Microcanonical functional integral for the gravitational field

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The gravitational field in a spatially finite region is described as a microcanonical system. The density of states ν is expressed formally as a functional integral over Lorentzian metrics and is a functional of the geometrical boundary data that are fixed in the corresponding action. These boundary data are the thermodynamical extensive variables, including the energy and angular momentum of the system. When the boundary data are chosen such that the system is described semiclassically by *any* real stationary axisymmetric black hole, then in this same approximation $\ln\nu$ is shown to equal $\frac{1}{4}$ the area of the black-hole event horizon. The canonical and grand canonical partition functions are obtained by integral transforms of ν that lead to "imaginary-time" functional integrals. A general form of the first law of thermodynamics for stationary black holes is derived. For the simpler case of nonrelativistic mechanics, the density of states is expressed as a real-time functional integral and then used to deduce Feynman's imaginary-time functional integral for the canonical partition function.

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I. INTRODUCTION

The energy of a physical system is reflected in the gravitational field it produces, so the gravitational field at a (spatially finite or infinite) closed surface that bounds the system encodes information about the energy content. By fixing appropriate components of the gravitational field, the energy of a self-gravitating system can be specified as *boundary data*. In statistical mechanics and thermodynamics where the concept of energy plays a central role, this circumstance allows for the direct specification of microcanonical boundary conditions in which the thermodynamical extensive variables (including energy) are held fixed. Here, we exploit this special property of the gravitational field in a direct construction of a "microcanonical function integral," a formal functional integral expression for the density of states, which characterizes a system with microcanonical boundary conditions.

The canonical partition function for nonrelativistic mechanics was first expressed as an imaginary-time functional integral by Feynman [1]. This prescription was later generalized to flat-space field theory [2], then to self-gravitating systems by Gibbons and Hawking [3]. As an alternative to this line of development, we present a direct expression of the density of states for nonrelativistic mechanics as a real-time functional integral. The generalization of this result to flat-space field theory is not immediate, because in that case fixing the energy involves a restriction on the integral of the Hamiltonian over the entire spatial extent of the system. However, the generalization to gravitating systems is quite natural, because the presence of gravity allows the energy to be fixed directly by the boundary data. In this paper, we consider the functional integral expression for the density of states for systems consisting only of the gravitational field. The inclusion of various matter fields will be given elsewhere [4]. Inasmuch as all systems are self-gravitating, even if only weakly, the formalism developed here is, in principle, completely general.

One of the key features of the present analysis is the use of finite boundaries in space. There are a number of advantages to be gained by imposing boundary conditions at a spatially finite location, as opposed to spatial infinity. For example, with finite spatial boundaries, there is no need to assume asymptotic flatness in spacelike directions. This is important, because a self-gravitating thermodynamical system generically does not satisfy asymptotic flatness. In particular, the system semiclassically approximated by a black hole in equilibrium with Hawking radiation is not asymptotically flat when the back reaction of radiation on the geometry is taken into account [5]. Another advantage of using finite spatial boundaries appears in the treatment of rotation. Since any system in thermal equilibrium must rotate rigidly (if at all) [6,7], such systems necessarily have finite spatial extent. Thus, a Kerr black hole surrounded by Hawking radiation can be treated as the semiclassical approximation to a thermal equilibrium system only if a spatially finite boundary is employed. As a final example, observe that the usual thermodynamic limit requiring infinite spatial extent does not exist for an equilibrium self-gravitating system at nonzero temperature. This is because the system is unstable to gravitational collapse, or recollapse if a black hole is already present. The instability of such a spatially infinite system at fixed temperature is reflected in a formally negative value for the heat capacity, which, in turn, implies that the canonical partition function diverges. (See, for example, Ref. [8].) On the other hand, a spatially finite system can avoid such difficulties. For

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the gravitational field at relatively low temperature, the system is approximated by flat space filled with dilute gravitational radiation; at relatively high temperature the system is approximated semiclassically by a large black hole surrounded by sparse gravitational radiation [5,9].

Self-gravitating systems in thermal equilibrium are typically spatially inhomogeneous because of gravitational "clumping." In particular, the temperature of an equilibrium system may vary in space due to gravitational redshifting [10]. As a consequence, such systems are characterized not by a single temperature, but can be described by a temperature field which is a local function defined on the spatial two-boundary [11,12]. Correspondingly, the thermodynamical conjugate of inverse temperature is not simply the total energy, but rather an energy surface density which is a local function on the spatial two-boundary. The microcanonical or canonical descriptions of a self-gravitating system are obtained by fixing the energy surface density or surface temperature (respectively) as boundary data. Generally, all thermodynamic intensive and extensive variables are functions defined on the spatial boundary. The appropriate definitions of energy density as well as angular momentum density are discussed in detail in Ref. [13], and are reviewed in Sec. III.

The density of states for the gravitational field is defined here as a functional of the energy surface density, momentum surface density, and the two-metric on the spatial boundary of the system. It is expressed formally as a functional integral over Lorentzian metrics satisfying the boundary conditions, and includes contributions from manifolds of various topologies. We evaluate the density of states in a "zero-order" approximation in which the functional integral is approximated by its integrand evaluated at an appropriate saddle point. When the boundary conditions are chosen such that the system is approximated classically by a stationary axisymmetric black hole, then in the zero-order approximation the entropy (identified as the logarithm of the density of states in absolute units $G=c=\hbar=k_B=1$) equals $\frac{1}{4}$ the area of the black hole's event horizon. This result applies to *any* stationary axisymmetric black hole, including those that are distorted relative to the standard Kerr family by stationary external matter fields. This result also extends to black holes coupled to electromagnetic and Yang-Mills fields [4]. (The result also does not appear to depend on axisymmetry.) When the boundary conditions for the density of states are chosen such that the system is approximated classically by flat space-time, it is shown that the entropy vanishes in the zero-order approximation, as expected.

In nonrelativistic mechanics, the canonical partition function is defined by a sum over energy levels weighted by the Boltzmann factor and appropriate degeneracy factors. In the cases we shall treat, this is generalized and expressed as a (functional) integral transform of the density of states. At the level of thermodynamics, the change of boundary data amounts to a (functional) Legendre transformation between the energy density and the inverse temperature, which are thermodynamically conjugate variables. At the level of dynamics, this change of boundary data amounts to a canonical transformation and the energy density and inverse temperature are given by the boundary values of a canonically conjugate pair of variables. For this interpretation, canonical conjugacy is defined with respect to the history of the spatial boundary, not with respect to the usual spatial time slices. Analogous relationship hold for the angular momentum density and its conjugate, the angular velocity, as well as other pairs of conjugate variables. These results reveal an intimate connection between thermodynamical and canonical conjugacy for self-gravitating systems [14].

In Sec. II, we present a real-time functional integral expression for the density of states in nonrelativistic mechanics. The relevant action functional is Jacobi's action [15,16], in which the energy of the system is fixed. Details of the construction are given in an Appendix. In Sec. III, we draw on the analysis of Ref. [13] to obtain a "microcanonical action," an action functional for which the appropriate boundary conditions include fixed energy surface density, momentum surface density, and boundary two-metric. The microcanonical action is used in Sec. IV to express the density of states formally as a functional integral. The functional integral is then evaluated in the saddle point or zero-order approximation to show that the entropy of any stationary axisymmetric black hole is $\frac{1}{4}$ the area of its event horizon. In Sec. V, the canonical and grand canonical partition functions are derived from the microcanonical construction and the correspondence between thermodynamical and canonical conjugacy is described. The first law of thermodynamics is derived in Sec. VI by considering variations of the microcanonical action with respect to the boundary data.

