Quasilocal energy and conserved charges derived from the gravitational action

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The quasilocal energy of gravitational and matter fields in a spatially bounded region is obtained by employing a Hamilton-Jacobi analysis of the action functional. First, a surface stress-energy-momentum tensor is defined by the functional derivative of the action with respect to the three-metric on ^3B, the history of the system's boundary. Energy surface density, momentum surface density, and spatial stress are defined by projecting the surface stress tensor normally and tangentially to a family of spacelike two-surfaces that foliate ^3B. The integral of the energy surface density over such a two-surface B is the quasilocal energy associated with a spacelike three-surface Σ whose orthogonal intersection with ^3B is the boundary B. The resulting expression for quasilocal energy is given in terms of the total mean curvature of the spatial boundary B as a surface embedded in Σ. The quasilocal energy is also the value of the Hamiltonian that generates unit magnitude proper-time translations on ^3B in the timelike direction orthogonal to B. Conserved charges such as angular momentum are defined using the surface stress tensor and Killing vector fields on ^3B. For spacetimes that are asymptotically flat in spacelike directions, the quasilocal energy and angular momentum defined here agree with the results of Arnowitt, Deser, and Misner in the limit that the boundary tends to spatial infinity. For spherically symmetric spacetimes, it is shown that the quasilocal energy has the correct Newtonian limit, and includes a negative contribution due to gravitational binding.

I. INTRODUCTION

Considerable effort has been expended in attempts to define quasilocal energy in general relativity [1]. Some earlier efforts made use of pseudotensor methods, which led to coordinate-dependent expressions whose geometric meanings were not clear. Other approaches were based on the identification of certain symmetries of the gravitational action and the construction of associated Noether charges. Some of the more recent efforts have focused on constructing from the gravitational Cauchy data mathematical expressions that exhibit certain physical properties commonly associated with energy. Although these approaches have led to some interesting mathematical results, no definitive expression for quasilocal energy has emerged. In this paper we address the problem of quasilocal energy from a somewhat different perspective [2]. The action principle for gravity and matter is allowed to dictate the definition of quasilocal energy and its resulting properties through a Hamilton-Jacobi-type analysis. Because of its intimate connection to the action and the Hamiltonian, we believe our quasilocal energy is a natural choice. Furthermore, this quasilocal energy has arisen directly in the study of thermodynamics for self-gravitating systems, where it plays the role of the thermodynamic internal energy that is conjugate to inverse temperature [3].

The basic idea for our definition of quasilocal energy is

best presented by considering first an analogy. In nonrelativistic mechanics, the time interval T between initial and final configurations enters the action as fixed endpoint data. The classical action S_{cl}, the action functional evaluated on a history that solves the classical equations of motion, is an ordinary function of the time interval and is identified as Hamilton's principal function [4]. Therefore S_{cl} satisfies the Hamilton-Jacobi equation $H = -\partial S_{cl}/\partial T$, which expresses the energy (Hamiltonian) H of the classical solution as minus the time rate of change of its action. By a similar analysis, we shall define the quasilocal energy for gravitational and matter fields in a spatially bounded region as minus the time rate of change of the classical action.

We will deal with the physics of a spacetime region M that is topologically the product of a three-space Σ and a real line interval. The symbol Σ will be used informally to denote either a family of spacelike slices that foliates M or a particular leaf of the foliation, depending on the context in which it is used. The boundary of Σ is B, which need not be simply connected. The product of B with segments of timelike world lines orthogonal to Σ at B is denoted as ^3B, an element of the three-boundary of M. The end points of the world lines define three-boundary elements denoted t' and t''. The situation is depicted in Fig. 1 in which B is chosen to have the topology of a single two-sphere; because one spatial dimension is suppressed, B is drawn as a closed curve. Often we will refer to ^3B as a three-boundary, although the complete three-boundary of M actually consists of the sum of ^3B, t', and t''.

Consider the usual action functional S^1 for gravity and

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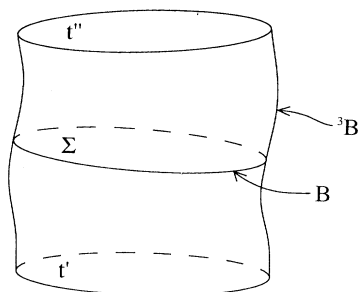
FIG. 1. Spacetime M with boundary ∂M, which consists of initial and final spacelike hypersurfaces t' and t'' and a timelike three-surface ^3B. A generic spacelike slice Σ has two-boundary B.

matter in which the three-metric is fixed on ^3B. This fixed three-metric plays a role analogous to the fixed time interval in nonrelativistic mechanics in that it determines the separation in time between initial and final configurations. The action evaluated on a classical solution is a functional of the boundary three-metric, and in particular it is a function of the proper time separation between a typical spacelike slice B and its neighboring slice within the three-boundary. The quasilocal energy associated with the spacelike hypersurface Σ is defined as minus the variation in the action with respect to a unit increase in proper time separation between B and its neighboring two-surface, as measured orthogonally to Σ at B. (See Fig. 2.)

The quasilocal energy defined in this way equals the value of the Hamiltonian that generates unit time translations orthogonal to Σ at the boundary two-surface B. This particular property might well be taken to be the definition of quasilocal energy. The advantage in following the procedure outlined above comes from considering changes in the classical action due to arbitrary variations in the boundary three-metric. This more general approach leads to a tensor that characterizes the stress-energy-momentum content of the bounded spacetime region. This stress tensor is a surface tensor defined locally on the three-boundary ^3B, and it has a simple relation to the gravitational momentum that is canonically conju-

gate to the boundary metric. Proper energy surface density is defined by the projection of the stress tensor normal to the spacelike surface B, while proper momentum surface density and spatial stress are (respectively) the normal-tangential and tangential-tangential projections of the stress tensor. The total quasilocal energy is then simply the integral over B of the energy surface density. Also, conserved charges are defined using the surface stress tensor and Killing vector fields (if any) on the boundary ^3B. In particular, angular momentum is the conserved charge associated with a rotational Killing vector, and is equal to the value of the Hamiltonian that generates spatial diffeomorphisms along that Killing vector field.

The surface stress tensor is not uniquely defined to the extent that the action S^1 for general relativity and matter is itself ambiguous—arbitrary functionals S^0 of the fixed boundary three-metric can always be subtracted from the action without affecting the equations of motion. This freedom is sometimes removed by imposing the condition that the action should vanish for flat empty spacetime [5]. However, such a criterion cannot be implemented except with special choices of boundary data for which the boundary three-metric can be embedded in flat spacetime. Because the embedding of a typical three-metric in a flat spacetime is not possible, this criterion is not generally applicable [2]. Instead, we partially remove the ambiguity by insisting that the energy and momentum surface densities, and therefore the quasilocal energy and angular momentum, depend only on the canonical variables defined on Σ. This restriction implies that (in particular) the quasilocal energy is a function on phase space.

The freedom that remains in the definition of quasilocal energy is just the freedom to choose the "zero" of energy. It is sometimes possible to choose the functional S^0 such that the quasilocal energy vanishes for a flat slice of flat spacetime. This can be achieved at least whenever the two-boundary B has two-sphere topology and positive curvature. With such a choice for S^0, the quasilocal energy in the limit that the boundary B is at spatial infinity agrees with the Arnowitt-Deser-Misner [6] energy at infinity for asymptotically flat spacetimes [7]. It should be emphasized that such a choice for S^0 is not from our point of view essential. For example, the functional $S^0 = 0$ yields a definition of quasilocal energy that is both practical and completely general. Ultimately, the choice of a functional S^0 may be physically irrelevant. In particular, the thermodynamical description of a self-gravitating system [3] is altered only superficially by different choices for S^0 [8].

For any allowed choice of the functional S^0, our quasilocal energy turns out to be determined from the two-boundary geometry and the total mean curvature of B, that is, the integral of the trace of the extrinsic curvature of B as embedded in Σ.[1] The quasilocal energy varies
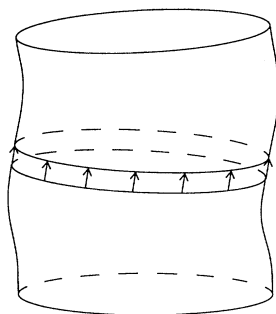

FIG. 2. Compare with Fig. 1. Here we depict a unit magnitude "stretch" of the boundary three-metric, normal to the boundary slice B. The change in the classical action induced by such a variation is identified using the Hamilton-Jacobi equation as minus the quasilocal energy.

