bar{m}^\alpha \bar{m}^\beta$; then one converts them into Fourier-harmonic components according to

$${}_0T_{\omega\ell m}(r) = \frac{1}{2\pi}\int dt\, d\Omega\, T_{\alpha\beta}n^\alpha n^\beta {}_0\bar{Y}_{\ell m}(\theta,\phi) e^{i\omega t}, \tag{2.5}$$

where $d\Omega = d\cos\theta\, d\phi$. Similar equations relate ${}_{-1}T_{\omega\ell m}$ and $T_{\alpha\beta}n^\alpha \bar{m}^\beta$, ${}_{-2}T_{\omega\ell m}$ and $T_{\alpha\beta}\bar{m}^\alpha \bar{m}^\beta$. The source can then be calculated from Teukolsky's equation (2.15) [18], or can be simply copied from Eq. (12) of Sasaki and Nakamura [21] (which, however, contains a sign error). The result is

$$T_{\omega\ell m}/2\pi = 2[(l-1)l(l+1)(l+2)]^{1/2}r^4{}_0T_{\omega\ell m} + 2[2(l-1)(l+2)]^{1/2}r^2 f\mathcal{L}\, r^3 f^{-1}{}_{-1}T_{\omega\ell m} + rf\mathcal{L}r^4 f^{-1}\mathcal{L}\, r_{-2}T_{\omega\ell m}, \tag{2.6}$$

where $\mathcal{L} = fd/dr + i\omega = d/dr^* + i\omega$; the variable r^* is the usual tortoise radius: $r^* = r + 2M\ln(r/2M - 1)$.

The standard way to integrate Eq. (2.3) is by means of a Green's function (I parallel here the discussion given in Detweiler [22]). To construct the Green's function, two linearly independent solutions of the homogeneous Teukolsky equation must be selected. The physically motivated choice is $R^H_{\omega\ell m}(r)$ and $R^\infty_{\omega\ell m}(r)$, which respectively describe purely ingoing waves falling into the black hole, and purely outgoing waves escaping to infinity. These functions have the following asymptotic behaviors [18]: $R^H_{\omega\ell m}(r) \sim r^4 f^2 e^{-i\omega r^*}$ for $r \to 2M$, while $R^H_{\omega\ell m}(r) \sim B^{\rm in}_{\omega\ell m} r^{-1} e^{-i\omega r^*} + B^{\rm out}_{\omega\ell m} r^3 e^{i\omega r^*}$ for $r \to \infty$; and $R^\infty_{\omega\ell m}(r) \sim r^3 e^{i\omega r^*}$ for $r \to \infty$. The Wronskian $W \equiv (R^H_{\omega\ell m}R^{\infty\prime}_{\omega\ell m} - R^\infty_{\omega\ell m}R^{H\prime}_{\omega\ell m})/r^2 f$ is a constant; $W = 2i\omega B^{\rm in}_{\omega\ell m}$ (the prime denotes differentiation with respect to r).

From the general theory of Green's functions (cf. Arfken [23], Sec. 16.5), if follows that the solution of Eq. (2.3), at large radii, with boundary conditions corresponding to no outgoing waves escaping from the black hole and no incident waves from infinity, is

$$2i\omega B^{\rm in}_{\omega\ell m} R_{\omega\ell m}(r\to\infty) \sim r^3 e^{i\omega r^*}\int_{2M}^\infty dr R^H_{\omega\ell m}(r) T_{\omega\ell m}(r)/r^4 f^2. \tag{2.7}$$

B. Source for circular orbits

Before working on Eq. (2.7) to extract physically interesting information, turn to the exact specification of the source $T_{\omega\ell m}(r)$.

The Schwarzschild geometry is taken to be perturbed by a point particle of rest mass $\mu \ll M$ in circular orbit at radius $r = r_0$. Its stress-energy tensor is

$$T^{\alpha\beta}(x) = \mu\int d\tau\, u^\alpha u^\beta \delta^{(4)}[x - z(\tau)], \tag{2.8}$$

where x is the spacetime point, and where $z(\tau)$ represents the particle's trajectory ($u^\alpha = dz^\alpha/d\tau$ is the particle's four-velocity). For the trajectory under consideration, Eq. (2.8) reduces to $T^{\alpha\beta} = (\mu/r_0{}^2)(u^\alpha u^\beta/u^t)\delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t)$, with $u^\alpha = (\tilde{E}/f_0, 0, 0, \tilde{L}/r_0{}^2)$. The particle's specific energy $\tilde{E}$, specific angular momentum $\tilde{L}$, and angular velocity Ω are

$$\begin{aligned}\tilde{E} &= f_0(1 - 3M/r_0)^{-1/2},\\ \tilde{L} &= (Mr_0)^{1/2}(1 - 3M/r_0)^{-1/2},\\ \Omega &= (M/r_0{}^3)^{1/2},\end{aligned} \tag{2.9}$$

where $f_0 \equiv 1 - 2M/r_0$.

It is straightforward to construct the tetrad projections of $T_{\alpha\beta}$; they are all proportional to $\delta(\cos\theta)\delta(\phi - \Omega t)$. It is equally straightforward to construct the Fourier-harmonic components by substituting the tetrad projections into Eq. (2.5) and its analogues. Integration over the solid angle yields something proportional to ${}_s\bar{Y}_{\ell m}(\frac{\pi}{2}, \Omega t) = {}_sY_{\ell m}(\frac{\pi}{2}, 0)e^{-im\Omega t}$, and integration over time then yields something proportional to $\delta(\omega - m\Omega)$. One therefore finds that the Fourier-harmonic components of the particle's stress-energy tensor are characterized by a frequency spectrum peaked at the harmonics of the orbital frequency,

$$\omega = m\Omega; \tag{2.10}$$

from Eq. (2.7), it is clear that the gravitational waves will share that spectrum.

C. Waveforms and luminosity

To calculate the gravitational waveforms and luminosity, it is useful to define a number $Z_{\ell m}$ (in fact a function of the orbital radius r_0) by

$$2i\omega B^{\rm in}_{\omega\ell m}\mu Z_{\ell m}\delta(\omega - m\Omega) = \int_{2M}^\infty dr R^H_{\omega\ell m}(r) T_{\omega\ell m}(r)/r^4 f^2. \tag{2.11}$$

Then, from Eq. (2.7), one finds that $R_{\omega\ell m}(r\to\infty) \sim \mu Z_{\ell m}\delta(\omega - m\Omega)r^3 e^{i\omega r^*}$. It follows from this and Eq. (2.2) that the Fourier-harmonic components of Ψ_4 are given by $\mu Z_{\ell m}\delta(\omega - m\Omega)r^{-1}e^{i\omega r^*}$. Knowing that for a monocromatic wave $\Psi_4 = \frac{1}{2}\omega^2(h_+ - ih_\times)$ [18], and then performing the summations over ω, ℓ, and m, the full gravitational waveforms (evaluated at infinity) may be obtained:

$$\begin{aligned}h_+ - ih_\times &= \sum_{\ell m}(h^+_{\ell m} - ih^\times_{\ell m})\\ &= (2\mu/r)\sum_{\ell m}(Z_{\ell m}/\omega^2)_{-2}Y_{\ell m}(\theta,\phi)e^{-i\omega(t-r^*)},\end{aligned} \tag{2.12}$$

where ω is given by Eq. (2.10); h_+ and $h_\times$, which are real quantities, denote the two fundamental polarizations of the gravitational waves. The energy radiated per unit time may then be calculated from Eq. (2.12), and the result is $dE/dt = \mu^2\sum_{\ell m}|Z_{\ell m}|^2/4\pi\omega^2$.