II. DENSITY OF STATES IN NONRELATIVISTIC QUANTUM MECHANICS

In this section, the formal expression of the density of states as a real-time functional integral is derived for nonrelativistic systems with a finite number of degrees of freedom. Our starting point is the density of states expressed as

$$\nu(E)=\mathrm{Tr}\,\delta(E-\hat{H}) \ , \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian operator for the system. The number of quantum states between E_1 and E_2 is

$$\int_{E_1}^{E_2} dE\, \nu(E) \ , \tag{2.2}$$

as seen by taking the trace in a basis of energy eigenstates. Using a coordinate basis for the trace, the density of states becomes

$$\nu(E)=\int dx \langle x|\delta(E-\hat{H})|x\rangle, \tag{2.3}$$

where x represents a set of configuration coordinates, x^1, x^2, ... The matrix elements in the integrand above are the diagonal entries of the matrix $\langle x''|\delta(E-\hat{H})|x'\rangle$, which can be expressed as

$$\langle x''|\delta(E-\hat{H})|x'\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dT\, e^{iET/\hbar}\langle x''|e^{-i\hat{H}T/\hbar}|x'\rangle \ . \quad (2.4)$$

In turn, the matrix elements in the integrand of Eq. (2.4) can be expressed as a functional integral [1]:

$$\langle x''|e^{-i\hat{H}T/\hbar}|x'\rangle = \int_{x(0)=x'}^{x(T)=x''} \mathcal{D}H e^{iS_T/\hbar}. \quad (2.5)$$

This functional integral is a sum over histories $x(t)$ that begin at $x(0)=x'$ and end at $x(T)=x''$, with $\mathcal{D}H$ denoting a measure for the space of histories. The histories are weighted by $\exp(iS_T/\hbar)$, where $S_T[x]$ is Hamilton's action with a fixed time interval T.

Collecting together the above results, the density of states becomes

$$\nu(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dT \int dx \int_{x(0)=x}^{x(T)=x} \mathcal{D}H e^{i(S_T+ET)/\hbar}. \quad (2.6)$$

This expression for $\nu(E)$ is an integral over all histories $x(t)$ that are periodic for some real time interval. In the Appendix, it is shown that this functional integral is precisely the sum over periodic histories constructed from Jacobi's action S_E [15,16]. In Jacobi's action, the energy E is fixed rather than the time interval T. Consequently the path integral for the density of states can be written as

$$\nu(E) = \int \mathcal{D}H_p e^{iS_E/\hbar} \ , \quad (2.7)$$

where it is understood that the histories H_p contributing to this path integral are periodic in real time. This is the key result that will be generalized to the case of self-gravitating systems: the density of states is a sum over periodic histories, weighted by a phase that is given by the action appropriate for describing the system at fixed energy.

The canonical partition function is obtained by summing the Boltzmann factor over each energy level. In terms of the density of states, the partition function is given by a Laplace transform:

$$Z(\beta) = \int_0^{\infty} dE\, \nu(E) e^{-\beta E} \ , \quad (2.8)$$

where $\beta^{-1} = k_B \times$(temperature) and k_B is Boltzmann's constant. Using the expression (2.1) for $\nu(E)$ and assuming the energy spectrum is positive gives the familiar result $Z(\beta) = \mathrm{Tr}\exp(-\hat{H}\beta)$. Alternatively, with the density of states expressed as the path integral (2.6), the partition function becomes

$$Z(\beta) = \int_0^{\infty} dE\, e^{-\beta E}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dT\, e^{iET/\hbar}\int \mathcal{D}H_p e^{iS_T/\hbar}, \quad (2.9)$$

where again H_p refers to periodic histories. With the change of variables $T=-i\tau$, the partition function becomes

$$Z(\beta) = \int_0^{\infty} dE\, e^{-\beta E}\frac{1}{2\pi i} \times \int_{-i\infty}^{i\infty} d(\tau/\hbar) e^{E\tau/\hbar} \times \left[\int \mathcal{D}H_p e^{iS_T/\hbar}\right]\Bigg|_{T=-i\tau} . \quad (2.10)$$

This expression is simply the Laplace transform of the inverse Laplace transform of the functional integral in parentheses. To be precise, the identification of the integral over $\tau/\hbar$ with an inverse Laplace transform assumes that the integration contour passes to the right of any poles in the complex plane. Such points will be ignored in the present formal analysis. Then the result of the successive inverse Laplace and Laplace transforms is to set τ equal to $\hbar\beta$ in the path integral factor, leaving

$$Z(\beta) = \int \mathcal{D}H_p e^{iS_T/\hbar}\Bigg|_{T=-i\hbar\beta} . \quad (2.11)$$

This is Feynman's result [1], that the canonical partition function can be written as an "imaginary-time" functional integral.

The expression (2.11) for $Z(\beta)$ is often taken as the starting point for a treatment of thermodynamics by functional integral methods. Observe that the existence of the canonical partition function depends on the convergence of the Laplace transform (2.8). If the density of states increases too rapidly for large E, then $Z(\beta)$ is not defined. This occurs when the heat capacity for the system is formally negative, signaling a thermodynamical instability. (The relationship between the convergence of the Laplace transform for $Z(\beta)$ and the sign of the heat capacity is spelled out in Ref. [8].) We regard the real-time functional integral (2.7) for the density of states as a more fundamental expression. In the following sections, this result is generalized to the case of self-gravitating systems.

III. MICROCANONICAL ACTION

We begin by summarizing the results of Ref. [13]. Start with the action for gravity:

$$S[g] = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,(\mathcal{R}-2\Lambda) + \frac{1}{\kappa}\int_{t'}^{t''} d^3x\sqrt{h}K - \frac{1}{\kappa}\int_{{}^3B} d^3x\sqrt{-\gamma}\Theta - S^0 \ , \quad (3.1)$$

where $\kappa=8\pi$ and Newton's constant is set to unity. The spacetime manifold is $M=\Sigma\times I$, the product of a space manifold Σ and a real line interval I. The two-boundary of space Σ is B, and the history of B is ${}^3B = B\times I$. The submanifolds of M that coincide with the end points of the line interval I are the hypersurfaces t' and t''. The notation $\int_{t'}^{t''} d^3x$ represents an integral over t'' minus an integral over t'. We also use the following notational conventions. The metric and curvature tensor on spacetime M are $g_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\alpha\beta}$, respectively, the metric and extrinsic curvature on the hypersurfaces Σ are h_{ij} and K_{ij}, respectively, and the metric and extrinsic curvature on ${}^3B$ are γ_{ij} and Θ_{ij}, respectively. (Latin letters i, j, etc., are used as indices for tensors on both ${}^3B$ and Σ. No cause for confusion arises from this convention.) The term S^0 in Eq. (3.1) is a functional of the metric γ_{ij} on ${}^3B$; however, it will be seen that such a term is unnecessary.

The action S is written in canonical form by foliating M into spacelike hypersurfaces Σ. Without loss of physical generality, we restrict these hypersurfaces to be orthogonal to the boundary element ${}^3B$. That is, on the boundary element ${}^3B$, the timelike unit normal u^μ of each surface Σ is required to be orthogonal to the spacelike unit normal of ${}^3B$. The result is [13]

$$S=\int_M d^4x(P^{ij}\dot{h}_{ij}-N\mathcal{H}-V^i\mathcal{H}_i)\\ -\int_{{}^3B} d^3x\sqrt{\sigma}(N\varepsilon-V^ij_i)\ , \tag{3.2}$$

where N is the lapse function, V^i is the shift vector, and the gravitational momentum conjugate to the metric h_{ij} is

$$P^{ij}=-\frac{1}{4\kappa}\frac{\sqrt{h}}{N}(h^{ij}h^{kl}-h^{ik}h^{jl})(\dot{h}_{kl}-2D_{(k}V_{l)})\ . \tag{3.3}$$

The gravitational contributions to the Hamiltonian and momentum constraints are

$$\mathcal{H}=\frac{\kappa}{\sqrt{h}}[2P^{ij}P_{ij}-(P^i_i)^2]-\frac{\sqrt{h}}{2\kappa}(R-2\Lambda)\ , \tag{3.4a}$$

$$\mathcal{H}_i=-2D_jP^j_i\ , \tag{3.4b}$$

where R and D_i are the curvature scalar and covariant derivative on Σ, respectively. In the surface term of the action (3.2), σ denotes the determinant of the metric tensor on B, and the energy surface density ε and momentum surface-density j_i are defined by

$$\varepsilon=\frac{1}{\kappa}k+\frac{1}{\sqrt{\sigma}}\frac{\delta S^0}{\delta N}\ , \tag{3.5a}$$

$$j_i=-\frac{2}{\sqrt{h}}\sigma_{ij}n_kP^{jk}-\frac{1}{\sqrt{\sigma}}\frac{\delta S^0}{\delta V^i}\ . \tag{3.5b}$$

Here, σ_{ij}, n_i, and k denote (respectively) the induced metric, the unit normal, and the trace of the extrinsic curvature for B as a surface embedded in Σ. In writing the Hamiltonian action, we have assumed that S^0, if present, is a linear functional of the lapse and shift on ${}^3B$, in accordance with the discussion of Ref. [13].