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[1]Such an expression for quasilocal energy also has been obtained by Polishchuk (see Ref. [1]).

among different two-surfaces whose timelike unit normals are contained in a common three-boundary ^3B. Thus, the quasilocal energy is not necessarily conserved. On the other hand, conserved charges defined from the surface stress tensor and Killing vector fields on ^3B are independent of the slice B within the three-boundary ^3B that is used for their evaluation.

In this paper, the Hamilton-Jacobi-type analysis leading to the stress-energy-momentum tensor, quasilocal energy, and conserved charges is explicitly carried out for general relativity and matter, with the restriction that the matter should be nonderivatively coupled to the gravitational field. The method we use can be applied to any generally covariant action that describes spacetime geometry and matter. The sign conventions of Misner, Thorne, and Wheeler [9] are used throughout, and κ denotes 8π times Newton's constant. In Sec. II, we present some notation and a preliminary discussion of the kinematical relationships needed for describing general relativity in the presence of spatial boundaries. Section III contains the analysis leading to the stress-energy-momentum tensor. There it is shown that when the bounded spacetime region is the history of a thin surface layer, the stress tensor yields the Lanczos-Israel tensor [10] describing the stress-energy-momentum content of a thin surface layer. In Sec. IV, the energy surface density, momentum surface density, spatial stress, and total quasilocal energy are defined. Also, the Hamiltonian describing general relativity on a manifold with boundary is derived. Conserved charges are defined in Sec. V, with special attention given to the description of angular momentum. When the boundary B is at spatial infinity, the angular momentum agrees with the definition of Arnowitt, Deser, and Misner (ADM) [6]. Section VI is devoted to an exploration of various properties of the quasilocal energy, and includes explicit calculations for spherically symmetric fluid stars and black holes. It is shown that the quasilocal energy of a spherical star agrees in the Newtonian limit with the energy deduced from Newtonian gravity. In addition, the first law of black-hole mechanics (thermodynamics) for Schwarzschild black holes is obtained directly by varying the quasilocal energy. Some of the mathematical details of our analysis are collected in the Appendix.

II. PRELIMINARIES

Our notation is summarized in Table I. The spacetime metric is $g_{\mu\nu}$, and n^μ is the outward pointing spacelike unit normal to the three-boundary ^3B. The metric and extrinsic curvature of ^3B are denoted by $\gamma_{\mu\nu}$ and $\Theta_{\mu\nu}$, respectively. These spacetime tensors are defined on ^3B only, and satisfy $n^\mu\gamma_{\mu\nu}=0$ and $n^\mu\Theta_{\mu\nu}=0$. In addition, γ^μ_ν serves as the projection tensor onto ^3B. $\gamma_{\mu\nu}$ and $\Theta_{\mu\nu}$ can be viewed alternatively as tensors on ^3B, denoted by γ_{ij} and Θ_{ij}, where the indices i,j refer to coordinates on ^3B. The boundary momentum is π^{ij}, and is conjugate to γ_{ij} where canonical conjugacy is defined with respect to the boundary ^3B (see the Appendix).

Let u^μ denote the future pointing timelike unit normal to a family of spacelike hypersurfaces Σ that foliate spacetime. The metric and extrinsic curvature for Σ are given by the spacetime tensors $h_{\mu\nu}$ and $K_{\mu\nu}$, respectively, and h^μ_ν is the projection tensor onto Σ. These tensors also can be viewed as (time-dependent) tensors on Σ, denoted by h_{ij}, K_{ij}, and $h^i_j=\delta^i_j$. The momentum canonically conjugate to the spatial metric h_{ij} is denoted by P^{ij}. Also, the spacetime metric can be written according to the usual ADM decomposition [6]:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu$$
$$=-N^2dt^2+h_{ij}(dx^i+V^idt)(dx^j+V^jdt), \quad (2.1)$$

where N is the lapse function and V^i is the shift vector.

Observe that lower case latin letters such as i,j,k,l refer both to coordinates on ^3B and to coordinates on space Σ. When used as tensor indices, the meaning of these latin letters is usually clear from the nature of the tensor. On occasions in which these two index types must be distinguished, we will underline the indices corresponding to coordinates on Σ; for example, $h_{\underline{kl}}$.

Throughout the analysis we assume that the hypersurface foliation Σ is "orthogonal" to ^3B, meaning that on the boundary ^3B the hypersurface normal u^μ and the three-boundary normal n^μ satisfy $(u\cdot n)|_{^3B}=0$. Thus, n^μ also can be viewed as a vector n^i in Σ with unit length in the hypersurface metric: $n^ih_{ij}n^j|_{^3B}=1$. So the unit normal in spacetime to the three-boundary ^3B is also the unit

TABLE I. A summary of notation. Some spaces are left blank, either because they are not applicable, or because they are not needed. The symbol $\mathcal{R}$ is used for the Riemann tensor on M and is not a tensor density. The unit normal for ^3B embedded in M is also the unit normal for B embedded in Σ by virtue of the condition $(u\cdot n)=0$ on ^3B.

	Metric	Covariant derivative	Unit normal	Intrinsic curvature	Extrinsic curvature	Momentum
Spacetime M	$g_{\mu\nu}$	∇_μ		$\mathcal{R}_{\mu\nu\sigma\rho}$		
Hypersurfaces Σ embedded in M	h_{ij}	D_i	u_μ	R_{ijkl}	K_{ij}	P^{ij}
Three-boundary ^3B embedded in M	γ_{ij}	$\mathcal{D}_i$	n_μ		Θ_{ij}	π^{ij}
Two-boundary B embedded in Σ	σ_{ab}		n_i		k_{ab}	

normal in Σ to the two-boundary B. This restriction simplifies enormously the technical details of our analysis, and has the following logical basis as well. In the canonical formalism, the boundary B is specified as a fixed surface in Σ. The Hamiltonian must evolve the system in a manner consistent with the presence of this boundary, and cannot generate transformations that map the canonical variables across B. This means that the component of the shift vector normal to the boundary must be restricted to vanish, $V^i n_i|_B = 0$. From a space-time point of view, this is the condition that the two-boundary evolves into a three-surface that contains the unit normal u^μ to the hypersurfaces Σ. Therefore, u^μ and n^μ are orthogonal on ^3B.

Because of the restriction $(u \cdot n)|_{3_B} = 0$, the metric on ^3B can be decomposed as

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt),$$
$$(2.2)$$

where x^a, $a = 1,2$, are coordinates on B and σ_{ab} is the two-metric on B. The extrinsic curvature of B as a surface embedded in Σ is denoted by k_{ab}. These tensors can be viewed as spacetime tensors $\sigma_{\mu\nu}$ and $k_{\mu\nu}$, or as tensors on Σ or ^3B by using indices i,j,k,l. Also, σ^μ_ν is the projection tensor onto B.

III. STRESS-ENERGY-MOMENTUM TENSOR

Hamilton-Jacobi theory provides a formal basis for identifying the stress-energy-momentum as dictated by the action. Before presenting this analysis, it will be useful to provide a quick review of Hamilton-Jacobi theory as applied to nonrelativistic mechanics. Start with the action in canonical form:

$$S^1 = \int dt \left[p\frac{dx}{dt} - H^1(x,p,t) \right].$$
$$(3.1)$$

Now parametrize the system by introducing a coordinate λ for the system path in state space (phase space and time). The action becomes

$$S^1 = \int_{\lambda'}^{\lambda''} d\lambda[p\dot{x} - iH^1(x,p,t)],$$
$$(3.2)$$

where the dot denotes a derivative with respect to λ. Varying this action gives

$$\delta S^1 = \text{(terms giving the equations of motion)}$$
$$+ p\delta x |_{\lambda'}^{\lambda''} - H^1 \delta t |_{\lambda'}^{\lambda''}.$$
$$(3.3)$$

Observe that by fixing x and t at the end points λ' and λ'', the end-point (boundary) terms vanish in δS^1. With these boundary conditions, solutions to the equations of motion extremize S^1.

Now restrict the variations in the action to variations among classical solutions. In this case, the terms in Eq. (3.3) giving the equations of motion vanish, leaving

$$\delta S^1_{cl} = p_{cl}\delta x |_{\lambda'}^{\lambda''} - H^1_{cl}\delta t |_{\lambda'}^{\lambda''},$$
$$(3.4)$$

where "cl" denotes evaluation at a classical solution. The Hamilton-Jacobi equations follow from this expression:

the classical momentum and energy at the final boundary λ'' are

$$p_{cl}|_{\lambda''} = \frac{\partial S^1_{cl}}{\partial x''},$$
$$(3.5a)$$

$$H^1_{cl}|_{\lambda''} = -\frac{\partial S^1_{cl}}{\partial t''}.$$
$$(3.5b)$$

This latter equation says that for a classical history, the energy (Hamiltonian) at the boundary λ'' is minus the change in the classical action due to a unit increase in the final time $t(\lambda'') = t''$. [Similarly, variation of the initial boundary time $t(\lambda') = t'$ leads to the energy at λ', but with no minus sign because positive changes in t' decrease, rather than increase, the time interval.]