The number $Z_{\ell m}$ still remains to be evaluated from Eq. (2.11). The calculation is straightforward, for the source $T_{\omega\ell m}(r)$ has support only at $r = r_0$; the result is

$$\begin{aligned}Z_{\ell m} = \pi\{ &\left[{}_0b_{\ell m} + 2i_{-1}b_{\ell m}(1 + \tfrac{1}{2}i\omega r_0/f_0) - i_{-2}b_{\ell m}\omega r_0 f_0^{-2}(1 - M/r_0 + \tfrac{1}{2}i\omega r_0)\right] R^H_{\omega\ell m}(r_0)\\ &- \left[i_{-1}b_{\ell m} - {}_{-2}b_{\ell m}(1 + i\omega r_0/f_0)\right] r_0 R^{H\prime}_{\omega\ell m}(r_0) - \tfrac{1}{2}{}_{-2}b_{\ell m}r_0{}^2 R^{H\prime\prime}_{\omega\ell m}(r_0)\}/i\omega r_0{}^2 B^{\rm in}_{\omega\ell m}.\end{aligned} \tag{2.13}$$

The b coefficients are all dimensionless and given by

$$\begin{aligned} {}_0b_{\ell m} &= \tfrac{1}{2}\left[(\ell-1)\ell(\ell+1)(\ell+2)\right]^{1/2} {}_0Y_{\ell m}(\tfrac{\pi}{2},0)\tilde{E}/f_0, \\ {}_{-1}b_{\ell m} &= \left[(\ell-1)(\ell+2)\right]^{1/2} {}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)\tilde{L}/r_0, \\ {}_{-2}b_{\ell m} &= {}_{-2}Y_{\ell m}(\tfrac{\pi}{2},0)\tilde{L}\Omega. \end{aligned} \tag{2.14}$$

An explicit expression for ${}_sY_{\ell m}(\frac{\pi}{2},0)$ can be derived from Eq. (3.1) of Goldberg *et al.* [20]; it reads

$$\begin{aligned} {}_sY_{\ell m}(\tfrac{\pi}{2},0) = {}& (\tfrac{1}{2})^{\ell+1}\left[(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!(2\ell+1)/\pi\right]^{1/2} \\ & \times \sum_{p=p_0}^{p_1} (-1)^{s+l+m+p}\left[p!(\ell-s-p)!(p+s-m)!(\ell+m-p)!\right]^{-1}, \end{aligned} \tag{2.15}$$

where $p_0 = \max(0, m-s)$ and $p_1 = \min(\ell-s, \ell+m)$.

Finally, it follows from Eq. (2.15) that ${}_sY_{\ell,-m}(\frac{\pi}{2},0) = (-1)^{s+\ell}{}_sY_{\ell m}(\frac{\pi}{2},0)$; using this together with Eqs. (2.10) and (2.13) implies that $Z_{\ell,-m} = (-1)^\ell \bar{Z}_{\ell m}$. This property reveals that a mode with a given value of ℓ and (say) negative value of m will contribute exactly as much to the luminosity as the mode with same value of ℓ and opposite value of m. Therefore the final expression for the luminosity shall be

$$dE/dt = \mu^2 \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell} |Z_{\ell m}|^2/2\pi\omega^2, \tag{2.16}$$

where $\omega = m\Omega$.

D. The Teukolsky function via the Regge-Wheeler equation

As expressed by Eqs. (2.13) and (2.16), the problem of calculating the gravitational waveforms and luminosity for a given ℓ and m reduces to that of evaluating essentially two things: the Teukolsky function $R^H_{\omega\ell m}(r)$ and its derivatives at the point $r = r_0$, as well as the Wronskian coefficient $B^{\rm in}_{\omega\ell m}$. In this subsection, I shall describe how these quantities can be obtained by integrating the Regge-Wheeler equation [19], which is handled more easily than the Teukolsky equation.

The Regge-Wheeler equation

$$\left\{\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\right\} X_{\omega\ell m}(r) = 0, \tag{2.17}$$

with the potential

$$V(r) = f\left[\ell(\ell+1)/r^2 - 6M/r^3\right], \tag{2.18}$$

was originally shown to govern the axial metric perturbations (cf. Chandrasekhar [15]) of the Schwarzschild geometry. Chandrasekhar [24] later showed that if $X^H_{\omega\ell m}(r)$ is a solution of the Regge-Wheeler equation, then

$$R^H_{\omega\ell m}(r) = r^2 f\mathcal{L}f^{-1}\mathcal{L}\, rX^H_{\omega\ell m}(r), \tag{2.19}$$

(where $\mathcal{L} = d/dr^* + i\omega$) is a solution of the homogeneous Teukolsky equation (I use the notation of Sasaki and Nakamura [21]). So rather than working with the Teukolsky equation directly, it will be convenient to solve instead the Regge-Wheeler equation, and then to apply the Chandrasekhar transformation to obtain the Teukolsky function.

The relevant solution $X^H_{\omega\ell m}(r)$ corresponds to purely ingoing waves at the black-hole horizon,

$$X^H_{\omega\ell m}(r\to 2M) \sim e^{-i\omega r^*}, \tag{2.20}$$

and, at infinity, to a superposition of ingoing and outgoing waves with comparable amplitudes

$$X^H_{\omega\ell m}(r\to\infty) \sim A^{\rm in}_{\omega\ell m}e^{-i\omega r^*} + A^{\rm out}_{\omega\ell m}e^{i\omega r^*}. \tag{2.21}$$

The Chandrasekhar transformation then implies

$$4\omega^2 B^{\rm in}_{\omega\ell m} = -\left[(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega\right]A^{\rm in}_{\omega\ell m}, \tag{2.22}$$

and $B^{\rm out}_{\omega\ell m} = -4\omega^2 A^{\rm out}_{\omega\ell m}$.

III. LUMINOSITY TO LEADING ORDER

In this and the following sections, I shall formulate appropriate approximations to the formalism of the previous section, which will lead to approximate expressions for the gravitational waveforms and luminosity. I begin in this section by calculating, to leading order, the power radiated by a given mode (ℓ, m); this leading-order calculation will be generalized in the following sections.

A. The nature of the approximation

Two dimensionless quantities appear naturally within the general formalism of Sec. II; the first is M/r_0, the other $M\omega$. These quantities are not independent, for the wave frequency is restricted to be a harmonic of the orbital frequency, as expressed by Eq. (2.10):

$$M\omega/m = M\Omega = (M/r_0)^{3/2} = v^3, \tag{3.1}$$

where Eq. (2.9) has been used. Throughout the remainder of this paper, I shall assume that M/r_0 is always much smaller than unity; Eq. (3.1) then ensures that $M\omega$ is an even smaller number.

Einstein's quadrupole formula implies that the gravitational waves emitted by a binary system have a frequency equal to twice the orbital frequency Ω. Moreover, it predicts that their luminosity is given by [8]

$$(dE/dt)_N = \tfrac{32}{5}(\mu/M)^2(M\Omega)^{10/3}, \tag{3.2}$$

where the subscript N stands for "Newtonian".

Since Eq. (3.2) defines a natural (dimensionless) unit for the luminosity of binary system, I shall find it convenient to express my results in terms of that natural unit. I shall therefore define $\eta_{\ell m}$ to be the power radiated by the mode (ℓ, m), divided by the Newtonian expression (3.2):

$$\eta_{\ell m} = \frac{\mu^2 |Z_{\ell m}|^2}{2\pi\omega^2} \frac{1}{(dE/dt)_N}; \tag{3.3}$$

in terms of this, the total luminosity is

$$dE/dt = (dE/dt)_N \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \eta_{\ell m}. \tag{3.4}$$

Compatibility of my results with linearized theory will be expressed by the fact that $\eta_{2,2} = 1 + O(v^2)$, while every other $\eta_{\ell m}$ will be at most $O(v^2)$.

B. The Regge-Wheeler and Teukolsky functions

To determine what the appropriate leading-order Regge-Wheeler function should be, it is useful to rewrite Eq. (2.17) in terms of the dimensionless variable $z = \omega r$; it becomes

$$\left\{ f^2 \frac{d^2}{dz^2} + \frac{2M\omega}{z^2} f \frac{d}{dz} + 1 - \tilde{V}(z) \right\} X_{\omega\ell m}(z) = 0, \tag{3.5}$$

with $\tilde{V}(z) = f\left[\ell(\ell+1)/z^2 - 6M\omega/z^3\right]$ and $f = 1 - 2M\omega/z$.

Having agreed that $M\omega$ is a small number, I shall simply set $M\omega = 0$ in Eq. (3.5) to obtain the leading-order Regge-Wheeler equation (which describes wave propagation in flat spacetime):

$$\left\{ \frac{d^2}{dz^2} + 1 - \frac{\ell(\ell+1)}{z^2} \right\} X_{\omega\ell m}(z) = O(M\omega). \tag{3.6}$$

This equation (the free Schrödinger equation) can be solved exactly in terms of spherical Bessel functions (cf. Arfken [23], Sec. 11.7); the general solution is $k_1 z j_\ell(z) + k_2 z n_\ell(z)$, where k_1 and k_2 are two arbitrary constants.