The Hamiltonian obtained from the action (3.2) is

$$H=\int_\Sigma d^3x(N\mathcal{H}+V^i\mathcal{H}_i)+\int_B d^2x\sqrt{\sigma}(N\varepsilon-V^ij_i). \tag{3.6}$$

The shift vector at the boundary B must satisfy $n_iV^i|_B=0$, so that the Hamiltonian does not generate spatial diffeomorphisms that map the field variables across the boundary B of the space manifold Σ. This restriction implies that the Hamiltonian evolves the initial data into a spacetime whose foliation by spacelike slices is orthogonal to the boundary element ${}^3B$. With the surface terms that appear in Eq. (3.6), the Hamiltonian has well-defined functional derivatives with respect to the canonical variables under the conditions that N, V^i, and σ_{ab} are fixed on the boundary B.

We use indices a, b, etc., to denote components of tensors on B. Such tensors also can be viewed as tensors on Σ that are orthogonal to the unit normal n^i of B. Thus, for example, we write the two-metric on B as σ_{ab} or σ_{ij}, the extrinsic curvature of B embedded in Σ as k_{ab} or k_{ij}, the shift vector on B as V^a or V^i, and the momentum surface density as j_a or j_i. We also have occasion to view these tensors as tensors on space-time, and will then use space-time indices μ, ν, etc.

One can calculate, as in Ref. [13], that a general variation of the action S with respect to the canonical variables h_{ij}, P^{ij}, lapse N, and shift V^i is given by

$$\delta S=(\text{terms giving the equations of motion})\\ +\int_{t'}^{t''} d^3x\, P^{ij}\delta h_{ij}\\ -\int_{{}^3B} d^3x\sqrt{\sigma}[\varepsilon\delta N-j_a\delta V^a-(N/2)s^{ab}\delta\sigma_{ab}]\ . \tag{3.7}$$

The term s^{ab} is the surface stress tensor on B, defined by [13]

$$s^{ab}=\frac{1}{\kappa}[k^{ab}+(n_ia^i-k)\sigma^{ab}]-\frac{2}{\sqrt{-\gamma}}\frac{\delta S^0}{\delta\sigma_{ab}}, \tag{3.8}$$

where a^i is the acceleration of the timelike unit normal of the spacelike hypersurfaces. The expression (3.7) shows that suitable boundary conditions for S are found by fixing the induced metric on the boundary ∂M. That is, fix the three-metric components h_{ij} on t' and t'', and fix the three-metric components N, V^a, and σ_{ab} on ${}^3B$. Then the surface terms in the variation δS vanish, and solutions of the equations of motion extremize the action S with respect to variations that obey these boundary conditions.

What we define as the *microcanonical action* S_m is obtained from S by adding boundary terms that change the appropriate boundary conditions on ${}^3B$ from fixed metric components N, V^a, and σ_{ab} to fixed energy surface density ε, momentum surface density j_a, and boundary metric σ_{ab}. Thus, define

$$S_m=S+\int_{{}^3B} d^3x\sqrt{\sigma}(N\varepsilon-V^aj_a) \tag{3.9a}$$

$$=\int_M d^4x(P^{ij}\dot{h}_{ij}-N\mathcal{H}-V^i\mathcal{H}_i)\ , \tag{3.9b}$$

and from Eq. (3.7), the variation of S_m is

$$\delta S_m=(\text{terms giving the equations of motion})\\ +\int_{t'}^{t''} d^3x\, P^{ij}\delta h_{ij}\\ +\int_{{}^3B} d^3x[N\delta(\sqrt{\sigma}\varepsilon)-V^a\delta(\sqrt{\sigma}j_a)\\ +(N\sqrt{\sigma}/2)s^{ab}\delta\sigma_{ab}]\ . \tag{3.10}$$

This result shows that solutions of the equations of motion extremize S_m under variations in which ε, j_a, and σ_{ab} are held fixed on the boundary ${}^3B$. Observe that the unspecified subtraction term S^0 does not appear in the action S_m, so in this sense the microcanonical action is unique. Nevertheless, the variation (3.10) of S_m is expressed in terms of the surface stress-energy-momentum components ε, j_a, and s^{ab}, which do depend on S^0 for their definitions. However, since S^0 is a linear functional of the lapse and shift, the S^0 dependences contained in the various terms of δS_m actually cancel.

The boundary terms in the variation (3.10) of S_m show that N and $\sqrt{\sigma}\varepsilon$ are canonically conjugate, where canonical conjugacy is defined with respect to the boundary element ${}^3B$. Likewise, V^a and $-\sqrt{\sigma}j_a$ are canonically conjugate, as are $(N\sqrt{\sigma}/2)s^{ab}$ and σ_{ab}. The boundary terms added to S in Eq. (3.9a) to obtain the microcanonical action S_m amount to the addition of terms of the form "pq" at ${}^3B$. These terms have the effect of changing the appropriate boundary conditions from fixed "q" to fixed conjugate "p."

The microcanonical action (3.9) can be written in spacetime covariant form by using expression (3.1) for S and the decomposition of the scalar curvature:

$$\mathcal{R}=R+K_{\mu\nu}K^{\mu\nu}-(K)^2-2\nabla_\mu(Ku^\mu+a^\mu) \ . \tag{3.11}$$

The extra boundary terms in S_m are written covariantly by using the decomposition of the extrinsic curvature $\Theta_{\mu\nu}$ found in Ref. [13]. That analysis yields the relationships

$$k=(g^{\mu\nu}+u^\mu u^\nu)\Theta_{\mu\nu}, \tag{3.12a}$$

$$-2V_iP^{ij}n_j/\sqrt{h}=-V^\mu u^\nu\Theta_{\mu\nu}/\kappa \ , \tag{3.12b}$$

for the corresponding terms in ε in j_i [see Eqs. (3.5)]. The microcanonical action in spacetime covariant form is therefore

$$S_m[g]=\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,(\mathcal{R}-2\Lambda)+\frac{1}{\kappa}\int_{t'}^{t''}d^3x\sqrt{h}\,K -\frac{1}{\kappa}\int_{{}^3B}d^3x\sqrt{-\gamma}\,t_\mu\Theta^{\mu\nu}\partial_\nu t \ . \tag{3.13}$$

Here, t is the scalar field defined on ${}^3B$ that labels the foliation on which ε, j_a, and σ_{ab} are fixed, $\Theta^{\mu\nu}$ is the extrinsic curvature tensor of ${}^3B$, and t^μ is the time vector field defined on ${}^3B$ that specifies the time direction. In terms of the timelike unit normal u^μ of the slices $B\subset{}^3B$, these quantities are given by $u_\mu=-N\partial_\mu t=(t_\mu-V_\mu)/N$.

IV. MICROCANONICAL FUNCTIONAL INTEGRAL

In Sec. II we showed that for nonrelativistic mechanics the density of states is given by a sum over periodic, real-time histories, where each history contributes a phase determined by the action that describes the system at fixed energy. In the case of nonrelativistic mechanics, the energy is just the value of the Hamiltonian that generates unit time translations. For a self-gravitating system, the Hamiltonian has a "many-fingered" character: space can be pushed into the future in a variety of ways, governed by different choices of lapse function N and shift vector V^i. The value of the Hamiltonian (3.6) depends on this choice. More precisely, the value of the Hamiltonian is determined by the choice of lapse and shift on the boundary B, since the lapse and shift on the domain of Σ *interior* to B are Lagrange multipliers for the (vanishing) Hamiltonian and momentum constraints. Accordingly, the energy surface density ε and momentum surface density j_a for a self-gravitating system play a role that is analogous to energy for a nonrelativistic mechanical system. In particular, the energy surface density ε is the value (per unit boundary area) of the Hamiltonian that generates unit magnitude proper time translations of the boundary B, in the spacetime direction orthogonal to Σ. Likewise, the momentum surface density j_a is the value (per unit boundary area) of the Hamiltonian that generates spatial diffeomorphisms in the $\partial/\partial x^a$ direction on the boundary B.