The action for any system is ambiguous in the sense that arbitrary functions of the fixed boundary data can be added to the action without changing the resulting equations of motion. For example, for nonrelativistic mechanics introduce a subtraction term

$$S^0 = \int_{\lambda'}^{\lambda''} d\lambda \frac{dh(t)}{d\lambda} = h(t'') - h(t'),$$
$$(3.6)$$

where h is an arbitrary function of t. The full action is now defined by $S = S^1 - S^0$. The subtraction just shifts the value of S^1 by a (boundary-data-dependent) constant, and S has the standard canonical form with Hamiltonian $H = H^1 + dh/dt$. The variation δS is just as in Eq. (3.3) but with H^1 replaced by H. The Hamilton-Jacobi equation for the energy at λ'' becomes

$$H_{cl}|_{\lambda''} = -\frac{\partial S_{cl}}{\partial t''},$$
$$(3.7)$$

so that different subtraction terms lead to different values of energy. If a particular physical system allows for a subtraction S^0 that gives a t-independent Hamiltonian, such a choice is usually preferred.

For general relativity coupled to matter, consider first the action suitable for fixation of the metric on the boundary [11]:

$$S^1 = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g}\, \mathcal{R} + \frac{1}{\kappa} \int_{t'}^{t''} d^3x \sqrt{h}\, K$$
$$- \frac{1}{\kappa} \int_{3_B} d^3x \sqrt{-\gamma}\, \Theta + S^m,$$
$$(3.8)$$

where S^m is the matter action, including a possible cosmological constant term. S^1 is a functional of the four-metric $g_{\mu\nu}$ and matter fields on M. The notation $\int_{t'}^{t''} d^3x$ represents an integral over the three-boundary t'' minus an integral over the three-boundary t'. The variation in S^1 due to arbitrary variations in the metric and matter fields is

$$\delta S^1 = \text{(terms giving the equations of motion)}$$
$$+ \text{(boundary terms coming}$$
$$\text{from the matter action)}$$
$$+ \int_{t'}^{t''} d^3x\, P^{ij}\delta h_{ij} + \int_{3_B} d^3x\, \pi^{ij}\delta\gamma_{ij}.$$
$$(3.9)$$

Here, P^{ij} denotes the gravitational momentum conjugate

to h_{ij}, as defined with respect to the spacelike hypersurfaces t' and t'', while π^{ij} is the gravitational momentum conjugate to γ_{ij}, defined with respect to the three-boundary ^3B. We assume that the matter fields are minimally coupled to gravity, so the matter action contains no derivatives of the metric. In this case, the gravitational momenta have the same form [Eqs. (A5) and (A8) of the Appendix] as in vacuum general relativity.

The gravitational and matter fields must be restricted by appropriate boundary conditions, so that the boundary terms in δS^1 vanish. This is required for the action to have well-defined functional derivatives that yield the classical equations of motion, and in turn implies that the action functional is extremized by solutions to those equations of motion. A natural set of boundary conditions consists in fixing on the boundaries the fields whose variations appear in the boundary terms of δS^1, so that the variations of those fields indeed vanish. We will adopt such boundary conditions. For the gravitational variables in particular, the boundary three-metric γ_{ij} is fixed on ^3B, and the hypersurface metric h_{ij} is fixed on t' and t''. (One alternative to S^1 is the action that differs from S^1 by the exclusion of the boundary term involving K. In that case, the term $P^{ij}\delta h_{ij}$ in δS^1 is replaced by $-h_{ij}\delta P^{ij}$ and the natural boundary conditions include fixed P^{ij} at t' and t''. Such a change does not affect the definition of the stress-energy-momentum tensor.)

The ambiguity in S^1 is taken into account by subtracting an arbitrary function of the fixed boundary data. Thus, define the action

$$S = S^1 - S^0, \tag{3.10}$$

where S^0 is a functional of γ_{ij}. Of course, S^0 can depend on the initial and final metrics $h'_{ij} = h_{ij}(t')$ and $h''_{ij} = h_{ij}(t'')$ as well, but for present purposes we find no advantage in allowing for such generality. The variation in S just differs from the result in Eq. (3.9) by the term

$$-\delta S^0 = -\int_{^3B} d^3x \frac{\delta S^0}{\delta \gamma_{ij}} \delta \gamma_{ij} \equiv -\int_{^3B} d^3x\, \pi_0^{ij} \delta \gamma_{ij}, \tag{3.11}$$

where π_0^{ij} is defined as the functional derivative of S^0. Therefore π_0^{ij} is a function of the metric γ_{ij} only.

The classical action S_{cl}, the action S evaluated at a classical solution, is a functional of the fixed boundary data consisting of γ_{ij}, h'_{ij}, h''_{ij}, and matter fields. The dependence of S_{cl} on this boundary data is obtained by restricting the general variation (3.9)–(3.11) to variations among classical solutions, which gives

$$\delta S_{cl} = (\text{terms involving variations in the matter fields})$$
$$+ \int_{t'}^{t''} d^3x\, P_{cl}^{ij}\delta h_{ij} + \int_{^3B} d^3x (\pi_{cl}^{ij} - \pi_0^{ij})\delta \gamma_{ij}. \tag{3.12}$$

The analogues of the Hamilton-Jacobi equation (3.5a) are the relationships

$$P_{cl}^{ij}\big|_{t''} = \frac{\delta S_{cl}}{\delta h''_{ij}} \tag{3.13}$$

for the gravitational momentum at the boundary t'', and corresponding relationships for the matter variables at

t''. (The notation is slightly awkward: for nonrelativistic mechanics, λ is a coordinate while t and x are dynamical variables; for gravity, t and x are coordinates.)

The analogue of the Hamilton-Jacobi equation (3.7) is more subtle. In the gravitational action, the three-metric components γ_{ij} are the fixed boundary data that determine the time between spacelike hypersurfaces. Then the analogue of the boundary data t'' from the example of nonrelativistic mechanics is *included* in γ_{ij}; but, of course, the boundary metric provides more than just information about time. It gives the metrical distance for all spacetime intervals in the boundary manifold ^3B. Correspondingly, the simple notion of energy in nonrelativistic mechanics becomes generalized to a surface stress-energy-momentum tensor for spacetime and matter, defined by

$$\tau^{ij} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \gamma_{ij}}. \tag{3.14}$$

The functional derivative of S_{cl} is determined from the variation Eq. (3.12) to be

$$\frac{\delta S_{cl}}{\delta \gamma_{ij}} = \pi_{cl}^{ij} - \pi_0^{ij}, \tag{3.15}$$

so the stress tensor becomes

$$\tau^{ij} = \frac{2}{\sqrt{-\gamma}} (\pi_{cl}^{ij} - \pi_0^{ij}). \tag{3.16}$$

It is interesting to note the similarity between definition (3.14) and the standard definition

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S^m}{\delta g_{\mu\nu}} \tag{3.17}$$

for the matter stress tensor $T^{\mu\nu}$. It should be emphasized that τ^{ij} *characterizes the entire system, including contributions from both the gravitational field and the matter fields.*

The Hamilton-Jacobi equation (3.15) and the corresponding matter equations relate the coordinates and momenta of the gravitational and matter fields as defined at the boundary ^3B. These boundary variables satisfy the constraints of general relativity, as well as gauge constraints associated with invariances of the matter action. In particular, the boundary momentum constraint reads

$$0 = -2\mathcal{D}_i \pi^{ij} - \sqrt{-\gamma}\, T^{nj}, \tag{3.18}$$

which is equivalent to the Einstein equation with one index projected normally to ^3B and the other index projected tangentially to ^3B. Here, $T^{nj} \equiv T^{\mu\nu}n_\mu \gamma_\nu^j$ is the matter stress tensor (3.17), with indices projected normally and tangentially to ^3B. The momentum constraint (3.18) implies that S_{cl} depends on the boundary data only to within diffeomorphism of ^3B. It also implies the relationship

$$\mathcal{D}_i \tau^{ij} = -T^{nj} \tag{3.19}$$

for the surface stress tensor. This result has a form similar to the equation of motion $\nabla_\mu T^{\mu\nu} = 0$ for the matter stress tensor, the key difference being the appearance of a

source term $-T^{nj}$ for the divergence of τ^{ij}. The consequences of Eq. (3.19) are explored in Sec. V, where conserved charges are defined.