According to Sec. II, the relevant particular solution $X^H_{\omega\ell m}(z)$ must be selected by imposing certain boundary conditions at the black-hole horizon, $z = 2M\omega$. But since $M\omega$ was put to zero previously, boundary conditions must instead be imposed at the origin, $z = 0$. The appropriate requirement, in the context of this leading-order calculation, is regularity of $X^H_{\omega\ell m}(z)$ at the origin. This can be justified as follows: Suppose that $X^H_{\omega\ell m}(z)$ were allowed to be singular at $z = 0$. This would mean that $X^H_{\omega\ell m}(z_0)$ could be made arbitrary large by choosing a z_0 sufficiently small. But $z_0 = \omega r_0 = m(M/r_0)^{1/2}$, and it follows that $X^H_{\omega\ell m}(r_0)$ could be made arbitrarily large by choosing an r_0 sufficiently *large*. This would imply that the power radiated by a point particle in a circular orbit at a sufficiently large radius could be arbitrarily large. This situation can obviously be judged unacceptable on physical grounds, and I shall therefore require $X^H_{\omega\ell m}(z)$ to be regular at the origin. Choosing its normalization arbitrarily, I shall write

$$X^H_{\omega\ell m}(z) = z j_\ell(z) + O(M\omega). \tag{3.7}$$

As pointed out in the previous paragraph, z_0 is a small number of order v; it will, therefore, be sufficient to know the behavior of $X^H_{\omega\ell m}(z)$ near the origin. Recalling the small-argument behavior of spherical Bessel functions,

$$\begin{aligned}(2\ell+1)!!\, X^H_{\omega\ell m}(z \to 0)& \\ \sim z^{\ell+1}\left[1 - \frac{1}{2}z^2/(2\ell+3) + O(z^4)\right] &+ O(M\omega); \end{aligned}\tag{3.8}$$

the second term within the square brackets, when evaluated at $z = z_0$, is $O(v^2)$ and will be ignored.

The parameter $A^{\text{in}}_{\omega\ell m}$ can be extracted from the behavior of the Regge-Wheeler function near $z = \infty$. Using the large-argument behavior of spherical Bessel functions, $X^H_{\omega\ell m}(z \to \infty) \sim \sin(z - \ell\pi/2) + O(M\omega)$, it follows that

$$A^{\text{in}}_{\omega\ell m} = \tfrac{1}{2} e^{i(\ell+1)\pi/2} + O(M\omega). \tag{3.9}$$

The Teukolsky function can now be obtained by applying the Chandrasekhar transformation (2.19) to Eq. (3.8). Setting $f = 1 + O(M\omega)$, one obtains

$$\begin{aligned}(2\ell+1)!!\, \omega R^H_{\omega\ell m}(z \to 0)& \\ \sim (\ell+1)(\ell+2) z^{\ell+2}\left[1 + O(z)\right] &+ O(M\omega); \end{aligned}\tag{3.10}$$

as indicated, the higher powers of z may be ignored. The Wronskian coefficient $B^{\text{in}}_{\omega\ell m}$ can be obtained from Eqs. (2.22) (in which, as usual, $M\omega$ is set to zero) and (3.9). The result is

$$\begin{aligned}B^{\text{in}}_{\omega\ell m} = &-\tfrac{1}{8}(\ell-1)\ell(\ell+1)(\ell+2) e^{i(\ell+1)\pi/2}/\omega^2 \\ &+ O(M\omega). \end{aligned}\tag{3.11}$$

C. Calculation of $\eta_{\ell m}$

Inspection of Eqs. (2.9), (2.14) reveals that ${}_0b_{\ell m} \sim 1$, while ${}_{-1}b_{\ell m} = O(v)$ and ${}_{-2}b_{\ell m} = O(v^2)$. It follows that $Z_{\ell m}$, Eq. (2.13), will be dominated by the term involving ${}_0b_{\ell m}$, unless ${}_0b_{\ell m}$ happens to vanish identically. This will be the case whenever ${}_0Y_{\ell m}(\frac{\pi}{2}, 0)$ vanishes, which happens when $\ell + m$ is an odd number. When $\ell + m$ is odd, the dominant contribution to $Z_{\ell m}$ comes from the terms involving ${}_{-1}b_{\ell m}$. For a given ℓ, therefore, the luminosity will be dominated by the modes for which $\ell + m$ is even; the power radiated by the modes for which $\ell + m$ is odd will be suppressed by a factor v^2.

Considering only the modes for which $\ell + m$ is an even number, the leading-order (LO) expression for $Z_{\ell m}$ is

$$Z^{\text{LO}}_{\ell m} = \pi\, {}_0b_{\ell m} R^H_{\omega\ell m}(r_0)/i\omega {r_0}^2 B^{\text{in}}_{\omega\ell m}, \tag{3.12}$$

which eventually becomes

$$Z_{\ell m}^{\rm LO} = \frac{4\pi m^{\ell+2} e^{-i\ell\pi/2}}{(2\ell+1)!!} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell}\right]^{1/2} {}_0Y_{\ell m}(\tfrac{\pi}{2},0)\, \Omega^{\ell+2} r_0{}^{\ell}. \tag{3.13}$$

Substituting this into Eq. (3.3), using exercise 12.6.3 of Arfken [23], finally yields

$$\eta_{\ell m}^{\rm LO} = \frac{5m^{2(\ell+1)}(\ell+1)(\ell+2)(2\ell+1)(\ell-m)!(\ell+m)!}{16(\ell-1)\ell\,[(2\ell+1)!!(\ell-m)!!(\ell+m)!!]^2}\, v^{2(\ell-2)}, \tag{3.14}$$

for $\ell+m$ even. For $\ell+m$ odd, $\eta_{\ell m}^{\rm LO} \propto v^{2(\ell-1)}$.

From Eq. (3.14) one may verify that $\eta_{2,2}^{\rm LO} = 1$, as claimed previously. Equation (3.14) also reveals that the power radiated by a mode of given ℓ is suppressed from the Newtonian result (3.2) by at least a factor v^2 to the $(\ell-2)$th power; as expected, higher multipoles contribute to the luminosity with increasing powers of v^2.

IV. A STUDY OF THE $O(M\omega)$ REGGE-WHEELER EQUATION

The leading-order calculation of the preceding section was a useful warmup exercise for what comes next — the calculation, to order v^3, of the gravitational waveforms and luminosity. This calculation is slightly involved, and will be the subject of this and the following section.

The first step is to solve the Regge-Wheeler equation with better accuracy than what was afforded in the previous section. It is clear from Eq. (3.6) that corrections of order $M\omega$ should be sought for the Regge-Wheeler function; this section is devoted to this topic.

A. The equation and its solution

I start by recalling the result of the preceding section, that to leading order, $X_{\omega\ell m}^H(z) \simeq zj_\ell(z)$, where $z=\omega r$. To first order in $M\omega$, I shall postulate the form

$$X_{\omega\ell m}^H(z) = zj_\ell(z) + 2M\omega X_\ell^{(1)}(z) + O[(M\omega)^2]. \tag{4.1}$$

The idea is then to substitute Eq. (4.1) into Eq. (3.5), and expand in powers of $M\omega$, up to first order. The result is an inhomogeneous spherical Bessel equation for $X_\ell^{(1)}(z)$:

$$\left\{\frac{d^2}{dz^2} + 1 - \frac{\ell(\ell+1)}{z^2}\right\} X_\ell^{(1)}(z) = -W_\ell(z), \tag{4.2}$$

with

$$W_\ell(z) = \left\{\frac{1}{z^2}\frac{d}{dz} + \frac{2}{z} - \frac{\ell(\ell+1)-3}{z^3}\right\} zj_\ell(z). \tag{4.3}$$

Equation (4.2) can be integrated by adding the general solution of the homogeneous Bessel equation, $\alpha_\ell zj_\ell(z) + \beta_\ell zn_\ell(z)$, to any particular solution of the inhomogeneous equation. This particular solution can be obtained by integrating the source term $W_\ell(z)$ over any choice of Green's function. The regularity considerations of Sec. III B apply to this situation as well. As a result, any contribution from $zn_\ell(z)$ to the Regge-Wheeler function must be forbidden; the coefficient β_ℓ must therefore be set to zero. Regularity considerations also indicate that

$$G_\ell(z,z') = z_< j_\ell(z_<) z_> n_\ell(z_>), \tag{4.4}$$

for $z_< = \min(z,z')$ and $z_> = \max(z,z')$, is an appropriate choice of Green's function. From the general theory of Green's functions, it follows that the most general — however regular at the origin — solution to Eq. (4.2) is

$$X_\ell^{(1)}(z) = zj_\ell(z)\left[\alpha_\ell - \int_z^\infty dz'\, z' n_\ell(z') W_\ell(z')\right] - zn_\ell(z)\int_0^z dz'\, z' j_\ell(z') W_\ell(z'). \tag{4.5}$$

The arbitrary constant α_ℓ has no physical consequence. It could be set equal to zero by renormalizing the Regge-Wheeler function according to $X_{\omega\ell m}^H \to (1-2M\omega\alpha_\ell)X_{\omega\ell m}^H$. It can also be kept, which is what I shall do; it will be interesting to see how α_ℓ eventually drops out of the calculation.