The above considerations lead us to propose that the density of states for a spatially finite, self-gravitating system is a functional of the energy surface density ε and momentum surface density j_a. In addition to these energylike quantities, the density of states is also a functional of the metric σ_{ab} on the boundary B, which specifies the size and shape of the system. In the absence of matter fields, these make up the complete set of variables and $\nu[\varepsilon,j_a,\sigma_{ab}]$ is interpreted as the density of quantum states of the gravitational field with energy density, momentum density, and boundary metric having the values ε, j_a, and σ_{ab}. The action to be used in the functional integral representation of ν is S_m, which describes the gravitational field with fixed ε, j_a, and σ_{ab}. Note that ε, j_a, and σ_{ab} play the role of thermodynamical extensive variables. These variables are all constructed from the dynamical phase-space variables (h_{ij},P^{ij}) for the system, where the phase-space structure is defined using the foliation of M into spacelike hypersurfaces. (We expect this to be a defining feature of extensive variables for general systems of gravitational and matter fields.) On the other hand, the variables N, V^a, and $(N\sqrt{\sigma}/2)s^{ab}$ are not constructed from phase-space variables. However, these variables are canonically conjugate to $\sqrt{\sigma}\varepsilon$, $-\sqrt{\sigma}j_a$, and σ_{ab} where canonical conjugacy is defined with respect to the boundary element ${}^3B$. In Sec. VI, the relations of N, V^a, and s^{ab} to the intensive variables thermodynamically conjugate to $\sqrt{\sigma}\varepsilon$, $-\sqrt{\sigma}j_a$, and σ_{ab} are given.

By analogy with the functional integral (2.7) for the density of states in nonrelativistic mechanics, the density of states for the gravitational field is expressed formally as

$$\nu[\varepsilon,j,\sigma]=\sum_M\int\mathcal{D}H\,\exp(iS_m) \ . \tag{4.1}$$

(Planck's constant has been set to unity.) The sum over M refers to a sum over manifolds of different topologies. The three-boundary for each M is required to have a topology $\partial M=B\times S^1$. If B has a two-sphere topology, then the sum over topologies includes $M=(\text{ball})\times S^1$, with $\partial M=\partial(\text{ball})\times S^1=S^2\times S^1$. Another example is $M=(\text{disk})\times S^2$, with $\partial M=\partial(\text{disk})\times S^2=S^1\times S^2$. The action S_m that appears in Eq. (4.1) is the microcanonical action (3.13) of the previous section, but with the t' and t'' terms dropped because the manifolds considered here have a single boundary component $\partial M={}^3B$:

$$S_m[g]=\frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,(\mathcal{R}-2\Lambda) -\frac{1}{\kappa}\int_{\partial M}d^3x\sqrt{-\gamma}\,t_\mu\Theta^{\mu\nu}\partial_\nu t \ . \tag{4.2}$$

The functional integral (4.1) for ν is a sum over Lorentzian metrics $g_{\mu\nu}$. Note that the action (4.2) may require the addition of a term that depends on the topology of M, such as the Euler number.

In the boundary conditions on $\partial M = B \times S^1$, the two-metric σ_{ab} that is fixed on the hypersurfaces B is typically real and spacelike. Likewise, the energy density ε is real, which requires the unit normal to ∂M to be spacelike. Therefore, the Lorentzian metrics on M must induce a Lorentzian metric on ∂M, where the timelike direction coincides with the periodically identified S^1. Note, however, that there are no nondegenerate Lorentzian metrics on a manifold with topology $M = (\text{disk}) \times S^2$ that also induce such a Lorentzian metric on ∂M. This implies that the formal functional integral (4.1) for the density of states must include degenerate metrics. (For a discussion of the role of degenerate metrics in classical and quantum gravity, see Ref. [17].)

Now consider the evaluation of the functional integral (4.1) for fixed boundary data ε, j_a, and σ_{ab} that correspond to a stationary, axisymmetric black hole. That is, start with a real Lorentzian, stationary, axisymmetric, black hole solution of the Einstein equations, and let $T = \text{const}$ be stationary time slices that contain the closed orbits of the axial Killing vector field. Next, choose a topologically spherical two-surface B that contains the orbits of the axial Killing vector field, and is contained in a $T = \text{const}$ hypersurface. From this surface B embedded in a $T = \text{constant}$ slice obtain the data ε, j_a, and σ_{ab}. In the functional integral for $\nu[\varepsilon, j, \sigma]$, fix this data on each $t = \text{constant}$ slice of ∂M. Observe that, to the extent that the physical system can be approximated by a single classical configuration, that configuration will be the real stationary black hole that is used to induce the boundary data.

The functional integral (4.1) can be evaluated semiclassically by searching for four-metrics $g_{\mu\nu}$ that extremize S_m and satisfy the specified boundary conditions. Observe that the Lorentzian black hole geometry that was used to motivated the choice of boundary conditions is *not* an extremum of S_m, because it has the topology [Wheeler (spatial) wormhole]$\times$[time] and cannot be placed on a manifold M with a single boundary $S^2 \times S^1$. However, there is a related complex four-metric that does extremize S_m, and is described as follows. Let the Lorentzian black hole be given by

$$ds^2 = -\tilde{N}^2 dT^2 + \tilde{h}_{ij}(dx^i + \tilde{V}^i dT)(dx^j + \tilde{V}^j dT), \tag{4.3}$$

where $\tilde{N}$, $\tilde{V}^i$, and $\tilde{h}_{ij}$ are T-independent functions of the spatial coordinates x^i. The horizon coincides with $\tilde{N} = 0$. For convenience, choose spatial coordinates that are "corotating" with the horizon [18,12]. Then the proper spatial velocity of the spatial coordinate system relative to observers at rest in the $T = \text{constant}$ slices vanishes on the horizon, $(\tilde{V}^i / \tilde{N}) = 0$, and the Killing vector field $\partial / \partial T$ coincides with the null generator of the horizon [18,19]. By assumption, the metric (4.3) satisfies the Einstein equations, which are analytic differential equations in T. Therefore the Einstein equations are satisfied by the above metric with T imaginary, or equivalently, with the replacement $T \to -iT$. This leads to the complex black hole metric

$$ds^2 = -(-i\tilde{N})^2 dT^2 + \tilde{h}_{ij}(dx^i - i\tilde{V}^i dT)(dx^j - i\tilde{V}^j dT), \tag{4.4}$$

where the coordinate T is real.

The complex metric (4.4) satisfies the Einstein equations everywhere on a manifold with topology $M = (\text{disk}) \times S^2$, with the possible exception of the points $\tilde{N} = 0$ where the foliation $T = \text{constant}$ degenerates. The locus of those points $\tilde{N} = 0$ is a two-surface called the "bolt" [20]. Near the bolt, the metric becomes

$$ds^2 \approx \tilde{N}^2 dT^2 + \tilde{h}_{ij} dx^i dx^j, \tag{4.5}$$

and describes a Euclidean geometry. The sourceless Einstein equations are not satisfied at the bolt if this geometry has a conical singularity in the two-dimensional submanifold that contains the unit normals $\tilde{n}^i$ to the bolt for each of the $T = \text{constant}$ hypersurfaces. However, there is no conical singularity if the circumferences of circles surrounding the bolt initially increase as 2π times proper radius. The circumference of such circles is given by $P\tilde{N}$, where P is the period in coordinate time T. Therefore the absence of conical singularities is ensured if the condition

$$P(\tilde{n}^i \partial_i \tilde{N}) = 2\pi \tag{4.6}$$

holds at each point on the bolt, where $\tilde{n}^i$ is the unit normal to the bolt in one of the $T = \text{constant}$ surfaces. Because the unit normal is proportional to $\partial_i \tilde{N}$ at the bolt, condition (4.6) restricts the period in coordinate time T to be $P = 2\pi / \kappa_H$, where

$$\kappa_H = [(\partial_i \tilde{N}) \tilde{h}^{ij} (\partial_j \tilde{N})]^{1/2}|_H$$

is the surface gravity of the Lorentzian black hole (4.3) [not to be confused with the constant $\kappa = 8\pi$ that appears in the action (3.1)]. Note that the surface gravity of a stationary axisymmetric black hole is a constant on its horizon [19], so the period $P = 2\pi / \kappa_H$ satisfies the condition (4.6) at each point on the bolt.

The lapse function and shift vector for the metric (4.4) are $N = -i\tilde{N}$ and $V^i = -i\tilde{V}^i$. Thus, the complex metric (4.4) and the Lorentzian metric (4.3) differ only by a factor of $-i$ in their lapse functions and shift vectors. In particular, the three-metric $\tilde{h}_{ij}$ and its conjugate momentum $\tilde{P}^{ij}$ [see Eq. (3.3)] coincide for the stationary metrics (4.3) and (4.4) [12]. Since the boundary data ε, j_a, and σ_{ab} are constructed from the canonical variables only, the complex metric (4.4) satisfies the boundary conditions imposed on the functional integral for $\nu[\varepsilon, j, \sigma]$.