It is useful to keep in mind that the boundary B need not be simply connected. An interesting application arises when, for example, B consists of two concentric, topologically spherical surfaces, B_1 and B_2. There are stress tensors associated with each connected part of the boundary: τ_1^{ij} and τ_2^{ij}. Consider the limit in which B_1 and B_2 coincide, so that the three-geometries on the histories of B_1 and B_2 are identical. The total stress-energy-momentum τ_{SL}^{ij} of the surface layer is just the sum of τ_1^{ij} and τ_2^{ij}, which can be written as the difference $\tau_{SL}^{ij} = \tau_2^{ij} - \tau_1^{ij}$ with the understanding that τ_1^{ij} is now computed using the outward normal to B_2. Inserting expression (3.16) for the stress tensors, the terms involving π_0^{ij} cancel and the surface layer stress tensor is given by

$$\tau_{SL}^{ij} = \frac{2}{\sqrt{-\gamma}} (\pi_2^{ij} - \pi_1^{ij})|_{cl}. \tag{3.20}$$

This equation embodies the results of Lanczos and Israel [10] on junction conditions in general relativity, which relate the jump in momentum π^{ij} to the *matter* stress-energy-momentum tensor of the surface layer. Equation (3.20) actually shows that the jump in momentum gives the *total* stress-energy-momentum for the layer. Evidently, with the assumption that the geometries on each side of the infinitesimally thin surface layer coincide, the gravitational contribution to τ_{SL}^{ij} vanishes, and τ_{SL}^{ij} equals the matter stress tensor. Physically, this result reflects the well-known absence of a local gravitational energy-momentum [9].

IV. ENERGY SURFACE DENSITY, MOMENTUM SURFACE DENSITY, AND SPATIAL STRESS

From the stress-energy-momentum tensor, the proper energy surface density ε, proper momentum surface density j_a, and spatial stress s^{ab} are defined by the normal and tangential projections of τ^{ij} on a two-surface B:

$$\varepsilon \equiv u_i u_j \tau^{ij} = -\frac{1}{\sqrt{\sigma}} \frac{\delta S_{cl}}{\delta N}, \tag{4.1a}$$

$$j_a \equiv -\sigma_{ai} u_j \tau^{ij} = \frac{1}{\sqrt{\sigma}} \frac{\delta S_{cl}}{\delta V^a}, \tag{4.1b}$$

$$s^{ab} \equiv \sigma_i^a \sigma_j^b \tau^{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \sigma_{ab}}. \tag{4.1c}$$

The second equalities in Eqs. (4.1a)–(4.1c) follow from definition (3.14) for τ^{ij} and the relationships

$$\partial \gamma_{ij} / \partial N = -2 u_i u_j / N, \tag{4.2a}$$

$$\partial \gamma_{ij} / \partial V^a = -2 \sigma_{a(i} u_{j)} / N, \tag{4.2b}$$

$$\partial \gamma_{ij} / \partial \sigma_{ab} = \sigma_{(i}^a \sigma_{j)}^b. \tag{4.2c}$$

The quantities ε, j_a, and s^{ab} are tensors defined on a generic two-surface B. They represent the energy surface density, momentum surface density, and spatial stress associated with matter and gravitational fields on the space-

like hypersurface Σ with boundary B. In particular, the total quasilocal energy for Σ is given by

$$E = \int_B d^2 x \sqrt{\sigma} \varepsilon = -\int_B d^2 x \frac{\delta S_{cl}}{\delta N}. \tag{4.3}$$

This expression is the closest analogue of the Hamilton-Jacobi equation (3.7), which gives the energy in nonrelativistic mechanics as minus the change in the classical action due to a unit change in the boundary time t''. Here, the energy of Σ is written as minus the change in S_{cl} due to a uniform, unit increase in the proper time between the boundary surface B and its neighboring two-surface in ${}^3B$, as measured normally to B. (See Fig. 2.) The change in the classical action due to such a uniform variation is expressed in Eq. (4.3) as the integral over B of the local variation $\delta S_{cl} / \delta N$.

From the form (3.16) for the stress tensor, the energy surface density, momentum surface density, and spatial stress are each seen to consist of two terms. The first term is proportional to projections of the classical gravitational momentum π_{cl}^{ij}, and the second term is proportional to projections of $\pi_0^{ij} \equiv \delta S^0 / \delta \gamma_{ij}$. The projections of π_{cl}^{ij} can be written in terms of the canonical variables h_{ij}, P^{ij}, lapse N, and shift V^i by making use of Eq. (A18) of the Appendix, while the projections of π_0^{ij} can be written as functional derivatives of S^0 by invoking the relationships (4.2). Using these results, the proper energy surface density (4.1a), momentum surface density (4.1b), and spatial stress (4.1c) become

$$\varepsilon = \frac{1}{\kappa} k|_{cl} + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N}, \tag{4.4a}$$

$$j_a = -2(\sigma_{ak} n_l P^{kl} / \sqrt{h})|_{cl} - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V^a}, \tag{4.4b}$$

$$s^{ab} = \frac{1}{\kappa} [k^{ab} + (n \cdot a - k) \sigma^{ab}]|_{cl} - \frac{2}{\sqrt{-\gamma}} \frac{\delta S^0}{\delta \sigma_{ab}}. \tag{4.4c}$$

The momentum surface density in Eq. (4.4b) can be viewed as a vector in ${}^3B$ by changing the a indices to i, or as a vector in Σ by changing the a indices to i.

Thus far, the subtraction term S^0 has been treated as an unspecified functional of the fixed boundary data γ_{ij}, which arises from an inherent ambiguity in the action. We now restrict the form of S^0 by demanding that *the energy surface density ε and momentum surface density j_a of a particular spacelike hypersurface Σ should depend only on the canonical variables h_{ij}, P^{ij} defined on Σ*. This requirement implies that ε and j_a are functions on phase space. Observe that no such restriction is placed on the spatial stress: s^{ab} is interpreted as the flux of the a component of momentum in the b direction, so s^{ab} should depend on the way the canonical data evolve in time. This dependence is already clear from the presence of the acceleration a of the timelike unit normal in the expression (4.4c) for the spatial stress. On the other hand, the first terms in Eqs. (4.4a) and (4.4b), those that do not involve S^0, are functions only of the canonical variables.

An obvious choice for a subtraction term S^0 that satisfies the above criterion is simply $S^0 = 0$. More generally, the complete expressions for energy surface densi-

ty and momentum surface density will be functions only of the canonical variables if S^0 is a linear functional of the lapse N and shift V^a on the boundary. With such a choice for S^0, the functional derivatives appearing in Eqs. (4.4a) and (4.4b) are functions only of the two-boundary metric σ_{ab}, which is the projection of the hypersurface metric h_{ij} onto the boundary B.

Restricted to a linear functional of the lapse and shift, S^0 can be written as

$$S^0 = -\int_{3_B} d^3x \, [N\sqrt{\sigma}(k/\kappa)|_0$$
$$+ 2\sqrt{\sigma} V^a(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0]. \quad (4.5)$$

Here, $k|_0$ and $(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0$ are arbitrary functions of the two-metric σ_{ab}. As suggested by their notation, one method of specifying these functions is to choose a reference space, that is, a fixed spacelike slice of some fixed spacetime, and then consider a surface in the slice whose induced two-metric is σ_{ab}. If such a two-surface exists, it can be used to evaluate k and $\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h}$, yielding the desired functions of σ_{ab}. With the subtraction term (4.5), the energy surface density and momentum surface density become

$$\varepsilon = (k/\kappa)|_0^{cl}, \quad (4.6a)$$

$$j_a = -2(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0^{cl}, \quad (4.6b)$$

where $|_0^{cl}$ denotes evaluation for the classical solution minus evaluation for the chosen reference space. Note that "evaluation for the classical solution" is actually evaluation for a particular spacelike hypersurface in the spacetime, that is, evaluation at a particular point in phase space. By definition, ε and j_a vanish for the reference space, so the freedom to choose different reference spaces is just the freedom to choose the zero of energy and momentum for the system. Observe that Eq. (4.6b) can also be written as a tensor equation in Σ:

$$j^i = -2(\sigma^i_k n_l P^{kl}/\sqrt{h})|_0^{cl}, \quad (4.7)$$

where we have returned to the practice of omitting the underbars on indices for tensors in Σ.