B. Asymptotic behaviors and comparison with the exact Regge-Wheeler function

From Eqs. (4.3) and (4.5), the behavior of $X_\ell^{(1)}(z)$ near $z=0$ can be determined. Recalling that $zj_\ell(z\to 0) \sim z^{\ell+1}/(2\ell+1)!!$ and $zn_\ell(z\to 0) \sim -(2\ell-1)!!/z^\ell$, it follows that

$$W_\ell(z\to 0) \sim -\frac{(\ell-2)(\ell+2)}{(2\ell+1)!!} z^{\ell-2}. \tag{4.6}$$

Using these results, the integration over $z' j_\ell(z') W_\ell(z')$, for small z, can easily be performed. The integration over $z' n_\ell(z') W_\ell(z')$ is not quite as easy to evaluate, for the range of integration covers almost the whole half-line. However it can be verified that the integrand is divergent as z' approaches zero. Therefore the integral, for small z, will be dominated by contributions from the vicinity of its lower bound, and

$$\int_{z\to 0}^\infty dz'\, z' n_\ell(z') W_\ell(z') \sim \frac{(\ell-2)(\ell+2)}{(2\ell+1)z}. \tag{4.7}$$

Gathering the results, it follows that

$$(2\ell+1)!!\, X_\ell^{(1)}(z\to 0) \sim (2\ell)^{-1}(\ell-2)(\ell+2)z^\ell + (\alpha_\ell - \Gamma_\ell)z^{\ell+1} + O(z^{\ell+2}), \tag{4.8}$$

where

$$\Gamma_\ell = \lim_{z\to 0}\left\{ \int_z^\infty dz'\, z' n_\ell(z') W_\ell(z') - (\ell-2)(\ell+2)/\left[(2\ell+1)z\right]\right\} \tag{4.9}$$

is a finite quantity. Finally, the small-argument behavior of the $O(M\omega)$ Regge-Wheeler function can be obtained by substituting Eq. (4.8) into Eq. (4.1), which yields

$$\begin{aligned}(2\ell+1)!!\, X^H_{\omega\ell m}(z\to 0) &\sim z^{\ell+1}\left\{1 + 2M\omega(\alpha_\ell - \Gamma_\ell) + O[(M\omega)^2]\right\} \\ &\quad\times\left\{1 - [(\ell-2)(\ell+2)/\ell]\, M\omega/z + O(z)\right\}.\end{aligned} \tag{4.10}$$

The extraction of the large z behavior of $X^{(1)}_\ell(z)$ goes along similar lines. First recalling that $zj_\ell(z\to\infty) \sim \sin(z-\ell\pi/2)$ and $zn_\ell(z\to\infty) \sim -\cos(z-\ell\pi/2)$, it follows from Eq. (4.3) that

$$W_\ell(z\to\infty) \sim 2z^{-1}\sin(z-\ell\pi/2). \tag{4.11}$$

Using this to evaluate the integration over $z'n_\ell(z')W_\ell(z')$ yields something proportional to $1/z$, and hence negligible. The integration over $z'j_\ell(z')W_\ell(z')$ may be performed by taking advantage of the fact that it is dominated by contributions from the vicinity of the higher bound. A quick calculation yields

$$\int_0^{z\to\infty} dz'\, z' j_\ell(z') W_\ell(z') \sim \ln z. \tag{4.12}$$

Gathering the results, one obtains

$$\begin{aligned}X^{(1)}_\ell(z\to\infty) &\sim \alpha_\ell \sin(z-\ell\pi/2) \\ &\quad + (\ln z + \tilde\Gamma_\ell)\cos(z-\ell\pi/2),\end{aligned} \tag{4.13}$$

where

$$\tilde\Gamma_\ell = \lim_{z\to\infty}\left\{\int_0^z dz'\, z' j_\ell(z') W_\ell(z') - \ln z\right\} \tag{4.14}$$

is a finite quantity. The large-argument behavior of the $O(M\omega)$ Regge-Wheeler function can now be obtained by substituting Eq. (4.13) into Eq. (4.1), which yields

$$\begin{aligned}X^H_{\omega\ell m}(z\to\infty) &\sim (1+2M\omega\alpha_\ell)\sin(z-\ell\pi/2) \\ &\quad + 2M\omega(\ln z + \tilde\Gamma_\ell)\cos(z-\ell\pi/2) \\ &\quad + O[(M\omega)^2].\end{aligned} \tag{4.15}$$

The presence of the $\ln z$ term in Eq. (4.15) might appear quite mysterious, for it suggests that $X^H_{\omega\ell m}(z)$ would grow arbitrarily large at large distances. A detailed comparison with the exact, large z, expression for the Regge-Wheeler function will clarify this point. A short manipulation brings Eq. (2.21) to the form

$$\begin{aligned}X^H_{\omega\ell m}(z\to\infty) &\sim (1+2M\omega a_-)\sin(z^*-\ell\pi/2) \\ &\quad + 2M\omega a_+ \cos(z^*-\ell\pi/2),\end{aligned} \tag{4.16}$$

where $z^* = \omega r^*$, and $1 + 2M\omega a_- \equiv i(A^{\text{out}}_{\omega\ell m}e^{i\ell\pi/2} - A^{\text{in}}_{\omega\ell m}e^{-i\ell\pi/2})$, $2M\omega a_+ \equiv A^{\text{out}}_{\omega\ell m}e^{i\ell\pi/2} + A^{\text{in}}_{\omega\ell m}e^{-i\ell\pi/2}$. Expanding Eq. (4.16) in powers of $M\omega$ then yields

$$\begin{aligned}X^H_{\omega\ell m}(z\to\infty) &\sim (1+2M\omega a_-)\sin(z-\ell\pi/2) \\ &\quad + 2M\omega(\ln z - \ln 2M\omega + a_+) \\ &\quad \times\cos(z-\ell\pi/2) + O[(M\omega)^2],\end{aligned} \tag{4.17}$$

which has exactly the same form as Eq. (4.15), including the suspect $\ln z$ contribution. This term is therefore seen to originate from within the phase of the Regge-Wheeler function. Climbing the gravitational well from the near zone to infinity induces a phase shift in the Regge-Wheeler function, with respect to the flat spacetime expression; this phase difference is expressed by the use of r^*, rather than r, in Eq. (2.21). After expanding in powers of $M\omega$, this gravitational effect is manifested no longer in the phase of the Regge-Wheeler function, but in its amplitude, which becomes divergent. Therefore the appearance of the $\ln z$ term in Eq. (4.15) is quite legitimate, and is a manifestation of an interesting physical effect.

The comparison between Eq. (4.15) and Eq. (4.17) reveals that the following identifications can be made: $\alpha_\ell = a_-$, $\tilde\Gamma_\ell + \ln 2M\omega = a_+$. It then follows that

$$A^{\text{in}}_{\omega\ell m} = \tfrac{1}{2}e^{i(\ell+1)\pi/2}\left\{1 + 2M\omega\left[\alpha_\ell - i(\tilde\Gamma_\ell + \ln 2M\omega)\right] + O[(M\omega)^2]\right\}. \tag{4.18}$$

C. The special case $\ell = 2$

This last subsection will provide an example of how the formalism of this section may be used. It deals with the rather important special case of quadrupole waves.