The complex metric (4.4) with the periodic identification given by Eq. (4.6) extremizes the action S_m and satisfies the chosen boundary conditions for the density of states $\nu[\varepsilon, j, \sigma]$. Although this metric is not included in the sum over Lorentzian geometries (4.1), it can be used for a steepest descents approximation to the functional integral by distorting the contours of integration for the lapse N and shift V^i in the complex plane. Then the density of states is approximated by

$$\nu[\varepsilon,j,\sigma]\approx\exp(iS_m[-i\tilde{N},-i\tilde{V},\tilde{h}])\ , \tag{4.7}$$

where $S_m[-i\tilde{N},-i\tilde{V},\tilde{h}]$ is the microcanonical action (4.2) evaluated at the complex extremum (4.4). The density of states can be expressed approximately as

$$\nu[\varepsilon,j,\sigma]\approx\exp(\mathcal{S}[\varepsilon,j,\sigma])\ , \tag{4.8}$$

where $\mathcal{S}[\varepsilon,j,\sigma]$ is the entropy of the system. Then the result (4.7) shows that the entropy is

$$\mathcal{S}[\varepsilon,j,\sigma]\approx iS_m[-i\tilde{N},-i\tilde{V},\tilde{h}] \tag{4.9}$$

for the gravitational field with microcanonical boundary conditions.

In order to evaluate S_m for the metric (4.4), first perform a canonical decomposition for the action (4.2) under the assumption that the manifold M has the topology of a punctured disk$\times S^2$. That is, the spacelike hypersurfaces Σ have topology $I\times S^2$, and the timelike direction is periodically identified (S^1). The outer boundary of the disk corresponds to the three-boundary element ${}^3B$ of M [denoted ∂M in Eq. (4.2)] on which the boundary conditions ε, j_a, and σ_{ab} are imposed. The inner boundary of the disk, the boundary of the puncture, appears as another boundary element ${}^3H$ for M. (No data are specified at ${}^3H$.) The canonical decomposition is largely a reversal of the steps that lead from the form (3.9) for S_m to expression (3.13), which applies when Σ has a single boundary B. In the present case, boundary terms appear at ${}^3H$ from the volume integral of the term $\nabla_\mu(Ku^\mu+a^\mu)$ in $\mathcal{R}$, and from an integration by parts on the term involving the shift vector and momentum constraint. The result is[1]

$$S_m=\int_M d^4x(P^{ij}\dot{h}_{ij}-N\mathcal{H}-V^i\mathcal{H}_i)+\int_{{}^3H}d^3x\sqrt{\sigma}[n^i(\partial_iN)/\kappa+2n_iV_jP^{ij}/\sqrt{h}\ ], \tag{4.10}$$

where the expression $a_i=(\partial_iN)/N$ for the acceleration of the timelike unit normal has been used.

Now evaluate the action S_m on the punctured disk$\times S^2$ for the complex metric (4.4), and take the limit as the puncture disappears to obtain a manifold topology $M=(\text{disk})\times S^2$. In this limit, the smoothness of the complex geometry is assured by the regularity condition (4.6). Since the metric satisfies the Einstein equations, the Hamiltonian and momentum constraints in Eq. (4.10) vanish, and the terms $P^{ij}\dot{h}_{ij}$ also vanish by stationarity. Moreover, the second boundary term at ${}^3H$ is zero because the shift vector vanishes at the horizon. Thus, only the first of the boundary terms at ${}^3H$ survives. Evaluating this term for the complex metric (4.4), that is, for the lapse function $N=-i\tilde{N}$, and using the regularity condition (4.6), the microcanonical action becomes

$$\begin{aligned}S_m[-i\tilde{N},-i\tilde{V},\tilde{h}]&=-\frac{i}{\kappa}\int_0^P dT\int d^2x\sqrt{\tilde{\sigma}}\tilde{n}^{\,i}\partial_i\tilde{N}\\&=-\frac{2\pi i}{P\kappa}\int_0^P dT\int d^2x\sqrt{\tilde{\sigma}}\\&=-\frac{2\pi i}{\kappa}A_H\\&=-\frac{i}{4}A_H\ .\end{aligned} \tag{4.11}$$

Here, A_H is the area of the event horizon for the Lorentzian black hole (4.3).

The result (4.11) for the microcanonical action evaluated at the extremum (4.4) leads to an approximation for the entropy (4.9), which is

$$\mathcal{S}[\varepsilon,j,\sigma]\approx\tfrac{1}{4}A_H\ . \tag{4.12}$$

The generality of the result (4.12) should be emphasized: The boundary data ε, j, and σ were chosen from a general stationary, axisymmetric black hole that solves the vacuum Einstein equations within a spatial region with boundary B. Outside the boundary B, the black hole space-time need not be free of matter or be asymptotically flat. Thus, for example, the black hole can be distorted relative to the standard Kerr family. Furthermore, recall that the quantum-statistical system with this boundary data must be classically approximated by the physical black-hole solution that matches that boundary data. The result (4.12) shows that the entropy of the system is approximately $\frac{1}{4}$ the area of the event horizon of the physical black-hole configuration that classically approximates the contents of the system.

It also should be emphasized that the microcanonical action S_m is independent of the term S^0 in Eq. (3.1); thus the entropy is independent of S^0 as has been shown in the framework of the canonical partition function [9]. Moreover, by setting $S^0=0$ in the definitions (3.5a) and (3.5b), the boundary data can be taken to be $\varepsilon=k/\kappa$, $j_i=-2\sigma_{ij}n_kP^{jk}/\sqrt{h}$, and σ_{ab}.

The calculations above have been carried out in the "zero-order" or classical approximation. Beyond this approximation, the density of states will acquire a contribution arising from integration over quadratic terms in the functional integral. Correspondingly, the entropy will acquire corrections to the zero-order result $A_H/4$. Physically, the system can be viewed in the zero-order approximation as consisting of a "vacuum" black hole. The next-order contribution to the functional integral is viewed as arising from thermal gravitons surrounding the black hole. It is known that any stationary, axisymmetric system in thermodynamical equilibrium must rotate rigidly [6,7]. Therefore, the average distribution of graviton radiation surrounding the black hole must rotate rigidly with an angular velocity equal to that of the black-hole horizon. As a consequence, an equilibrium thermodynamical system cannot have an infinite spatial extent, because the graviton flux would then exceed the speed of light beyond some speed-of-light surface surrounding the black hole. This conclusion is supported by the analysis of Frolov and Thorne [7], who show that for a quantum field in the Hartle-Hawking vacuum state on a Kerr

[1]The boundary term at ${}^3H$ has been given in Ref. [12].

black-hole background, the Hadamard function is singular on and outside the speed-of-light surface. The above observations indicate that the density of states calculation of this section is not valid if the two-boundary B used to generate the boundary data ε, j_a, and σ_{ab} is too far from the (rotating) black hole. The difficulty should show itself in the calculation of the quadratic contribution to the functional integral for the density of states. One possibility is that for a too-large boundary B the contour for the functional integral cannot be distorted from Lorentzian metrics to pass through the extremum (4.4) along a path of steepest descents, but only along a path of steepest ascents. In this case, there may be no (generally complex) classical solution that dominates the functional integral for the density of states.

Finally, consider the steepest descents evaluation of the density of states (4.1) for boundary data ε, j_a, and σ_{ab} that correspond to flat Lorentzian spacetime. That is, use a two-boundary B in a stationary time slice of flat space-time to induce the boundary data, then fix this data on each t=constant slice of ∂M. In this case, the same flat space-time that motivates the boundary conditions can be periodically identified and placed on a manifold with boundary topology $B\times S^1$. It therefore constitutes a saddle point for the functional integral for ν. More precisely, continuously many saddle points are obtained since the periodic identification can be made with any proper period. Since these saddle points all arise in a topological sector with $M=\Sigma\times S^1$, the action S_m can be written in the form of Eq. (3.9). This shows that S_m vanishes at each of these saddle points, so the entropy (4.9) vanishes in this "zero-order" approximation.