The construction of $k|_0$ and $(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0$ described above is sensible only if the two-metric σ_{ab} indeed can be embedded in the reference space, and if the embedding is unique. As a concrete example, choose a flat three-dimensional slice of flat spacetime as the reference space. In this case, there are a considerable number of existence and uniqueness results concerning the embedding of a surface in R^3. For example, it is known that any Riemannian manifold with two-sphere topology and everywhere positive curvature can be globally immersed in R^3 [12]. (An immersion differs from an embedding by allowing for self-intersection of the surface.) The Cohn-Vossen theorem [12] states that any compact surface contained in R^3 whose curvature is everywhere nonnegative is unwarpable. (Unwarpable means the surface is uniquely determined by its two-metric, up to translations or rotations in R^3.) From these results it follows that the functions $k|_0$ and $(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0$ are uniquely determined by the flat reference space, at least for all pos-

itive curvature two-metrics with two-sphere topology. Note in particular that for a flat slice of flat spacetime, $(\sigma_{a\underline{k}}n_{\underline{l}}P^{\underline{kl}}/\sqrt{h})|_0$ vanishes since $P^{\underline{kl}}$ is identically zero.

Henceforth, we shall adopt the flat reference space subtraction term, assuming its existence and uniqueness for the two-metrics of interest. The quasilocal energy (4.3) then becomes

$$E = \frac{1}{\kappa} \int_B d^2x \sqrt{\sigma}(k - k|_0), \quad (4.8)$$

which is $(1/\kappa$ times) the total mean curvature of B as embedded in Σ, minus the total mean curvature of B as embedded in flat space. In this equation, the superfluous "cl" has been dropped. For spacetimes that are asymptotically flat in spacelike directions, the energy (4.8) with B at spatial infinity agrees with the ADM energy [6]. In the more usual expression for the ADM energy [6], the flat space subtraction is also present, but it is hidden in the use of ordinary derivatives acting on the metric tensor components in the asymptotically flat space with Cartesian coordinates.

Now consider the action (3.10) written in canonical form. For simplicity, we will omit the matter field (and cosmological constant) contribution S^m, although its inclusion is straightforward. Using the space-time split (A20) of the curvature $\mathcal{R}$ and the trace Θ of the three-boundary extrinsic curvature from Eq. (A17), the action becomes

$$S = \frac{1}{2\kappa} \int_M d^4x \, N\sqrt{h} \, [R + K_{\mu\nu}K^{\mu\nu} - (K)^2]$$
$$- \frac{1}{\kappa} \int_{3_B} d^3x \, N\sqrt{\sigma}k - S^0. \quad (4.9)$$

Next, use expressions (A4) and (A5) for the hypersurface extrinsic curvature and momentum to obtain

$$N\sqrt{h} \, [K_{\mu\nu}K^{\mu\nu} - (K)^2] = 2\kappa[P^{ij}\dot{h}_{ij} - 2P^{ij}D_i V_j$$
$$- 2\kappa N G_{ijkl}P^{ij}P^{kl}], \quad (4.10)$$

where $G_{ijkl} = (h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})/(2\sqrt{h})$ is the inverse superspace metric. Inserting this result into the action (4.9) and using the explicit form (4.5) for S^0 gives the action in canonical form:

$$S = \int_M d^4x \, [P^{ij}\dot{h}_{ij} - N\mathcal{H} - V^i\mathcal{H}_i]$$
$$- \int_{3_B} d^3x \sqrt{\sigma}[N\varepsilon - V^i j_i], \quad (4.11)$$

where the gravitational contributions to the Hamiltonian and momentum constraints are

$$\mathcal{H} = (2\kappa)G_{ijkl}P^{ij}P^{kl} - \sqrt{h}R/(2\kappa), \quad (4.12a)$$

$$\mathcal{H}_i = -2D_j P_i^j. \quad (4.12b)$$

In Eq. (4.11), ε and j_i denote the energy surface density (4.6a) and momentum surface density (4.7), but with "cl" omitted in favor of evaluation at the generic phase space point h_{ij}, P^{ij}. From the action (4.11), the Hamiltonian is explicitly determined to be

$$H = \int_\Sigma d^3x (N\mathcal{H} + V^i\mathcal{H}_i) + \int_B d^2x \sqrt{\sigma}(N\varepsilon - V^i j_i).$$

(4.13)

The quasilocal energy is seen to equal the value of the Hamiltonian that generates unit time translations orthogonal to the boundary B, that is, the value of H with $N = 1$ and $V^i = 0$ on the boundary.

V. CHARGE

As mentioned in Sec. III, the stress-energy-momentum tensor describing a solution to the equations of motion for gravity and matter will satisfy the relationship

$$\mathcal{D}_i \tau^{ij} = -T^{nj},$$

(5.1)

where $T^{nj} \equiv T^{\mu\nu} n_\mu \gamma^j_\nu$. This expression is similar to the familiar equation of motion for the matter stress tensor, namely $\nabla_\mu T^{\mu\nu} = 0$, and has a similar interpretation as an approximate local conservation law [9]. The primary difference is that the divergence of τ^{ij} in Eq. (5.1) contains a source term, $-T^{nj}$. To interpret Eq. (5.1) as a conservation law, consider a sufficiently small "box" ΔB contained in B, over a sufficiently short time Δt, such that the timelike unit normal u^i is approximately constant, $\mathcal{D}_i u_j \approx 0$. Contracting Eq. (5.1) with u^i and integrating over the spacetime region $\Delta B \Delta t$ gives the approximate conservation law for the energy-momentum current density $-u_j \tau^{ij}$, whose components are the proper energy surface density ε and proper momentum surface density j_a. This conservation law states that the increase in time in the total energy-momentum contained in ΔB equals the net energy-momentum that enters ΔB from within ^3B, plus a contribution from the source $-T^{nj}$. That contribution is the *matter* energy-momentum $-u_j T^{nj}$ that passes through $\Delta B \Delta t$ as it flows across the boundary ^3B into M.

The conservation law described above is approximate because typically u^i is not a Killing vector field on ^3B, so $\mathcal{D}_{(i} u_{j)}$ is not zero. If the boundary three-metric γ_{ij} does possess an isometry, then global conserved charges can be defined as follows. Let ξ^i denote a Killing vector field, $\mathcal{D}_{(i} \xi_{j)} = 0$, associated with an isometry of the boundary three-metric. Contract expression (5.1) with ξ^i and integrate over ^3B to obtain

$$-\int_{t'\cap^3B}^{t''\cap^3B} d^2x \sqrt{\sigma}(u_i \tau^{ij}\xi_j) = -\int_{^3B} d^3x \sqrt{-\gamma} T^{ni}\xi_i. \quad (5.2)$$

This equation naturally motivates the definition

$$Q_\xi(B) \equiv -\int_B d^2x \sqrt{\sigma}(u_i \tau^{ij}\xi_j) \quad (5.3)$$

for the global "charge" Q_ξ, where the component $T^{ni}\xi_i$ of the matter stress tensor serves as its source:

$$Q_\xi(t''\cap^3B) - Q_\xi(t'\cap^3B) = -\int_{^3B} d^3x \sqrt{-\gamma} T^{ni}\xi_i. \quad (5.4)$$

Note that when the surface Killing vector field ξ_i can be extended to a Killing vector field ξ_μ throughout M, Eq. (5.4) can be written as

$$Q_\xi(t''\cap^3B) - Q_\xi(t'\cap^3B) = -\int_{t'}^{t''} d^3x \sqrt{h} (u_\mu T^{\mu\nu}\xi_\nu).$$

(5.5)

This result follows from expressing the identity $\int d^4x \sqrt{-g} \nabla_\mu(T^{\mu\nu}\xi_\nu) = 0$ in terms of surface integrals over ^3B, t', and t''.

For many applications, the source term on the right-hand side of Eq. (5.4) will vanish either because there is no matter in a neighborhood of the boundary ^3B, or more specifically because the component of T^{ni} in the ξ_i direction vanishes. In this case, Eq. (5.4) describes the conservation of charge: because $t'\cap^3B$ and $t''\cap^3B$ are arbitrary surfaces within ^3B, Q_ξ is independent of the two-surface B used for its evaluation. This independence applies to arbitrary spacelike surfaces B, not just to the slices constituting some given foliation of ^3B. On the other hand, the total energy E, defined by Eq. (4.3), is never conserved in this strong sense. Although E may have the same value on each slice of a carefully chosen foliation, this value will generally differ from the energy for other two-surfaces.

The distinction between the charges Q_ξ and energy E is clarified by using definitions (4.1a) and (4.1b) to write the energy-momentum current density as

$$-u_j \tau^{ij} = \varepsilon u^i + j^i. \quad (5.6)$$

The charge (5.3) then becomes

$$Q_\xi = \int_B d^2x \sqrt{\sigma}(\varepsilon u^i + j^i)\xi_i. \quad (5.7)$$

Now consider the situation in which a Killing vector field ξ_i exists that is timelike, has unit length ($\xi^i\xi_i = -1$), and is also surface forming. Then ξ_i is the unit normal to a particular foliation of the three-boundary ^3B, and on each slice of this foliation, the conditions $\xi_i = u_i$ and $\xi_i j^i = 0$ hold. Comparing the energy expression Eqs. (4.3) with the charge (5.7) shows that the energy E associated with such a slice equals minus the charge Q_ξ. For other slices that are not orthogonal to the Killing vector field ξ_i, the associated energy will generally differ from $-Q_\xi$.