For $\ell = 2$, Eq. (4.5) may be expressed in closed form. Using $zj_2(z) = (3/z^2-1)\sin z - (3/z)\cos z$, $zn_2(z) = -(3/z^2-1)\cos z - (3/z)\sin z$, $W_2(z) = -(15/z^5 - 12/z^3 + 2/z)\sin z + (15/z^4 - 7/z^2)\cos z$, the integrations $J(z) \equiv \int_0^z dz'\, z' j_2(z') W_2(z')$ and $I(z) \equiv \int_z^\infty dz'\, z' n_2(z') W_2(z')$ may be performed explicitly. The integrals can be written in terms of simple trigonometric functions, as well as sine and cosine integrals: $\mathrm{si}(2z) = -\int_{2z}^\infty dz' \sin z'/z'$, $\mathrm{ci}(2z) = -\int_{2z}^\infty dz' \cos z'/z'$; they are

$$\begin{aligned}J(z) &= \tfrac{15}{4}(1-\cos 2z)/z^6 - \tfrac{15}{2}\sin 2z/z^5 \\ &\quad - \tfrac{3}{4}(1 - 11\cos 2z)/z^4 + \tfrac{13}{2}\sin 2z/z^3 \\ &\quad - \tfrac{1}{4}(3 + 13\cos 2z)/z^2 + \ln z - \mathrm{ci}(2z) \\ &\quad + \ln 2 + \gamma - \tfrac{5}{3},\end{aligned} \tag{4.19}$$

where $\gamma \simeq 0.5772$ is the Euler number, and

$$\begin{aligned}I(z) &= \tfrac{15}{4}\sin 2z/z^6 - \tfrac{15}{2}\cos 2z/z^5 \\ &\quad - \tfrac{33}{4}\sin 2z/z^4 + \tfrac{13}{2}\cos 2z/z^3 \\ &\quad + \tfrac{13}{4}\sin 2z/z^2 - \tfrac{1}{2}/z + \mathrm{si}(2z).\end{aligned} \tag{4.20}$$

The Regge-Wheeler function can then be expressed as

$$X^H_{\omega 2m}(z) = \left\{1 + 2M\omega\left[\alpha_2 - I(z)\right]\right\} zj_2(z) - 2M\omega J(z) zn_2(z) + O[(M\omega)^2]. \tag{4.21}$$

Its behavior at small z is given by

$$15\, X^H_{\omega 2m}(z\to 0) \sim z^3\left\{1+2M\omega(\alpha_2+\tfrac{\pi}{2})+O[(M\omega)^2]\right\} \times\left\{1-\tfrac{13}{21}M\omega z-\tfrac{1}{14}z^2+O(z^3)\right\}, \tag{4.22}$$

which reveals that $\Gamma_2 = \int_0^\infty dz\, zn_2(z)W_2(z) = -\frac{\pi}{2}$, as would the direct evaluation of the definite integral.

At infinity, Eq. (4.21) reduces to

$$X^H_{\omega 2m}(z\to\infty) \sim -\,(1+2M\omega\alpha_2)\sin z - 2M\omega(\ln z+\ln 2+\gamma-\tfrac{5}{3})\cos z, \tag{4.23}$$

which implies that $\tilde{\Gamma}_2 = \ln 2+\gamma-\frac{5}{3}$. Substitution of this result into Eq. (4.18) finally yields

$$A^{\rm in}_{\omega 2m} = -\tfrac{1}{2}i\left\{1+2M\omega\left[\alpha_2-i(\ln 4M\omega+\gamma-\tfrac{5}{3})\right] + O[(M\omega)^2]\right\}. \tag{4.24}$$

V. CALCULATION OF THE WAVEFORMS AND LUMINOSITY, TO ORDER v^3

With the foundations now laid, the calculations of this section are perfectly straightforward; however, some of the steps are quite tedious. The calculations are divided into four subsections, corresponding to each value of ℓ which needs be considered (although the luminosity can be calculated, to order v^3, by only including multipoles up to $\ell = 3$, the waveforms make use of all the multipoles up to $\ell = 5$). Necessarily, the content of this section will be found to be quite dry; this section, however, may be skipped altogether, and the reader will find the final answers summarized in Sec. VI.

The results of this section are most conveniently expressed in terms of $\eta_{\ell m}$, as defined in Eq. (3.3), and in terms of the rescaled partial waveforms $\zeta^{+,\times}_{\ell m}$ defined by

$$h^{+,\times}_{\ell m} + h^{+,\times}_{\ell,-m} = -(\mu/r)(M\Omega)^{2/3}\zeta^{+,\times}_{\ell m}; \tag{5.1}$$

it is also convenient to introduce the phase

$$\psi = \Omega(t-r^*)-\phi. \tag{5.2}$$

The calculations of the $\zeta^{+,\times}_{\ell m}$ are simplified by the use of the property $Z_{\ell,-m} = (-1)^\ell \bar{Z}_{\ell m}$ derived in Sec. II C, and require the evaluation of the spherical harmonics ${}_{-2}Y_{\ell,\pm m}(\theta,\phi)$.

Each subsection begins by calculating the Teukolsky function for the given ℓ. Then the $Z_{\ell m}$, $\eta_{\ell m}$, and $\zeta^{+,\times}_{\ell m}$ are calculated for each m (in the case of $\ell = 5$, only the odd values of m need be considered; the even ones would only contribute to order higher than v^3). I shall now present those calculations without much further comment.

A. Calculations for $\ell = 2$

The $O(M\omega)$ Regge-Wheeler function was obtained for $\ell = 2$ in the preceding section. It is then a simple matter to obtain the Teukolsky function at the required level of accuracy. It is given by

$$R^H_{\omega 2m}(r) = \tfrac{4}{5}\omega^3 r^4\left\{1+2M\omega(\alpha_2+\tfrac{\pi}{2}-\tfrac{3}{4}i)+O[(M\omega)^2]\right\}\left\{-4M/r+1+\tfrac{2}{3}i\omega r-\tfrac{11}{42}(\omega r)^2-\tfrac{1}{14}i(\omega r)^3+\cdots\right\}. \tag{5.3}$$

It should be noted that the coefficients of ωr, $(\omega r)^2$, and $(\omega r)^3$ in Eq. (5.3) are only valid up to leading order; their $O(M\omega)$ corrections are, however, irrelevant for this subsection's purposes. An expression for $B^{\rm in}_{\omega 2m}$ can be obtained by substituting Eq. (4.24) into Eq. (2.22); one obtains

$$B^{\rm in}_{\omega 2m} = 3i\left\{1+2M\omega\left[\alpha_2-i(\ln 4M\omega+\gamma-\tfrac{17}{12})\right]+O[(M\omega)^2]\right\}/\omega^2. \tag{5.4}$$

It may be noted that the arbitrary constant α_2 finally disappears from sight at this stage. This happens because both $R^H_{\omega 2m}(r_0)$ and $B^{\rm in}_{\omega 2m}$ contain a term $2M\omega\alpha_2$; this term therefore undergoes cancellation when the division is performed to order $M\omega$.

Specialize now to the case $m = 2$, and calculate the b coefficients, as defined in Eq. (2.14). First of all the spin-weighted spherical harmonics may be evaluated using Eq. (2.15). One obtains ${}_0Y_{2,2}(\frac{\pi}{2},0) = \frac{1}{4}(15/2\pi)^{1/2}$, ${}_{-1}Y_{2,2}(\frac{\pi}{2},0) = \frac{1}{4}(5/\pi)^{1/2}$, and ${}_{-2}Y_{2,2}(\frac{\pi}{2},0) = \frac{1}{8}(5/\pi)^{1/2}$. It follows from Eqs. (2.14) and (3.1) that

$$\begin{aligned} {}_0b_{2,2} &= \tfrac{3}{4}(5/\pi)^{1/2}\left\{1+\tfrac{3}{2}v^2+O(v^4)\right\},\\ {}_{-1}b_{2,2} &= \tfrac{1}{2}(5/\pi)^{1/2}v\left\{1+\tfrac{3}{2}v^2+O(v^4)\right\},\\ {}_{-2}b_{2,2} &= \tfrac{1}{8}(5/\pi)^{1/2}v^2\left\{1+O(v^2)\right\}. \end{aligned} \tag{5.5}$$