V. CANONICAL PARTITION FUNCTIONS

The canonical partition function characterizes a system that is open to exchange of energy with its surroundings and has fixed inverse temperature. In the case of self-gravitating systems, the inverse temperature β is fixed on the boundary B that separates the system from its surroundings. Recall that β is not, in general, constant on B. The partition function is defined by an integral over energy densities $\sqrt{\sigma}\varepsilon$,

$$Z_c[\beta,j,\sigma]=\int \mathcal{D}(\sqrt{\sigma}\varepsilon)\,\nu[\varepsilon,j,\sigma]\exp\left[-\int_B d^2x\sqrt{\sigma}\varepsilon\beta\right], \tag{5.1}$$

where the exponential factor arises from a product of Boltzmann factors for each point of B. Using the approximate identification (4.8) of entropy $\mathcal{S}$ as the logarithm of the density of states, Z_c becomes

$$Z_c[\beta,j,\sigma]\approx\int \mathcal{D}(\sqrt{\sigma}\varepsilon)\exp\left[\mathcal{S}[\varepsilon,j,\sigma]-\int_B d^2x\sqrt{\sigma}\varepsilon\beta\right]. \tag{5.2}$$

The partition function can be evaluated approximately by performing the integration over $\sqrt{\sigma}\varepsilon$ in a steepest descents approximation. The stationary point in $\sqrt{\sigma}\varepsilon$ is given by the solution $\varepsilon(\beta)$ of the equation

$$\frac{\delta\mathcal{S}}{\delta(\sqrt{\sigma}\varepsilon)}=\beta\,, \tag{5.3}$$

which will be recognized as a generalized form of the usual relation between the entropy of a system and its thermodynamic temperature. The approximation for the canonical partition function becomes

$$\ln Z_c[\beta,j,\sigma]\approx\mathcal{S}[\varepsilon(\beta),j,\sigma]-\int_B d^2x\sqrt{\sigma}\beta\varepsilon(\beta)\,, \tag{5.4}$$

which expresses the Massieu function $\ln Z_c$ as a (functional) Legendre transform of the entropy $\mathcal{S}$. The expectation value of energy density is defined by

$$\langle\sqrt{\sigma}\varepsilon\rangle=-\frac{\delta\ln Z_c}{\delta\beta}=\frac{1}{Z_c}\int\mathcal{D}(\sqrt{\sigma}\varepsilon)\,\nu(\sqrt{\sigma}\varepsilon)\exp\left[-\int_B d^2x\sqrt{\sigma}\varepsilon\beta\right]. \tag{5.5}$$

This integral can be carried out in a steepest descents approximation, with the result $\langle\sqrt{\sigma}\varepsilon\rangle\approx\sqrt{\sigma}\varepsilon(\beta)$.

By inserting expression (4.1) for the density of states into Eq. (5.1), the canonical partition function can be written as

$$\begin{aligned}Z_c[\beta,j,\sigma]&=\sum_M\int\mathcal{D}H\exp\left[iS_m-\int_B d^2x\sqrt{\sigma}\beta\varepsilon\right]\\&=\sum_M\int\mathcal{D}H\exp\left[iS_m-i\int_{\partial M}d^3x\sqrt{\sigma}N\varepsilon\right]\Bigg|_{\int dt\,N|_B=-i\beta}\\&=\sum_M\int\mathcal{D}H\exp(iS_c)\Bigg|_{\int dt\,N|_B=-i\beta}\,.\end{aligned} \tag{5.6}$$

From the discussion of Sec. III, it is clear that S_c is the action appropriate for boundary conditions consisting of fixed two-metric σ_{ab}, fixed momentum density j_a, and fixed lapse N on ∂M. The functional integral (5.6) is a sum over Lorentzian metrics with these boundary conditions. Furthermore, the gauge invariant part of N on the boundary, namely, the proper distance $\int dt\,N|_B$, is analytically continued to the imaginary value $-i\beta$. The distance $\int dt\,N|_B$ denotes the proper length of curves in the boundary $\partial M=B\times S^1$ that are orthogonal to the slices B and begin and end on the same slice. If it is possible to rotate the contours of integration for the lapse function (at each point of M) to the imaginary axis, then the functional integral (5.6) for Z_c becomes a sum over Euclidean metrics with σ_{ab}, j_a, and $\int dt\, N|_B = \beta$ fixed on ∂M. This prescription for the functional integral representation of the canonical partition function generalizes the results of Gibbons and Hawking [3] to allow for a finite spatial boundary and the effects of rotation. Likewise, Eq. (5.10) below generalizes their results for the grand canonical partition function.

The inverse temperature β that appears in the canonical partition function is the *thermodynamic* temperature of the system. It is measured by the so-called "zero-angular-momentum observers" (ZAMOs) [21] at B, that is, by observers at rest on the spacelike slices $B \subset {}^3B$, and whose four-velocities were earlier denoted by u^μ. Likewise the "chemical potential" defined below is the angular velocity ω of the system at B as measured by these same ZAMOs. See Refs. [12,22] for discussions of β and ω as ZAMO-measured thermodynamical variables.

For the grand canonical partition function Z_g, the system is open to exchange of momentum as well as energy. In the self-gravitating case, assume as in the previous section that the fixed boundary metric σ_{ab} is axisymmetric, and let ϕ^a denote the axial Killing vector field on B. The momentum density in the ϕ^a direction is $\sqrt{\sigma} j_a \phi^a$, and its thermodynamical conjugate is $\beta\omega$ with ω denoting the chemical potential. Below, ω is identified as the angular velocity of the system in the ϕ^a direction with respect to the local proper time on B. (In Ref. [12] this proper-time angular velocity was denoted by $\hat{\omega}$.) The grand canonical partition function is defined by transforming both from fixed energy density $\sqrt{\sigma}\varepsilon$ to fixed inverse temperature β and from fixed angular momentum density $\sqrt{\sigma} j_a \phi^a$ to fixed $\beta\omega$:

$$Z_g[\beta,\beta\omega,\sigma] = \int \mathcal{D}(\sqrt{\sigma}\varepsilon)\mathcal{D}(\sqrt{\sigma} j_a\phi^a)\, \nu[\varepsilon,j,\sigma] \times \exp\left[-\int_B d^2x \sqrt{\sigma}\beta(\varepsilon - \omega j_a \phi^a)\right] . \tag{5.7}$$

(As defined here, Z_g is still a functional of the component $j_\perp$ of j_a in the direction orthogonal to ϕ^a. With axisymmetric boundary data, $j_\perp$ can be simply set equal to zero. Alternatively, Z_g could be defined to include an integral transformation of $j_\perp$ to a zero value of its conjugate.) With the density of states approximated by the exponential of the entropy $\mathcal{S}$, the stationary point for a steepest descents evaluation of Z_g is given by the simultaneous solution of Eq. (5.3) and

$$\frac{\delta\mathcal{S}}{\delta(\sqrt{\sigma} j_a \phi^a)} = -\beta\omega. \tag{5.8}$$

In the zero-order approximation, the grand partition function (5.7) becomes

$$\ln Z_g \approx \mathcal{S} - \int_B d^2x \sqrt{\sigma}\beta(\varepsilon - \omega j_a\phi^a) , \tag{5.9}$$

where ε and $j_a\phi^a$ are functions of β and ω that solve Eqs. (5.3) and (5.8). Equation (5.9) expresses the Massieu function $\ln Z_g$ as a (functional) Legendre transformation of the entropy $\mathcal{S}$. The expectation values of $\sqrt{\sigma}\varepsilon$ and $\sqrt{\sigma} j_a\phi^a$ are defined by derivatives of $\ln Z_g$ with respect to β and $\beta\omega$, respectively. The results are approximated by the solutions of Eqs. (5.3) and (5.8).