An important example of charge is angular momentum, which is defined whenever the boundary three-metric admits a rotational symmetry. In this case, denote the Killing vector field by ϕ^i and the charge by $J \equiv Q_\phi$. If the boundary B used to compute J is chosen to contain the orbits of ϕ^i, so ϕ^i is tangent to B, then according to Eq. (5.7) the angular momentum can be written as

$$J = \int_B d^2x \sqrt{\sigma} j_i \phi^i. \quad (5.8)$$

This expresses the total angular momentum as the integral over a two-surface B, with unit normal orthogonal to ϕ^i, of the momentum surface density in the ϕ^i direction. Observe that J is minus the value of the Hamiltonian (4.13) that generates a rotation along ϕ^i; that is, minus the value of the Hamiltonian with $N = 0$ and $V^i = \phi^i$ on the boundary. From Eq. (4.1b), which defines j_a as a functional derivative, J can be identified as the change in the classical action due to a "twist" in the boundary three-metric. The expression (5.8) for J also agrees with

the ADM angular momentum at infinity for asymptotically flat spacetimes [6].

According to the previous discussion, any change in angular momentum is governed by $T^{ni}\phi_i$, which is the flux across the boundary ^3B of the matter momentum in the ϕ^i direction. If $T^{ni}\phi_i$ vanishes, then the angular momentum is conserved. A related and important property of angular momentum holds whenever the Killing vector field ϕ^i on ^3B can be extended throughout M. In particular, choose a slice Σ containing the orbits of ϕ^μ, so that $u \cdot \phi = 0$ and ϕ^i is a Killing vector field on Σ. Next, write the momentum surface density (4.7) as simply $j^i = -2\sigma^i_k n_l P^{kl}/\sqrt{h}$, where "cl" has been omitted and we have used the flat reference space for which $P^{kl}|_0$ vanishes. Then from Eq. (5.8) the angular momentum becomes

$$J = -2\int_B d^2x\sqrt{\sigma}\, n_i P^{ij}\phi_j/\sqrt{h} = \int_\Sigma d^2x(-2D_iP^{ij})\phi_j.$$

$$(5.9)$$

The term in parentheses is just the gravitational contribution (4.12b) to the momentum constraint, which in general reads

$$0 = -2D_iP^{ij} - \sqrt{h}\, T^{uj},$$

$$(5.10)$$

with $T^{uj} \equiv -u_\mu T^{\mu\nu}h_\nu^{\ j}$ denoting the proper matter momentum density. Therefore, the total angular momentum is

$$J = \int_\Sigma d^3x\sqrt{h}\, T^{uj}\phi_j,$$

$$(5.11)$$

the matter momentum density in the ϕ^μ direction, integrated over a slice Σ that respects the spacetime symmetry ϕ^μ.

The above results reveal a similarity between angular momentum J and the total electric charge enclosed by a boundary B. With the matter momentum density in the ϕ^μ direction $T^{uj}\phi_j$ playing the role of electric charge density, Eq. (5.11) expresses the total charge J as an integral over space of the charge density. The surface integral expression (5.8) for J is then analogous to the integral form of Gauss's law; Gauss's law expresses the total charge as a surface integral of the (radial) electric field, whereas Eq. (5.8) expresses the total angular momentum as a surface integral of the gravitational field $j_i\phi^i$.

Expression (5.11) for J implies that the total angular momentum of any axisymmetric, vacuum spacetime region vanishes. This result applies in particular to the Kerr black-hole solution, but deserves further comment in that case. Recall that the spatial sections $t = $const of the Kerr geometry, where t is the Boyer-Lindquist stationary time coordinate, contain the axial Killing vector field. These slices have the topology of a Wheeler wormhole, $R \times S^2$. Therefore a single surface surrounding the black hole does not constitute a complete boundary for a region of space Σ contained in a $t = $const slice. That is, as assumed above, B should consist of two disjoint surfaces at different "radii," and expression (5.8) for J includes surface integral contributions from both surfaces. These contributions cancel, giving $J = 0$ in agreement with the result obtained from Eq. (5.11). The angu-

lar momentum of a Kerr black hole is more appropriately defined by a single surface integral $\int d^2x\sqrt{\sigma}\, j_i\phi^i$ over some topologically spherical surface surrounding the hole. This is the natural definition of J for, say, a star with spatial topology R^3, so with this definition the angular momentum of stars and black holes are treated on an equal footing.

VI. PROPERTIES OF THE ENERGY

One simple property that the *quasilocal* energy possesses is additivity. That is, consider space to consist of two possibly intersecting regions Σ_1 and Σ_2, and assume that Σ_1, Σ_2, $\Sigma_1 \cup \Sigma_2$, and $\Sigma_1 \cap \Sigma_2$ all have smooth boundaries so their energies can be computed from expression (4.8). It follows that the energy satisfies

$$E(\Sigma_1 \cup \Sigma_2) = E(\Sigma_1) + E(\Sigma_2) - E(\Sigma_1 \cap \Sigma_2),$$

$$(6.1)$$

because the contributions from the common boundary of any two adjacent regions will cancel. As a particular example, let Σ_1 be topologically a ball with a two-sphere boundary, and let Σ_2 be topologically a thick shell surrounding Σ_1. In this case $\Sigma_1 \cap \Sigma_2$ is empty and the total energy of the ball $\Sigma_1 \cup \Sigma_2$ is just the sum $E(\Sigma_1) + E(\Sigma_2)$.

In order to gain some intuition for the quasilocal energy, consider the case of a static, spherically symmetric spacetime

$$ds^2 = -N^2dt^2 + h^2dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$(6.2)$$

where N and h are functions of r only. Let Σ be the interior of a $t = $const slice with two-boundary B specified by $r = R = $const. For simplicity, set Newton's constant to unity, $\kappa = 8\pi$. A straightforward calculation of the extrinsic curvature k_{ab} yields

$$k^\theta_\theta = k^\phi_\phi = -\frac{1}{rh}\bigg|_R.$$

$$(6.3)$$

The acceleration of the timelike unit normal u^μ satisfies

$$n \cdot a = \frac{N'}{Nh}\bigg|_R,$$

$$(6.4)$$

where N' is the derivative of N with respect to r. The subtraction term Eq. (4.5) is given by

$$S^0 = \frac{1}{4\pi}\int dt\, d\theta\, d\phi\, NR\sin\theta,$$

$$(6.5)$$

which is obtained by using k from Eq. (6.3) with $h = 1$.

From the above results, the proper energy surface-density (4.4a) becomes

$$\varepsilon = \frac{1}{4\pi}\left[\frac{1}{r} - \frac{1}{rh}\right]\bigg|_R,$$

$$(6.6)$$

while the proper momentum surface density (4.4b) vanishes. The trace of the spatial stress (4.4c) is given by

$$\sigma_{ab}s^{ab} = \frac{1}{8\pi}[2(n \cdot a) - k]|_{cl} - \frac{2}{N\sqrt{\sigma}}\sigma_{ab}\frac{\delta S^0}{\delta\sigma_{ab}},$$

$$(6.7)$$

where the functional derivative of S^0 can be obtained from

$$\frac{\delta S^0}{\delta R} = \frac{\partial \sigma_{ab}}{\partial R} \frac{\delta S^0}{\delta \sigma_{ab}} = \frac{2}{R} \sigma_{ab} \frac{\delta S^0}{\delta \sigma_{ab}}. \tag{6.8}$$

Combining these last two equations gives

$$\sigma_{ab} s^{ab} = \frac{1}{4\pi} \left[\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r} \right] \Bigg|_R. \tag{6.9}$$

Also, the quasilocal energy (4.3) is

$$E = (r - r/h)|_R. \tag{6.10}$$

From the discussion in Sec. V, it follows that the conserved charge associated with the timelike static Killing vector field with unit normalization at the boundary is equal to minus the energy for a t=const slice; that is, minus the energy computed in Eq. (6.10). Similarly, the vanishing of j_a for the t=const slices shows that the angular momentum (5.8) is zero.