Substituting Eqs. (5.3) – (5.5) into Eq. (2.13) yields, after some algebra,

$$Z_{2,2} = -16(\pi/5)^{1/2}\Omega^4 {r_0}^2\left\{1-\tfrac{107}{42}v^2+\left[2\pi+4i(3\ln 2v+\gamma-\tfrac{17}{12})\right]v^3+O(v^4)\right\}. \tag{5.6}$$

By substituting Eq. (5.6) into Eq. (3.3), one then finds

$$\eta_{2,2} = 1-\tfrac{107}{21}v^2+4\pi v^3+O(v^4). \tag{5.7}$$

It should be noted that the v^2 term in Eq. (5.7) originates from post-Newtonian corrections to the source part of $Z_{2,2}$; the v^3 term is a consequence of wave-propagation effects. Now by using ${}_{-2}Y_{2,\pm 2}(\theta,\phi) = \frac{1}{8}(5/\pi)^{1/2}(1\pm 2\cos\theta +$

$\cos^2\theta)e^{\pm 2i\phi}$, and by substituting this and Eq. (5.6) into Eq. (2.12), one finally obtains

$$\begin{aligned}\zeta^{+}_{2,2} &= 2(1+\cos^2\theta)\left\{(1-\tfrac{107}{42}v^2+2\pi v^3)\cos 2\psi + 4(3\ln 2v+\gamma-\tfrac{17}{12})v^3\sin 2\psi + O(v^4)\right\},\\ \zeta^{\times}_{2,2} &= 4\cos\theta\left\{(1-\tfrac{107}{42}v^2+2\pi v^3)\sin 2\psi - 4(3\ln 2v+\gamma-\tfrac{17}{12})v^3\cos 2\psi + O(v^4)\right\}.\end{aligned} \tag{5.8}$$

The case $m=1$ is treated similarly. With ${}_0Y_{2,1}(\frac{\pi}{2},0)=0$, ${}_{-1}Y_{2,1}(\frac{\pi}{2},0)={}_{-2}Y_{2,1}(\frac{\pi}{2},0)=\frac{1}{4}(5/\pi)^{1/2}$, one obtains

$$Z_{2,1}=\tfrac{4}{3}i(\pi/5)^{1/2}\Omega^4 r_0{}^2\left\{v-\tfrac{17}{28}v^3+O(v^4)\right\}, \tag{5.9}$$

and

$$\eta_{2,1}=\tfrac{1}{36}v^2+O(v^4). \tag{5.10}$$

The waveforms are then calculated using ${}_{-2}Y_{2,\pm 1}(\theta,\phi)=\frac{1}{4}(5/\pi)^{1/2}\sin\theta(1\pm\cos\theta)e^{\pm i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{2,1} &= -\tfrac{4}{3}\sin\theta\left\{(v-\tfrac{17}{28}v^3)\sin\psi+O(v^4)\right\},\\ \zeta^{\times}_{2,1} &= \tfrac{4}{3}\sin\theta\cos\theta\left\{(v-\tfrac{17}{28}v^3)\cos\psi+O(v^4)\right\}.\end{aligned} \tag{5.11}$$

B. Calculations for $\ell = 3$

The Regge-Wheeler function is obtained from Eqs. (3.8) and (4.10), which imply

$$X^H_{\omega 3m}(z)=\tfrac{1}{105}z^4\left\{1-\tfrac{5}{3}M\omega/z-\tfrac{1}{18}z^2+\cdots\right\}, \tag{5.12}$$

while Eq. (3.9) yields $A^{\text{in}}_{\omega 3m}=\frac{1}{2}+O(M\omega)$. By using the Chandrasekhar transformation, one finds

$$R^H_{\omega 3m}(r)=\tfrac{4}{21}\omega^4 r^5\left\{1+\tfrac{1}{2}i\omega r-\tfrac{1}{6}(\omega r)^2-5M/r+\cdots\right\}, \tag{5.13}$$

and $B^{\text{in}}_{\omega 3m}=-15/\omega^2+O(M\omega)$.

Specialize now to the case $m=3$. With ${}_0Y_{3,3}(\frac{\pi}{2},0)=-\frac{1}{8}(35/\pi)^{1/2}$, ${}_{-1}Y_{3,3}(\frac{\pi}{2},0)=-\frac{1}{16}(105/\pi)^{1/2}$, and ${}_{-2}Y_{3,3}(\frac{\pi}{2},0)=-\frac{1}{8}(21/2\pi)^{1/2}$, one obtains

$$Z_{3,3}=-81i(\pi/42)^{1/2}\Omega^5 r_0{}^3\left\{1-4v^2+O(v^3)\right\}, \tag{5.14}$$

and

$$\eta_{3,3}=\tfrac{1215}{896}v^2+O(v^4). \tag{5.15}$$

The waveforms are then calculated using ${}_{-2}Y_{3,\pm 3}(\theta,\phi)=\mp\frac{1}{8}(21/2\pi)^{1/2}\sin\theta(1\pm 2\cos\theta+\cos^2\theta)e^{\pm 3i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{3,3} &= -\tfrac{9}{4}\sin\theta(1+\cos^2\theta)\left\{(v-4v^3)\sin 3\psi+O(v^4)\right\},\\ \zeta^{\times}_{3,3} &= \tfrac{9}{2}\sin\theta\cos\theta\left\{(v-4v^3)\cos 3\psi+O(v^4)\right\}.\end{aligned} \tag{5.16}$$

Consider next the case $m=2$. With ${}_0Y_{3,2}(\frac{\pi}{2},0)=0$, ${}_{-1}Y_{3,2}(\frac{\pi}{2},0)=-\frac{1}{8}(35/2\pi)^{1/2}$, and ${}_{-2}Y_{3,2}(\frac{\pi}{2},0)=-\frac{1}{4}(7/\pi)^{1/2}$, one obtains

$$Z_{3,2}=-\tfrac{16}{3}(\pi/7)^{1/2}\Omega^5 r_0{}^3\left\{v+O(v^3)\right\}, \tag{5.17}$$

and

$$\eta_{3,2}=O(v^4). \tag{5.18}$$

The waveforms are then calculated using ${}_{-2}Y_{3,\pm 2}(\theta,\phi)=\mp\frac{1}{8}(7/\pi)^{1/2}(2\pm\cos\theta-4\cos^2\theta\mp 3\cos^3\theta)e^{\pm 2i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{3,2} &= -\tfrac{4}{3}(1-2\cos^2\theta)\left\{v^2\cos 2\psi+O(v^4)\right\},\\ \zeta^{\times}_{3,2} &= -\tfrac{2}{3}\cos\theta(1-3\cos^2\theta)\left\{v^2\sin 2\psi+O(v^4)\right\}.\end{aligned} \tag{5.19}$$

Finally consider the case $m=1$. With ${}_0Y_{3,1}(\frac{\pi}{2},0)=\frac{1}{8}(21/\pi)^{1/2}$, ${}_{-1}Y_{3,1}(\frac{\pi}{2},0)=\frac{1}{16}(7/\pi)^{1/2}$, and ${}_{-2}Y_{3,1}(\frac{\pi}{2},0)=-\frac{1}{16}(70/\pi)^{1/2}$, one obtains

$$Z_{3,1}=\tfrac{1}{3}i(\pi/70)^{1/2}\Omega^5 r_0{}^3\left\{1-\tfrac{8}{3}v^2+O(v^3)\right\}, \tag{5.20}$$

and

$$\eta_{3,1}=\tfrac{1}{8064}v^2+O(v^4). \tag{5.21}$$

The waveforms are then calculated using ${}_{-2}Y_{3,\pm1}(\theta,\phi) = \mp\frac{1}{16}(70/\pi)^{1/2}\sin\theta(1\mp2\cos\theta-3\cos^2\theta)e^{\pm i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{3,1} &= \tfrac{1}{12}\sin\theta(1-3\cos^2\theta)\left\{(v-\tfrac{8}{3}v^3)\sin\psi+O(v^4)\right\},\\ \zeta^{\times}_{3,1} &= \tfrac{1}{6}\sin\theta\cos\theta\left\{(v-\tfrac{8}{3}v^3)\cos\psi+O(v^4)\right\}.\end{aligned} \tag{5.22}$$

C. Calculations for $\ell = 4$

For the calculations of this subsection, it will be sufficient to know the Regge-Wheeler function to leading order. Equation (3.8) implies $X^{H}_{\omega 4m}(z) = z^5/945$, which yields $R^{H}_{\omega 4m}(r) = \frac{2}{63}\omega^5 r^6\left\{1+\frac{2}{5}i\omega r+\cdots\right\}$. Then Eq. (3.11) gives $B^{\text{in}}_{\omega 4m} = -45i/\omega^2 + O(M\omega)$.