Combining the functional integral expression (4.1) for the density of states with the definition (5.7) for the grand canonical partition function yields

$$\begin{aligned} Z_g[\beta,\beta\omega,\sigma] &= \sum_M \int \mathcal{D}H \exp\left[iS_m - \int_B d^2x \sqrt{\sigma}\beta(\varepsilon - \omega j_a\phi^a)\right] \\ &= \sum_M \int \mathcal{D}H \exp(iS_g)\Big|_{\int dt\, N|_B = -i\beta \text{ and } \int dt\, V^\phi|_B = -i\beta\omega} . \end{aligned} \tag{5.10}$$

Here, V^ϕ is the component of the shift vector in the ϕ^a direction. For axisymmetric boundary data with $j_\perp = 0$, the action S_g is precisely the action S discussed in Sec. III for which the two-metric σ_{ab}, lapse N, and shift V^a are fixed on the boundary ∂M. In the functional integral (5.10), the gauge-invariant distance $\int dt\, N|_B$ is fixed to the value $-i\beta$, and $\int dt\, V^\phi|_B$ is fixed to the value $-i\beta\omega$. The quantity $\int dt\, V^\phi|_B$ gives the amount of "twist" in the periodic identification of the boundary $\partial M = B \times S^1$. More precisely, note that the curves on ∂M that are orthogonal to the slices B and begin and end on a single slice need not close. Then $\int dt\, V^{\hat{\phi}}|_B$ equals the proper distance separating the initial and final points of such a curve, as measured along a trajectory of the Killing vector field ϕ^a, where $V^{\hat{\phi}} = \sqrt{\sigma_{\phi\phi}} V^\phi$. If the contours of integration for the lapse N and shift V^ϕ are rotated to the imaginary axis of the complex plane, then the functional integral (5.10) becomes a sum over a set of complex metrics with σ_{ab}, $\int dt\, N|_B = \beta$, and $\int dt\, V^\phi|_B = \beta\omega$ fixed on ∂M.

Recall that the lapse function N and shift vector V^a are canonically conjugate to energy density $\sqrt{\sigma}\varepsilon$ and momentum density $-\sqrt{\sigma} j_a$, respectively, where canonical conjugacy is defined with respect to the boundary ∂M. The functional integral expressions (5.6) and (5.10) for the canonical and grand canonical partition functions show that the canonical and thermodynamical conjugates of the extensive variables $\sqrt{\sigma}\varepsilon$ and $-\sqrt{\sigma} j_a$ are related by

$$\int dt\, N|_B = -i\beta, \tag{5.11a}$$

$$\int dt\, V^{\phi}|_B = -i\beta\omega \ . \tag{5.11b}$$

Furthermore, consider the partition function that is appropriate when the system is open to fluctuations in the two-boundary metric σ_{ab}, and define an intensive variable $(\beta\sqrt{\sigma}/2)p^{ab}$ that is thermodynamically conjugate to σ_{ab}. Writing this partition function as a functional integral shows that

$$\int dt (N\sqrt{\sigma}/2)s^{ab}|_B = -i(\beta\sqrt{\sigma}/2)p^{ab}. \tag{5.12}$$

where $(N\sqrt{\sigma}/2)s^{ab}$ is the canonical conjugate of σ_{ab} and s^{ab} is the spatial stress tensor (3.8). Relations (5.11) and (5.12) show that canonical and thermodynamical conjugacy are intimately connected [14]. Specifically, the thermodynamical conjugate of an extensive variable equals i times the boundary value of the time integral of its canonical conjugate. These relations also hold when matter is minimally coupled to the gravitational field [4], and can be generalized straightforwardly to cases of nonminimal coupling.

Now consider choosing boundary data for the various partition functions such that the complex black hole solution (4.4) extremizes the corresponding action. In this case, the lapse and shift are given by $N=-i\tilde{N}$ and $V^a=-i\tilde{V}^a$. Equation (5.11a) shows that the inverse temperature β equals the proper length of a curve orthogonal to the slices B in the boundary $\partial M = B\times S^1$ of the complex black hole (4.4) [11,12]. From Eq. (5.11b) the chemical potential is $\omega=\tilde{V}^{\phi}/\tilde{N}$. Thus, ω is the proper angular velocity of the Lorentzian black hole (4.3) in the ϕ^a direction, as measured by the ZAMOs [11,12]. Similarly, Eq. (5.12) shows that p^{ab} equals the spatial stress tensor $\tilde{s}^{ab}$ for the Lorentzian black hole (4.3).

VI. THE FIRST LAW

The first law of thermodynamics expresses changes in the entropy of a system in terms of changes in the extensive variables. In the "zero-order" approximation, the first law follows from the general variation of the microcanonical action S_m. That variation includes terms that yield the classical equations of motion, plus boundary terms that arise from integrations by parts. Those boundary terms are just the ones displayed in Eq. (3.10) for the boundary ${}^3B$. Thus, the variation in S_m is

$$\begin{aligned}\delta S_m = &(\text{terms giving the equations of motion})\\ &+\int_{\partial M} d^3x [N\delta(\sqrt{\sigma}\varepsilon) - V^a\delta(\sqrt{\sigma}j_a)\\ &+(N\sqrt{\sigma}/2)s^{ab}\delta\sigma_{ab}] \ .\end{aligned} \tag{6.1}$$

If the variations are restricted to those described by complex black-hole solutions of the form (4.4) for different choices of boundary data (extensive variables) ε, j_a, and σ_{ab}, then the terms giving the equations of motion vanish and the variation becomes

$$\begin{aligned}\delta(iS_m) = \int dt \int_B d^2x [&\tilde{N}\delta(\sqrt{\sigma}\varepsilon) - \tilde{V}^a\delta(\sqrt{\sigma}j_a)\\ &+(\tilde{N}\sqrt{\sigma}/2)\tilde{s}^{ab}\delta\sigma_{ab}] \ .\end{aligned} \tag{6.2}$$

Using the identifications (5.11) and (5.12) for the complex black holes and the approximation $\mathcal{S}\approx iS_m$, this variation becomes

$$\begin{aligned}\delta\mathcal{S}[\varepsilon,j,\sigma] &\approx \delta(A_H/4)\\ &= \int_B d^2x [\beta\delta(\sqrt{\sigma}\varepsilon) - \beta\omega\delta(\sqrt{\sigma}j_a\phi^a)\\ &\quad +\beta(\sqrt{\sigma}p^{ab}/2)\delta\sigma_{ab}] \ .\end{aligned} \tag{6.3}$$

This is the first law of thermodynamics for the gravitational field in a spatially finite region. It is seen to have the form $d\mathcal{S}=$"$\beta\, dE - \beta\omega\, dJ + \beta p\, dV$," familiar from standard thermodynamical treatments of nongravitating systems. In fact, if the boundary data are chosen such that β is a constant on B, then the first term in Eq. (6.3) becomes $\beta\, dE$, where $E=\int_B d^2x\sqrt{\sigma}\varepsilon$ is the total (quasilocal) energy of the system [13]. If the boundary data are chosen such that $\beta\omega$ is a constant on B, then the second term in Eq. (6.3) becomes $-\beta\omega\, dJ$, where $J=\int_B d^2x\sqrt{\sigma}j_a\phi^a$ is the total angular momentum in the ϕ^a direction [13]. Likewise, if the boundary data are spherically symmetric, then the third term in Eq. (6.3) becomes $\beta p\, dA$, where p is the surface pressure and A is the surface area of B [9,14]. However, it should be emphasized that these simplifications hold simultaneously only when the formalism is restricted to static, spherically symmetric systems. In order to treat a system that is classically approximated by, say, a distorted Schwarzchild black hole or a rotating black hole, it is necessary to consider boundary data that are not constant functions on the boundary surface B [11,12].

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APPENDIX: PATH INTEGRAL FOR JACOBI'S ACTION

Consider a system described by the phase space x^1, p_1, x^2, p_2, . . ., and let σ denote a parameter along the phase-space path that increases monotonically from σ' at one end point to σ'' at the other end point. Suppressing the indices on x and p, Jacobi's action reads [16]

$$S_E = \int_{\sigma'}^{\sigma''} d\sigma[\dot{x}p - N\mathcal{H}(x,p)] \ , \tag{A1}$$

where N is a Lagrange multiplier and $\mathcal{H}(x,p)=H(x,p)-E$ is a constraint that sets the Hamiltonian $H(x,p)$ equal to E. When varied with $x(\sigma')=x'$ and $x(\sigma'')=x''$ held fixed, this action yields Newton's equations of motion with the restriction that the energy take the value E. The Lagrange multiplier N has the interpretation as the lapse in physical time:

$$dt = N\, d\sigma. \tag{A2}$$

Note that Jacobi's action is invariant under the gauge transformation

$$\delta x=[x,\epsilon\mathcal{H}],\quad \delta p=[p,\epsilon\mathcal{H}],\quad \delta N=\dot{\epsilon}, \tag{A3}$$

with $\epsilon(\sigma)$ vanishing at the end points σ',σ''. This transformation is just the canonical version of reparametrization invariance, which reflects the arbitrariness in the choice of a path parameter σ.