For a simple isentropic fluid with energy density $\rho(r)$ and pressure $p(r)$, the Hamiltonian constraint $G_t^t = -8\pi\rho$ implies [9]

$$h = (1 - 2m/r)^{-1/2}, \tag{6.11}$$

where

$$m(r) = 4\pi \int_0^r d\bar{r}\, \bar{r}^2 \rho(\bar{r}) + M. \tag{6.12}$$

Similarly, the Einstein equation $G_r^r = 8\pi p$ gives [9]

$$\frac{N'}{N} = \frac{m + 4\pi r^3 p}{r^2 - 2mr}. \tag{6.13}$$

The Schwarzschild black-hole solution is obtained by choosing $\rho = p = 0$ and $m = M$, whereas a fluid star solution with $\rho \neq 0$ must have $M = 0$ for the geometry to be smooth at the origin. In each case, the energy is

$$E = R \left[1 - \left[1 - \frac{2m(R)}{R} \right]^{1/2} \right] \tag{6.14}$$

with $m(R)$ defined in Eq. (6.12). Observe that for a compact star or black hole, $m(R)$ is finite as $R \to \infty$. Thus the energy (6.14) becomes $E \to m(\infty)$ in this limit, which is precisely the ADM energy at infinity [13].

The Newtonian approximation for E consists in assuming m/R to be small, which yields

$$E \approx m + m^2/2R. \tag{6.15}$$

In this same approximation the first term, $m(R)$, is just the sum of the matter energy density plus the Newtonian gravitational potential energy associated with assembling the ball of fluid by bringing the individual particles together from infinity [13]. The second term in Eq. (6.15), namely, $m^2/2R$, is just *minus* the Newtonian gravitational potential energy associated with building a spherical shell of radius R and mass m, by bringing the individual particles together from infinity. Thus, in the Newtonian approximation, the energy E has the natural interpretation as the sum of the matter energy density plus the potential energy associated with assembling the ball of fluid

by bringing the particles together from the boundary of radius R. In this sense, E is the total energy of the system contained within the boundary, reflecting precisely the energy needed to create the particles, place them in the system, and arrange them in the final configuration. Any energy that may be expended or gained in the process of bringing the particles to the boundary of the system, say, from infinity, is irrelevant.

A related example is obtained by solving expression (6.14) for $m(R)$, which yields

$$m(R) = E - E^2/2R. \tag{6.16}$$

If the boundary R is outside the matter, then $m(R) = m(\infty) = E(\infty)$ is the total energy at infinity. Then using the additivity of the quasilocal energy, Eq. (6.16) expresses the energy at infinity as the sum of the energy E within the radius R and the energy $-E^2/(2R)$ outside the radius R. The energy outside R is negative, and in fact equals the Newtonian gravitational binding energy associated with building a shell of mass E and radius R. For a charged spherically symmetric distribution of matter, the corresponding analysis yields

$$E(\infty) = E - \frac{E^2}{2R} + \frac{Q^2}{2R}, \tag{6.17}$$

where Q is the total electric charge. In this case, the energy outside R consists of two contributions, the negative gravitational binding energy $-E^2/(2R)$ and the positive electrostatic binding energy $+Q^2/(2R)$ associated with building a shell of charge Q and radius R.[2]

As a final example, consider a black-hole solution $m = M$. If the radius R and mass M are changed in such a solution, the energy (6.14) varies according to

$$dE = \left[1 - \frac{1 - M/R}{\sqrt{1 - 2M/R}} \right] dR + \frac{1}{\sqrt{1 - 2M/R}} dM. \tag{6.18}$$

Now define the surface pressure by

$$s \equiv \frac{1}{2} \sigma_{ab} s^{ab} = \frac{1}{8\pi R} \left[\frac{1 - M/R}{\sqrt{1 - 2M/R}} - 1 \right], \tag{6.19}$$

where Eq. (6.13) with $p = 0$ has been inserted into the trace (6.9) of the spatial stress. The change in energy now becomes

$$dE = -s\, d(4\pi R^2) + (8\pi M \sqrt{1 - 2M/R})^{-1} d(4\pi M^2), \tag{6.20}$$

which is the first law of mechanics for static, spherically symmetric black holes. This result also has an interpretation as the first law of thermodynamics for Schwarzschild

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[2]The quasilocal energy of Katz, Lynden-Bell, and Israel (see Ref. [1]) is similar to our expression (4.8), but includes a factor of the lapse function N on B. When applied to a static, charged, spherically symmetric matter distribution, the Newtonian limit of this quasilocal energy includes gravitational and electrostatic binding energy contributions of the *same* sign.

black holes [3,5]. Accordingly, the boundary surface area $4\pi R^2$ and the surface pressure s are thermodynamically conjugate variables, and $(4\pi M^2)$ is the Bekenstein-Hawking entropy of the black hole (with $\hbar$ and Boltzmann's constant set equal to unity). The quantity $(8\pi M\sqrt{1-2M/R})^{-1}$ is the Hawking black-hole temperature blueshifted from infinity to the finite radius R.

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APPENDIX: KINEMATICS

The spacetime metric is $g_{\mu\nu}$, and u^μ is the future pointing timelike unit normal for a family of spacelike hypersurfaces Σ that foliate spacetime. The normal is proportional to the gradient of a scalar function t that labels the hypersurfaces, so that $u_\mu = -Nt_{,\mu}$ where N is the lapse function fixed by the condition $u\cdot u = -1$. A vector field T^μ (or tensor field, in an obvious generalization) is said to be "spatial" (or tangent to Σ) if it satisfies $u\cdot T = 0$. The metric on Σ is defined as the spatial tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \tag{A1}$$

which is induced on Σ by the spacetime metric $g_{\mu\nu}$. Note that $h^\nu_\mu = g^{\nu\sigma}h_{\sigma\mu}$ is the identity for spatial tensors.

The covariant derivative D_μ compatible with the spatial metric $h_{\mu\nu}$ that acts on spatial tensors is defined by projecting the spacetime covariant derivative ∇_μ. That is, D_μ is defined by $D_\mu f = h^\alpha_\mu \nabla_\alpha f$ for any scalar function f, $D_\mu T^\nu = h^\alpha_\mu h^\nu_\beta \nabla_\alpha T^\beta$ for any spatial vector T^μ, and similarly for higher rank tensors. The extrinsic curvature of Σ as embedded in M is defined by

$$K_{\mu\nu} = -\tfrac{1}{2}\pounds_u h_{\mu\nu} \tag{A2a}$$

$$= -h^\alpha_\mu \nabla_\alpha u_\nu, \tag{A2b}$$

where $\pounds_u$ is the Lie derivative along u^μ. The expression (A2b) for $K_{\mu\nu}$ is symmetric in μ and ν because the unit normal u^μ is surface forming, and has vanishing vorticity $h^\alpha_\mu \nabla_\alpha u_\nu - h^\alpha_\nu \nabla_\alpha u_\mu = 0$.

It is convenient to introduce coordinates that are adapted to the foliation by choosing t as the time coordinate. Then the space coordinates x^i, $i=1,2,3$ lie in the surfaces Σ and $\partial/\partial x^i$ are spatial vectors. In these adapted coordinates, the normal satisfies $u_\mu = -N\delta^0_\mu$, and spatial vector fields T^μ have vanishing contravariant time components, $T^0 = 0$. From the definition (A1), it follows that the space components h^{ij} of the contravariant tensor $h^{\mu\nu}$ form the matrix inverse of the metric components h_{ij}, so that $h_{ik}h^{kj} = \delta^j_i$. The space components of a spatial vector are raised and lowered with h_{ij} and its inverse h^{ij}, since $T_i \equiv g_{i\nu}T^\nu = h_{ij}T^j$ and $T^i \equiv g^{i\nu}T_\nu = h^{ij}T_j$. In particular, the spacetime tensors $D_\mu f$, $D_\mu T^\nu$, and $K_{\mu\nu}$ are tangent to Σ, so $D_i f$, $D_i T^j$, and K_{ij} are tensors on Σ with indices raised and lowered by the metric h_{ij}.

Using the adapted coordinates, the spacetime metric can be written according to the usual ADM decomposition [6],

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= (h_{\mu\nu} - u_\mu u_\nu)dx^\mu dx^\nu$$
$$= -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt), \tag{A3}$$

where $V^i = h^i_0 = -Nu^i$ is the shift vector. Also, the space components of the extrinsic curvature (A2) become

$$K_{ij} = -\frac{1}{2N}[\dot{h}_{ij} - 2D_{(i}V_{j)}], \tag{A4}$$

where the dot in $\dot{h}_{ij}$ denotes a derivative with respect to the coordinate t. In addition, define the momentum for the hypersurfaces Σ as

$$P^{ij} = \frac{1}{2\kappa}\sqrt{h}\,(Kh^{ij} - K^{ij}), \tag{A5}$$

where $h = \det(h_{ij})$. This definition is appropriate if, as we assume, the matter fields are minimally coupled to gravity. (That is, the matter action does not contain derivatives of the metric tensor.)