Consider first the case $m=4$. With ${}_0Y_{4,4}(\frac{\pi}{2},0) = \frac{3}{32}(70/\pi)^{1/2}$ and ${}_{-1}Y_{4,4}(\frac{\pi}{2},0) = \frac{3}{16}(14/\pi)^{1/2}$, one obtains

$$Z_{4,4} = \tfrac{512}{9}(\pi/7)^{1/2}\Omega^6 {r_0}^4\left\{1+O(v^2)\right\}, \tag{5.23}$$

which implies that $\eta_{4,4} = O(v^4)$. The waveforms are then calculated using ${}_{-2}Y_{4,\pm4}(\theta,\phi) = \frac{3}{16}(7/\pi)^{1/2}(1\pm2\cos\theta\mp 2\cos^3\theta-\cos^4\theta)e^{\pm4i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{4,4} &= -\tfrac{8}{3}\sin^2\theta(1+\cos^2\theta)\left\{v^2\cos4\psi+O(v^4)\right\},\\ \zeta^{\times}_{4,4} &= -\tfrac{16}{3}\sin^2\theta\cos\theta\left\{v^2\sin4\psi+O(v^4)\right\}.\end{aligned} \tag{5.24}$$

Consider next the case $m=3$. With ${}_0Y_{4,3}(\frac{\pi}{2},0) = 0$ and ${}_{-1}Y_{4,3}(\frac{\pi}{2},0) = \frac{3}{16}(7/\pi)^{1/2}$, one obtains

$$Z_{4,3} = -\tfrac{81}{5}i(\pi/14)^{1/2}\Omega^6 {r_0}^4\left\{v+O(v^2)\right\}, \tag{5.25}$$

which implies that $\eta_{4,3} = O(v^5)$. The waveforms are then calculated using ${}_{-2}Y_{4,\pm3}(\theta,\phi) = \frac{3}{8}(7/2\pi)^{1/2}\sin\theta(1-3\cos^2\theta\mp2\cos^3\theta)e^{\pm3i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{4,3} &= \tfrac{27}{20}\sin\theta(1-3\cos^2\theta)\left\{v^3\sin3\psi+O(v^4)\right\},\\ \zeta^{\times}_{4,3} &= \tfrac{27}{10}\sin\theta\cos^3\theta\left\{v^3\cos3\psi+O(v^4)\right\}.\end{aligned} \tag{5.26}$$

Consider now the case $m=2$. With ${}_0Y_{4,2}(\frac{\pi}{2},0) = -\frac{3}{16}(10/\pi)^{1/2}$ and ${}_{-1}Y_{4,2}(\frac{\pi}{2},0) = -\frac{3}{16}(2/\pi)^{1/2}$, one obtains

$$Z_{4,2} = -\tfrac{16}{63}\pi^{1/2}\Omega^6 {r_0}^4\left\{1+O(v^2)\right\}, \tag{5.27}$$

which implies that $\eta_{4,2} = O(v^4)$. The waveforms are then calculated using ${}_{-2}Y_{4,\pm2}(\theta,\phi) = \frac{3}{8}(1/\pi)^{1/2}(1\mp5\cos\theta-6\cos^2\theta\pm7\cos^3\theta+7\cos^4\theta)e^{\pm2i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{4,2} &= \tfrac{2}{21}(1-6\cos^2\theta+7\cos^4\theta)\left\{v^2\cos2\psi+O(v^4)\right\},\\ \zeta^{\times}_{4,2} &= -\tfrac{2}{21}\cos\theta(5-7\cos^2\theta)\left\{v^2\sin2\psi+O(v^4)\right\}.\end{aligned} \tag{5.28}$$

Consider finally the case $m=1$. With ${}_0Y_{4,1}(\frac{\pi}{2},0) = 0$ and ${}_{-1}Y_{4,1}(\frac{\pi}{2},0) = -\frac{9}{16}(1/\pi)^{1/2}$, one obtains

$$Z_{4,1} = \tfrac{1}{105}i(\pi/2)^{1/2}\Omega^6 {r_0}^4\left\{v+O(v^2)\right\}, \tag{5.29}$$

which implies that $\eta_{4,1} = O(v^5)$. The waveforms are then calculated using ${}_{-2}Y_{4,\pm1}(\theta,\phi) = -\frac{3}{16}(2/\pi)^{1/2}\sin\theta(1\pm8\cos\theta-7\cos^2\theta\mp14\cos^3\theta)e^{\pm i\phi}$; one obtains

$$\begin{aligned}\zeta^{+}_{4,1} &= \tfrac{1}{140}\sin\theta(1-7\cos^2\theta)\left\{v^3\sin\psi+O(v^4)\right\},\\ \zeta^{\times}_{4,1} &= -\tfrac{1}{70}\sin\theta\cos\theta(4-7\cos^2\theta)\left\{v^3\cos\psi+O(v^4)\right\}.\end{aligned} \tag{5.30}$$

D. Calculations for $\ell = 5$

As mentioned previously, only the odd values of m will be considered in this subsection; a leading-order calculation will be sufficient, hence Eq. (3.13) may be used directly.

Consider first the case $m=5$. With ${}_0Y_{5,5}(\frac{\pi}{2},0) = -\frac{3}{32}(77/\pi)^{1/2}$, it follows that

$$Z_{5,5} = \tfrac{3125}{24}i(5\pi/66)^{1/2}\Omega^7 {r_0}^5\left\{1+O(v)\right\}. \tag{5.31}$$

Using ${}_{-2}Y_{5,\pm5}(\theta,\phi) = \mp\frac{1}{16}(165/2\pi)^{1/2}\sin\theta(1\pm2\cos\theta\mp2\cos^3\theta-\cos^4\theta)e^{\pm5i\phi}$, the waveforms are then obtained:

$$\begin{aligned}\zeta^{+}_{5,5} &= \tfrac{625}{192}\sin^3\theta(1+\cos^2\theta)\left\{v^3\sin5\psi+O(v^4)\right\},\\ \zeta^{\times}_{5,5} &= -\tfrac{625}{96}\sin^3\theta\cos\theta\left\{v^3\cos5\psi+O(v^4)\right\}.\end{aligned} \tag{5.32}$$

Consider next the case $m=3$. With ${}_0Y_{5,3}(\frac{\pi}{2},0) = \frac{1}{32}(385/\pi)^{1/2}$, it follows that

$$Z_{5,3} = -\tfrac{81}{40} i (3\pi/22)^{1/2} \Omega^7 r_0{}^5 \left\{1 + O(v)\right\}. \tag{5.33}$$

Using ${}_{-2}Y_{5,\pm 3}(\theta,\phi) = \mp\frac{1}{16}(33/2\pi)^{1/2}\sin\theta(1 \mp 10\cos\theta - 8\cos^2\theta \pm 18\cos^3\theta + 15\cos^4\theta)e^{\pm 3i\phi}$, the waveforms are then obtained:

$$\begin{aligned} \zeta^{+}_{5,3} &= -\tfrac{27}{320}\sin\theta(1 - 8\cos^2\theta + 15\cos^4\theta)\left\{v^3 \sin 3\psi + O(v^4)\right\}, \\ \zeta^{\times}_{5,3} &= -\tfrac{27}{160}\sin\theta\cos\theta(5 - 9\cos^2\theta)\left\{v^3\cos 3\psi + O(v^4)\right\}. \end{aligned} \tag{5.34}$$

Consider finally the case $m = 1$. With ${}_0Y_{5,1}(\frac{\pi}{2}, 0) = -\frac{1}{16}(165/2\pi)^{1/2}$, it follows that

$$Z_{5,1} = \tfrac{1}{360} i (\pi/77)^{1/2} \Omega^7 r_0{}^5 \left\{1 + O(v)\right\}. \tag{5.35}$$

Using ${}_{-2}Y_{5,\pm 1}(\theta,\phi) = \pm\frac{1}{16}(77/\pi)^{1/2}\sin\theta(1 \mp 2\cos\theta - 12\cos^2\theta \pm 6\cos^3\theta + 15\cos^4\theta)e^{\pm i\phi}$, the waveforms are then obtained:

$$\begin{aligned} \zeta^{+}_{5,1} &= -\tfrac{1}{1440}\sin\theta(1 - 12\cos^2\theta + 15\cos^4\theta)\left\{v^3 \sin\psi + O(v^4)\right\}, \\ \zeta^{\times}_{5,1} &= -\tfrac{1}{720}\sin\theta\cos\theta(1 - 3\cos^2\theta)\left\{v^3\cos\psi + O(v^4)\right\}. \end{aligned} \tag{5.36}$$

VI. SUMMARY AND CONCLUSION

The only remaining task of this paper is to gather the results of the preceding section for $\eta_{\ell m}$, $\zeta^{+}_{\ell m}$, and $\zeta^{\times}_{\ell m}$, and to perform the summation over ℓ and m. The result will be expressions for the gravitational waveforms and luminosity, valid to order v^3.