We will now construct the sum over histories associated with Jacobi's action and show that its trace is precisely the density of states $\nu(E)$. The gauge redundancy will be handled using Becchi-Rouet-Stora-Tyutin (BRST) methods [23]. Let π denote the conjugate to N, so the full set of constraints becomes $\pi=0$ and $\mathcal{H}=0$. Introduce the ghost coordinate C and momentum $\bar{\mathcal{P}}$ associated with the constraint $\mathcal{H}=0$, and the ghost coordinate $-i\mathcal{P}$ and momentum $i\bar{C}$ associated with the constraint $\pi=0$. The ghosts C, $\bar{\mathcal{P}}$, $\mathcal{P}$, and $\bar{C}$ are all anticommuting. The original phase-space variables, Lagrange multiplier and its conjugate, and ghost variables constitute an extended phase-space with fundamental Poisson brackets

$$[p,x]=-1, \tag{A4}$$

$$[\pi,N]=-1, \tag{A5}$$

$$[\bar{\mathcal{P}},C]=-1, \tag{A6}$$

$$[\bar{C},\mathcal{P}]=-1. \tag{A7}$$

The theory is rank zero [23] since the constraints π, $\mathcal{H}$ have vanishing Poisson brackets with one another. As a result the BRST generator is a simple sum of constraints multiplied by ghost coordinates:

$$\Omega=-i\pi\mathcal{P}+\mathcal{H}C. \tag{A8}$$

BRST transformations are defined by $[\cdot,\Omega\varepsilon]$, with ε an anticommuting parameter.

In the extended phase space, Jacobi's action becomes

$$S_E=\int_{\sigma'}^{\sigma''}d\sigma(\dot{x}p+\dot{N}\pi+\dot{\mathcal{P}}\bar{C}+\dot{C}\bar{\mathcal{P}}+[\psi,\Omega]), \tag{A9}$$

where ψ is an anticommuting gauge fixing function on the extended phase space. From the nilpotency of the BRST generator, $[\Omega,\Omega]=0$, the action (A9) is seen to be invariant under BRST transformations with $C(\sigma')=0$ and $C(\sigma'')=0$. The path integral associated with Jacobi's action is now written as

$$Z_E(x'',x')=\int\mathcal{D}He^{iS_E/\hbar}, \tag{A10}$$

where the measure $\mathcal{D}H$ is the product over time of the Liouville measure on the extended phase space. The conditions

$$\begin{aligned}&x(\sigma')=x',\quad x(\sigma'')=x'',\\&\pi(\sigma')=0,\quad \pi(\sigma'')=0,\\&C(\sigma')=0,\quad C(\sigma'')=0,\\&\bar{C}(\sigma')=0,\quad \bar{C}(\sigma'')=0\end{aligned} \tag{A11}$$

are BRST invariant and imply $\Omega(\sigma')=0=\Omega(\sigma'')$, and thus constitute a consistent set of boundary conditions [23] that we will adopt for the path integral (A10).

The Fradkin-Vilkovsiky theorem [23] states that the path integral $Z_E(x'',x')$ is independent of the choice of gauge fixing function ψ. For the purpose of evaluation, a convenient choice is $\psi=\bar{\mathcal{P}}N$, so the path integral (A10) becomes

$$Z_E(x'',x')=\int\mathcal{D}x\,\mathcal{D}p\,\mathcal{D}N\,\mathcal{D}\pi\,\mathcal{D}C\,\mathcal{D}\bar{\mathcal{P}}\,\mathcal{D}\bar{C}\,\mathcal{D}\mathcal{P}\times\exp\left[\frac{i}{\hbar}\int_{\sigma'}^{\sigma''}d\sigma(\dot{x}p+\dot{N}\pi+\dot{\mathcal{P}}\bar{C}+\dot{C}\bar{\mathcal{P}}-i\bar{\mathcal{P}}\mathcal{P}-N\mathcal{H})\right]. \tag{A12}$$

With this choice of ψ the ghost contribution to the path integral decouples, and can be independently evaluated using any among a variety of techniques. One method is to recognize that the ghost path integral equals the determinant of the operator $\partial^2/\partial\sigma^2$ acting in the space of functions that vanish at σ' and σ''. This determinant can be regularized [24], yielding the result $(\sigma''-\sigma')$. The integration over π in the path integral (A12) gives a formal infinite product (over σ) of δ functions of $\dot{N}$, restricting the lapse function N to be a constant. Thus, the result of the $\mathcal{D}\pi\mathcal{D}N$ integration is to leave a single integral $dN_0/2\pi\hbar$ over the constant value N_0 of the lapse.

Collecting together the above results, the path integral becomes

$$Z_E(x'',x')=\frac{\sigma''-\sigma'}{2\pi\hbar}\int dN_0\int\mathcal{D}x\,\mathcal{D}p\,\exp\left[\frac{i}{\hbar}\int_{\sigma'}^{\sigma''}d\sigma(\dot{x}p-N_0\mathcal{H})\right]. \tag{A13}$$

Using the identification (A2), the argument of the exponent can be expressed as an integral over t, while the integration variable $N_0(\sigma''-\sigma')$ is seen to equal the total time interval $T=\int dt$. This leads to

$$Z_E(x'',x')=\frac{1}{2\pi\hbar}\int dT\,e^{iET/\hbar}\int\mathcal{D}x\,\mathcal{D}p\,\exp\left\{\frac{i}{\hbar}\int_0^T dt\left[\left(\frac{dx}{dt}\right)p-H\right]\right\}, \tag{A14}$$

where the definition $\mathcal{H}=H-E$ has been used. The functional integral over x and p contained in Eq. (A14) gives the matrix elements of the evolution operator (2.5), so comparison with Eq. (2.4) shows that the path integral for Jacobi's action is

$$Z_E(x'',x')=\langle x''|\delta(E-\hat{H})|x'\rangle . \tag{A15}$$

The trace of this path integral is

$$\nu(E)=\int dx\, Z_E(x,x), \tag{A16}$$

the density of states.

The above analysis shows that the density of states is the sum over periodic histories constructed from Jacobi's action. Observe that this identification assumes the range of integration for T is over all real values. Thus, the path integral $Z_E(x'',x')$ differs from the causal Green function,[2] which is obtained from expression (A14) by integrating over just positive values of T [25].

By integrating the total time T over all real values, the path integral for $\nu(E)$ consists of a sum over *pairs* of histories, where the members of each pair are weighted with opposite phases. To see this, consider a typical periodic history $x(t)$, $p(t)$, with period $T>0$ that contributes with phase $\exp(iS/\hbar)$ to the path integral for the density of states. Another history that contributes to $\nu(E)$ is $\tilde{x}(t)\equiv x(-t)$, $\tilde{p}(t)\equiv -p(-t)$ with period $\tilde{T}=-T<0$. As t *decreases* from 0 to $\tilde{T}$, the history $\tilde{x}(t)$, $\tilde{p}(t)$ passes through the same sequence of configurations as obtained from the original history for t ranging from 0 to T. Observe that the history $\tilde{x}(t)$, $\tilde{p}(t)$ is closely related to the time reversed history $\bar{x}(t)\equiv x(T-t)$, $\bar{p}(t)\equiv -p(T-t)$, which has period T and consists of the original sequence of configurations taken in reversed order (for increasing t). Now, if the system is time reversal invariant, then the actions for the original history and the time reversed history are the same: denoting the Lagrangian by L,

$$\begin{aligned} S=\bar{S}&=\int_0^T dt\, L(\bar{x}(t),\bar{p}(t)) \\ &=\int_{\tilde{T}}^0 dt\, L(\tilde{x}(t),\tilde{p}(t)), \end{aligned} \tag{A17}$$

where the last equality follows from a change of dummy integration variables. But the phase in the path integral associated with the history $\tilde{x}(t)$, $\tilde{p}(t)$ is determined by $\tilde{S}=\int_0^{\tilde{T}} dt\, L(\tilde{x},\tilde{p})$, so that $\tilde{S}=-S$. Therefore, the history $\tilde{x}(t)$, $\tilde{p}(t)$ and the original history of $x(t)$, $p(t)$ represent the same sequence of configurations but contribute with phases of opposite signs to the path integral for $\nu(E)$. Consequently, each such pair of histories contributes to $\nu(E)$ with phase $2\cos(S/\hbar)$, confirming that the density of states is real.

[2]In this context, the causal Green function is the Green function for the time-independent Schrödinger equation defined by the Fourier transform of the retarded Green function for the time-dependent Schrödinger equation.

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