The intrinsic and extrinsic geometry of the three-boundary ^3B are defined analogously to the above definitions for the family of hypersurfaces Σ. However, in this case, the three-boundary ^3B is not treated as a member of a foliation of the spacetime M. (The spacetime topology may prohibit the extension of ^3B into a foliation throughout all of M.) Let n^μ denote the outward pointing spacelike normal to the boundary ^3B and define the metric on ^3B by

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \tag{A6}$$

Likewise, define the extrinsic curvature by

$$\Theta_{\mu\nu} = -\gamma^\alpha_\mu \nabla_\alpha n_\nu. \tag{A7}$$

Let $\mathcal{D}_\mu$ denote the induced covariant derivative for tensors that are tangent to ^3B, defined by projecting the spacetime covariant derivative onto ^3B. Introducing intrinsic coordinates x^i, $i=0,1,2$, on the three-boundary, the intrinsic metric becomes γ_{ij}. Tensors tangent to ^3B, such as the extrinsic curvature (A7), can be written as Θ_{ij} with indices raised and lowered by γ_{ij} and its inverse γ^{ij}. Also define the boundary momentum by

$$\pi^{ij} = -\frac{1}{2\kappa}\sqrt{-\gamma}(\Theta\gamma^{ij} - \Theta^{ij}), \tag{A8}$$

where $\gamma = \det(\gamma_{ij})$.

Recall that the hypersurface foliation is restricted by the condition $(u\cdot n)|_{^3B} = 0$. With this in mind, define the metric on the two-boundaries B, which are the intersections of ^3B and the family of slices Σ, as

$$\sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu$$
$$= \gamma_{\mu\nu} + u_\mu u_\nu$$
$$= g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu. \tag{A9}$$

Also define the extrinsic curvature of B as embedded in Σ by

$$k_{\mu\nu} = -\sigma_\mu^\alpha D_\alpha n_\nu, \tag{A10}$$

where D_α is the covariant derivative on Σ. In coordinates that are adapted to the foliation Σ and the three-boundary ^3B, the line element on ^3B is

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt), \tag{A11}$$

where x^a, $a = 1,2$, are the coordinates on B. Note that $\sigma_{\mu\nu}$ and $k_{\mu\nu}$ are defined only on ^3B.

We will now outline the steps involved in expressing the extrinsic curvature of the three-boundary ^3B in terms of the intrinsic and extrinsic geometry of spacetime foliated into spacelike hypersurfaces Σ. The derivation makes repeated use of the restriction that on ^3B the hypersurface normal u^μ and the three-boundary normal n^μ are orthogonal. With this condition, n^μ is a unit normal for both the three-boundary ^3B embedded in spacetime M, and for the two-boundaries B as embedded in the hypersurfaces Σ.

The identity tensor, expressed as $\delta_\mu^\alpha = (h_\mu^\alpha - u_\mu u^\alpha)$, can be used to split $\Theta_{\alpha\beta}$ into tensors whose free indices are projected tangentially or normally to the hypersurfaces Σ. This results in

$$\Theta_{\mu\nu} = -h_\mu^\alpha h_\nu^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta - u_\mu u_\nu u^\alpha u^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta$$
$$+ 2h_{(\mu}^\alpha u_{\nu)} u^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta. \tag{A12}$$

Because $u \cdot n = 0$ on ^3B, the projections onto ^3B and Σ commute, and the first term on the right-hand side of Eq. (A12) becomes

$$-h_\mu^\alpha h_\nu^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta = -\gamma_\mu^\alpha h_\nu^\beta h_\alpha^\rho \nabla_\rho n_\beta$$
$$= -\sigma_\mu^\alpha D_\alpha n_\nu$$
$$= k_{\mu\nu}. \tag{A13}$$

By definition (A10), this is the extrinsic curvature of the two-boundary B as embedded in Σ.

For the second term on the right-hand side of Eq. (A12), observe that the hypersurface normal u^μ lies in the three-boundary, so that on ^3B, $u^\alpha \gamma_\alpha^\rho = u^\rho$. Also use the relationship $u^\rho u^\beta \nabla_\rho n_\beta = -u^\rho n_\beta \nabla_\rho u^\beta$, which is derived by differentiating $u \cdot n |_{3_B} = 0$. Then the second term in Eq. (A12) becomes

$$-u_\mu u_\nu u^\alpha u^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta = u_\mu u_\nu n_\beta a^\beta, \tag{A14}$$

where $a^\beta \equiv u^\rho \nabla_\rho u^\beta$ is the acceleration of the timelike hypersurface normal u.

The third term on the right-hand side of Eq. (A12) is simplified by recognizing that $\gamma_\alpha^\rho u^\beta \nabla_\rho n_\beta = -\gamma_\alpha^\rho n^\beta \nabla_\rho u_\beta$, and by using the relationship $h_\mu^\alpha \gamma_\alpha^\rho = \sigma_\mu^\alpha h_\alpha^\rho$ on ^3B. This gives

$$2h_{(\mu}^\alpha u_{\nu)} u^\beta \gamma_\alpha^\rho \nabla_\rho n_\beta = -2\sigma_{(\mu}^\alpha u_{\nu)} n^\beta h_\alpha^\rho \nabla_\rho u_\beta$$
$$= 2\sigma_{(\mu}^\alpha u_{\nu)} n^\beta K_{\alpha\beta}, \tag{A15}$$

where $K_{\alpha\beta}$ is the extrinsic curvature, defined in Eq. (A2), for the hypersurfaces Σ.

Collecting these results together, the boundary extrinsic curvature is expressed as

$$\Theta_{\mu\nu} = k_{\mu\nu} + u_\mu u_\nu n_\beta a^\beta + 2\sigma_{(\mu}^\alpha u_{\nu)} n^\beta K_{\alpha\beta}. \tag{A16}$$

It immediately follows that the trace of the boundary extrinsic curvature is

$$\Theta = k - n_\beta a^\beta. \tag{A17}$$

Equation (A16) shows that the projection of $\Theta_{\mu\nu}$ onto B is the two-boundary extrinsic curvature $k_{\mu\nu}$, while the projection of $\Theta_{\mu\nu}$ along the normal u^μ is $n_\alpha a^\alpha$. The "off-diagonal" projection of $\Theta_{\mu\nu}$ is given by the "off-diagonal" projection of the hypersurface extrinsic curvature $K_{\mu\nu}$, according to the relationship $\sigma_\alpha^\mu u^\nu \Theta_{\mu\nu} = -\sigma_\alpha^\nu n^\nu K_{\mu\nu}$. The space-time split of $\Theta_{\mu\nu}$ also can be written in terms of the hypersurface momentum (A5) and the boundary momentum (A8) as

$$\pi^{ij} = \frac{N\sqrt{\sigma}}{2\kappa}[k^{ij} + (n \cdot a)\sigma^{ij} - k\gamma^{ij}] - \frac{2N\sqrt{\sigma}}{\sqrt{h}}\sigma_{\underline{k}}^{(i} u^{j)} P^{\underline{kl}} n_{\underline{l}}. \tag{A18}$$

In this equation, it is necessary to distinguish tensor indices that refer to coordinates on ^3B, which are denoted by i and j, from tensor indices that refer to coordinates on the slices Σ, which are denoted by $\underline{k}$ and $\underline{l}$.

The final mathematical ingredient needed for our analysis is the space-time split of the curvature scalar $\mathcal{R}$. This is obtained from the decomposition

$$\mathcal{R} = h^{\mu\nu} h^{\alpha\beta} \mathcal{R}_{\mu\alpha\nu\beta} - 2u^\mu u^\nu \mathcal{R}_{\mu\nu}. \tag{A19}$$

The Gauss-Codazzi relation [9] for the projection of the Riemann tensor onto Σ gives the first term of Eq. (A19) as $h^{\mu\nu} h^{\alpha\beta} \mathcal{R}_{\mu\alpha\nu\beta} = R + (K)^2 - K_{\mu\nu} K^{\mu\nu}$. The second term of Eq. (A19) is rewritten using the Ricci identity $\mathcal{R}_{\alpha\mu\beta\nu} u^\nu = 2\nabla_{[\alpha} \nabla_{\mu]} u_\beta$; contracting with $u^\mu g^{\alpha\beta}$ and rearranging derivatives gives $u^\mu u^\nu \mathcal{R}_{\mu\nu} = (K)^2 - K_{\mu\nu} K^{\mu\nu} + \nabla_\mu (K u^\mu + a^\mu)$. Together, these results yield

$$\mathcal{R} = R + K_{\mu\nu} K^{\mu\nu} - (K)^2 - 2\nabla_\mu (K u^\mu + a^\mu). \tag{A20}$$

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