The expression for the luminosity was given in Eq. (1.3); the waveforms may be written as

$$h_{+,\times} = -(\mu/r)(M\Omega)^{2/3}\left\{\zeta^{(0)}_{+,\times} + v\zeta^{(1)}_{+,\times} + v^2\zeta^{(2)}_{+,\times} + v^3\zeta^{(3)}_{+,\times} + O(v^4)\right\}, \tag{6.1}$$

where

$$\begin{aligned} \zeta^{(0)}_{+} &= 2(1+\cos^2\theta)\cos 2\psi, \\ \zeta^{(1)}_{+} &= -\tfrac{1}{4}\sin\theta\left[(5+\cos^2\theta)\sin\psi + 9(1+\cos^2\theta)\sin 3\psi\right], \\ \zeta^{(2)}_{+} &= -\tfrac{1}{3}\left[(19 + 9\cos^2\theta - 2\cos^4\theta)\cos 2\psi + 8\sin^2\theta(1+\cos^2\theta)\cos 4\psi\right], \\ \zeta^{(3)}_{+} &= 4\pi(1+\cos^2\theta)\cos 2\psi + 8(3\ln 2v + \gamma - \tfrac{17}{12})(1+\cos^2\theta)\sin 2\psi + \tfrac{1}{96}\sin\theta(57 + 20\cos^2\theta - \cos^4\theta)\sin\psi \\ &\quad + \tfrac{9}{64}\sin\theta(73 + 40\cos^2\theta - 9\cos^4\theta)\sin 3\psi + \tfrac{625}{192}\sin^3\theta(1+\cos^2\theta)\sin 5\psi, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \zeta^{(0)}_{\times} &= 4\cos\theta\sin 2\psi, \\ \zeta^{(1)}_{\times} &= \tfrac{3}{2}\sin\theta\cos\theta(\cos\psi + 3\cos 3\psi), \\ \zeta^{(2)}_{\times} &= -\tfrac{2}{3}\cos\theta\left[(17 - 4\cos^2\theta)\sin 2\psi + 8\sin^2\theta\sin 4\psi\right], \\ \zeta^{(3)}_{\times} &= \cos\theta\big[\,8\pi\sin 2\psi - 16(3\ln 2v + \gamma - \tfrac{17}{12})\cos 2\psi - \tfrac{1}{48}\sin\theta(63 - 5\cos^2\theta)\cos\psi \\ &\qquad - \tfrac{9}{32}\sin\theta(67 - 15\cos^2\theta)\cos 3\psi - \tfrac{625}{96}\sin^3\theta\cos 5\psi\big]. \end{aligned} \tag{6.3}$$

I recall that $\psi = \Omega(t - r^*) - \phi$, and $v = (M/r_0)^{1/2} = (M\Omega)^{1/3}$. The waveforms (6.1) – (6.3) are illustrated in Fig. 1, and the caption discusses some of their properties. It is worth noting that the angle θ (defined with respect to the z axis of the coordinate system, which is chosen in such a way that the orbital motion lies in the x-y plane and is right handed) is the orbital inclination with respect to the observer's line of sight.

It is immediate that the zeroth-order term in Eq. (6.1) agrees with the standard quadrupole-formula result [3]. Moreover, it may be verified that the expansion of the waveforms to order v^2 agrees with the $\mu \ll M$ limit of Wagoner and Will [11] (as expressed in Krolak [25]). In particular, $\zeta^{(1)}_{+,\times}$ originates from mass-difference effects; the post-Newtonian calculation reveals that this term would disappear should the companions have equal masses. The fourth term in the expansion, $\zeta^{(3)}_{+,\times}$, has never been calculated before. Together with Eq. (1.3), Eqs. (6.1) – (6.3) are the fundamental results of this paper.

The techniques developed in this paper are not sufficient to calculate the v^4 terms in the expansions for the gravitational waveforms and luminosity. The reason is simple. Since $v^4 = (M/r_0)^2$, which is quadratic in the black-hole mass M, the calculation of these terms would require higher-order corrections to the Teukolsky function, namely, corrections of order $(M\omega)^2$. Since I have limited the analysis to first order, this calculation can-

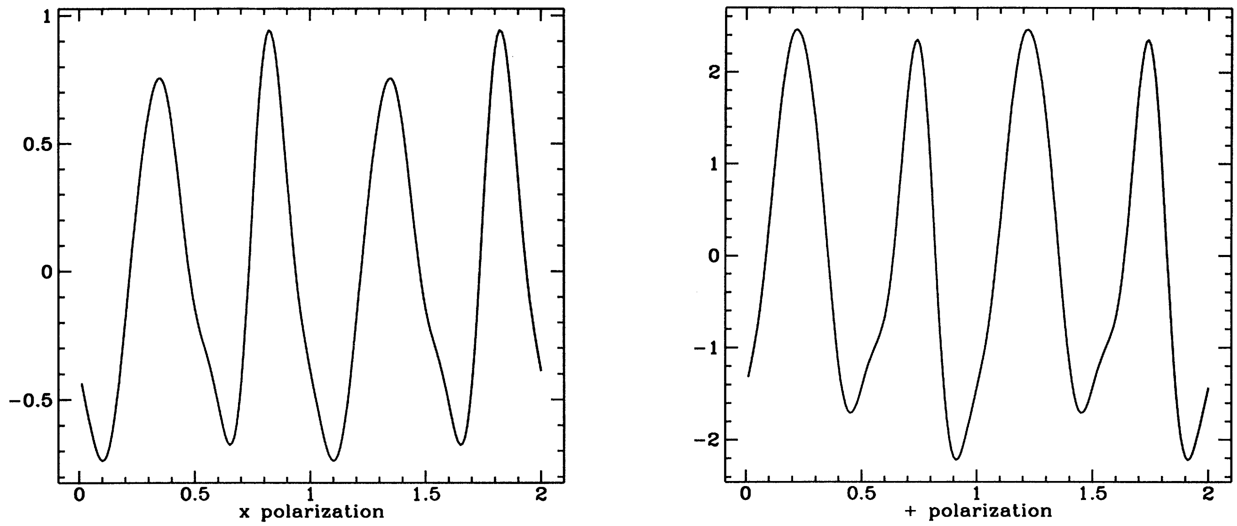

FIG. 1. Plots of the gravitational waveforms, $h_{+,\times}$ divided by the factor $(\mu/r)(M\Omega)^{2/3}$, as functions of the number of orbital cycles (compare with Detweiler, Ref. [22], and Lincoln and Will, Ref. [26]). The observation angle is 10 degrees above the orbital plane ($\theta = 4\pi/9$), and $M\Omega$ is set to 0.065, corresponding to an orbital radius of $r_0/M \simeq 6.2$. The presence of the higher-order terms in Eq. (6.1) is seen to significantly alter the shape of the waveforms, with respect to the Newtonian, monocromatic curves. With an observation angle of 90 degrees above the orbital plane ($\theta = 0$, face-on view), both polarizations have equal amplitudes, and the frequency spectrum is degenerate: $\omega = 2\Omega$ only — the waves are circularly polarized. That this property must hold to all orders in v follows from the observation that the radiative field of a circular binary will appear stationary when observed face on from a reference frame rotating at the orbital frequency. With an observation angle of zero degrees above the orbital plane ($\theta = \pi/2$, edge-on view), the radiation is entirely contained in the "+" polarization. This property should also hold to all orders in v.

not be performed here. However, the formalism developed in this paper could be used as a foundation for the second-order analysis; it is therefore conceivable that the higher-order terms might be calculable analytically (or semianalytically).

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